Revista Brasileira de Física, Vol. 11, nº 1, 1981

On-the-Energy-Shell Approximation for the Heavy Ion Coupled-Channels Problems (I)

Consequences on the Sub-Barrier Elastic Scattering †

BRETT V. CARLSON

Physics Department, University of Wisconsin-Madison, Madison, WI 53706 USA

and

MAHIR S. HUSSEIN

Physics Department, University of Wisconsin-Madison, Madison, WI 53706 USA
and
*Instituto de Física, Universidade de São Paulo, CP 20526, São Paulo, SP, Brasil**

Recebido em 7 de Agosto de 1980

Starting with the coupled channels equations describing multiple Coulomb excitations in heavy ion collisions we develop an approximation scheme based on replacing the channel Green's functions by their on-the-energy shell forms, which permits an exact analytic solution for the scattering matrix. We construct the trivially equivalent Coulomb polarization potential valid for strong coupling and small energy loss in the excitation processes. This potential is seen to have a very simple r-dependence. A simple formula for the sub-barrier elastic scattering cross section is then derived both by using the WKB approximation and by summing the Born series for the T-matrix. Comparison of the two forms for the elastic cross section shows that they give almost identical numerical results in the small coupling limit only. We also compare our results with the predictions of the Alder-Winther theory.

---

† Supported in part by the Tinker Foundation, the U.S.-N.S.F., the CNPq Brazil and FAPESP-Brazil

* Perrnanent address.

A partir das equações acopladas que descrevem as excitações múltiplas Coulombianas em íons pesados, desenvolvemos uma aproximação baseada no uso da forma "on-energy-shell" das funções de Green dos canais, que permitem achar uma solução analítica para a matriz ,de espalhamento. Construímos o potencial local equivalente Coulombiano válido para acoplamento forte e pequenas energias de excitação. Este potencial tem uma de~endenciaem ir bastante simples. Calculamos então uma fórmula simples para a secção de choque de espalhamento elástico usando tanto a WKB como a soma de Born. Os dois métodos (WKBe a soma de Born),apresentam resultados quase idênticos apenas para o caso de acoplamento fraco. Comparamos também os resultados com a teoria Alder-Winther.

1. INTRODUCTION

Multistep processes are the rule rather than the exceptíon in heavy ion collision phenomena[1]. To deal with these processes one has to perform a coupled channels calculation which becomes prohibitively costly as tne nuinber of channels increases. Severa1 approximation schemes have been deveioped which have in common the basic aim of reducing the computation time. In particular, due to the short wave lengths thot characterize these reactions, several forms of semiclassical approximations were developed and put into test recently giving overall reasonable results. Of these theories we mention that developed by the Copenhagem group[2] and the methods based on ttie work cf Miller[3] developed primarily by the Berkeley group[4]. However, the intrinsic difficulty of performing a many coupled channel calculation is not completely overcome by these methods, The Llinther-de Boer code for Coulomb excitation cari handle few channels; the method developed by the Berkeley group can, in principle, handle a larger number of coupled channels but is restricted to cases where a classical hamiltonian function can be constructed. Most of the numerical results given in Ref.4 were obtained for back angle scattering in which case the resulting geometrical simplification makes possible a speedy calculation. Extension of the methods of Ref.4 to three dimensions is still pending[5]

In the present series of papers we try an alternative method which is based on the use of the on-energy-shell form of the channel

Green's functions. Although several authors have discussed this method in several contexts[6], we believe that a study of the full consequence of the OES approximation is called for. Moreover, this approximation scheme has been adopted recently[7] to extend the closed formalism approach developed by Frahn[8] by including specific multistep processes in heavy ion collision phenomena. An important test is a comparison of the range of applicability of the OES approximation with the already established results of multiple Coulomb excitation theory[9].

The paper is organized as follows: In Section 2 we formulate the coupled-channels problem for multiple Coulomb excitation. In Section 3 we introduce the on-energy-shell approximation for the channel Green's functions and derive an expression for the locally equivalent Coulomb polarization potential (LECPP) in closed form. In Section 4 we demonstrate, through numerical calculation, that the cross section for sub-barrier elastic scattering calculated using the WKB approximation and using the LECPP is almost identical to that calculated by summing the Born series for the elastic amplitude without any references to the LECPP. Discussion of our results as well as suggestions for possible improvements and the conclusions are given in Section 5.

2. THE QUANTUM MECHAMICAL COUPLED CHANNEL EQUATIONS FOR MULTIPLE COULOMB EXCITATION

We consider the collision of a spherical nucleus 1 on a deformed tarqet nucleus 2, at sub-barrier energies. We study the Coulomb excitation of low-lying states IM of spin I and magnetic quantum number M with excitation energy E_I. We consider hucleus 1 as a point charge, $Z_1 e$. For the Total wave function in thé center of mass system we use the expansion[9]

$$|\psi> = \sum_{\substack{\ell_0 \ell \\ IJN}} \frac{\sqrt{\pi(2\ell_0+1)}}{k_{I_0} r} i^{\ell_0+1} e^{i\sigma_{\ell_0}(\eta_{I_0})} <\ell_0\ 0\ I_0 M_0 | JN> \times \psi^{\ell_0 I_0}_{(\ell I)J}(r)\ |(\ell I)JN> \qquad (1)$$

where the channel wave functions are given by

$$|(\ell I)JN\rangle = \sum_{Mm} \langle \ell m\ IM|JN\rangle\ |IM\rangle\ Y_{\ell m}(\hat{r}) \ . \tag{2}$$

Here J is the' total conserved channel angular momentum and N its projection in the Z-axis. The relative position of the centers of mass of 1 and 2 is denoted by r while the superindices $\ell_0 I_0$ indicate the initial condition. Inserting the wave function (1) into the time-independent Schrödinger equation

$$(H-E)|\psi\rangle = 0 \ . \tag{3}$$

we obtain a system of coupled equations for the radial wave functions

$$\left\{ \frac{d^2}{dr^2} + k_1^2 - \frac{2\mu}{\hbar^2} \frac{\ell(\ell+1)}{2\mu r^2} - \frac{2\mu}{\hbar^2} \frac{Z_1 Z_2 e^2}{r} \right\} \psi^{\ell_0 I_0}_{(\ell I)J}(r) =$$

$$= \sum_{\ell' I'} V^J_{\ell I,\ \ell' I'}(r)\ \psi^{\ell_0 I_0}_{(\ell' I')J}(r) \tag{4}$$

where

$$k_I^2 = \frac{2\mu}{\hbar^2} (E-E_I)$$

The coupling potentiai $V^J_{\ell I, \ell' I'}$ is given by

$$V^J_{\ell I, \ell' I'}(r) = \frac{4\sqrt{\pi} Z_1 e \mu}{\hbar^2} \sum_{\lambda=1} \frac{(2\ell+1)(2\ell'+1)}{(2\lambda+1)} (-1)^{J+I'+\lambda}$$

$$\times \langle I'||M(E\lambda)||I\rangle \begin{pmatrix} \ell & \ell' & \lambda \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} J & \ell' & I' \\ \lambda & I & \ell \end{Bmatrix} \frac{1}{r^{\lambda+1}} \tag{5}$$

where $\langle I'||M(E\lambda)||I\rangle$ is the reduced matrix element of the electric h-pole moment of the target. We shall, in the following, consider the specific case of a quadrupole-deformed even-even target nucleus 2. Assuming a rotational band structure for the low-lying excited state of 2, the coupling potential matrix $V^{\ell_0}_{\ell I, \ell' I'}(r)$ becomes

$$V^{\ell_0}_{\ell I,\ell' I'}(r) = \sqrt{a_I a_{I'} \eta_I \eta_{I'}}\; q_{I\to I'} \begin{Bmatrix} I & I' & 2 \\ 0 & 0 & 0 \end{Bmatrix} \sqrt{(2I+1)(2I'+1)}$$

$$\times \sqrt{(2\ell+1)(2\ell'+1)}\; (-1)^{\ell_0+I'} \begin{Bmatrix} \ell & \ell' & 2 \\ 0 & 0 & 0 \end{Bmatrix} \tag{6}$$

$$\begin{Bmatrix} \ell & \ell' & I \\ 2 & I & \ell \end{Bmatrix} \frac{1}{r^3} \equiv a_{\ell I \ell' I'} \frac{1}{r^3}$$

where a_I is half the distance of closest approach for head on collision in channel I, $a_I = Z_1 Z_2 e^2/2(E-E_I)$, $\eta_I = Z_1 Z_2 e/\hbar v_I = k_I a_I$ is the Sommerfeld parameter in channel I and $q_{I\to I'}$ is the symmetrized dimensionless quadrupole strength parameter for the coupling $I\to I'$ and is defined by

$$q_{I\to I'} = \sqrt{\frac{\pi}{5}}\; \frac{\sqrt{\eta_I \eta_{I'}}}{a_I\, a'_I} \quad \frac{\langle I||M(E2)||I'\rangle}{Z_2 e}$$

$$\times \frac{1}{\sqrt{(2I+1)(2I'+1)}\; \begin{pmatrix} I & I' & 2 \\ 0 & 0 & 0 \end{pmatrix}} \tag{7}$$

In the limit of a pure quadrupole rotational band and zero energy loss in the different excitation processes we have

$$q_{I\to I'} = (-1)^{\frac{I+I'+2}{2}}\; q_{0\to 2} \tag{8}$$

Once the $\psi^{\ell_0 I_0}_{(\ell I)J}(\mathbf{r})$ are obtained from Eq. (4) the corresponding T-matrices $T^J_{\ell I,\ell_0 I_0}$ are extrasted from their asymptotic forms, i.e.,

$$\psi^{\ell_0 I_0}_{(\ell I)J}(r) \overset{r\to\infty}{=} \delta_{II_0}\,\delta_{\ell\ell_0}\, e^{-i\phi_{\ell I}(r)} - \sqrt{\frac{k_{I_0}}{k_I}}\, e^{i\phi_{\ell I}(r)}\, T^J_{\ell I,\ell_0 I_0} \tag{9}$$

where

$$\phi_{\ell I}(r) = k_I r - \eta_I \ln(2k_I r) - \frac{\pi}{2}\ell + \sigma_L(\eta_I)$$

σ_E being the Coulomb phase shift. With the $T^J_{\ell I, \ell_0 I_0}$ one can then calculate the amplitudes $f_{I_0 M_0 \to IM}(\theta,\phi)$ for Coulomb excitation from the ground state $|I_0 M_0>$ to the final state $|IM>$ (in coordinate system C of Ref. (10))

$$f_{I_0 M_0 \to IM}(\theta,\phi) = \frac{i\sqrt{\pi}}{\sqrt{k_I k_{I_0}}} \sum_{\ell \ell_0 J} \sqrt{(2\ell_0+1)} <\ell_0\ 0\ I_0 M_0 | JM_0> \; i^{\ell_0-\ell} <\ell m\ IM | JM_0> \{\delta_{II_0}\delta_{\ell\ell_0} - e^{i(\sigma_\ell(\eta_I)+\sigma_{\ell_0}(\eta_{I_0}))} \; * \; T^J_{\ell I, \ell_0 I_0}\} Y_{\ell M}(\theta,\phi) \tag{10}$$

The differential cross sections are then calculated as usual.

3. THE ON-ENERGY-SHELL APPROXIMATION AND THE EQUIVALENT LOCAL COULOMB POLARIZATION POTENTIAL

In order to calculate the cross-section for inelastic Coulomb scattering one has to solve Eq.(4) with the appropriate boundary condition of an incorning wave present only in the elastic channel. For fuller details we refer the reader to Refs. 9 and 10. It is important to recognize that with an increasing number of coupled channels the solution to Eq. (4) becomes more and more complicated. Four our purposes we adopt the on-energy-shell approximation for the channel Coulomb Green's function[11]

$$G^{(+)}_{(\ell I)}(r,r') \cong -\frac{i}{k_I} F_\ell(k_I r_<)\, F_\ell(k_I r_>) \tag{11}$$

We shall assess the accuracy of this approximation later. In the above, $F_a(kr)$ is the regular Coulomb function and $r_<(r_>)$ corresponds to the smaller (larger) of r and r'.

The solution of Eq. (4) may be written as an integral, equation

$$\psi_{(\ell I)\ell_0}(r) = F_{\ell_0}(k_0 r)\,\delta_{II_0}\,\delta_{\ell\ell_0} + \sum_{\ell' I'} \int G_{\ell I}(r,r')\, a_{\ell I \ell' I'}\, \frac{1}{r'^3}\, \psi_{(\ell' I')\ell_0}(r')\,dr' \qquad (12)$$

We now use the approximation (11) in (12) and obtain

$$\psi_{(\ell I)\ell_0}(r) = F_{\ell_0}(k_0 r)\,\delta_{II_0}\,\delta_{\ell\ell_0} + \left(-\frac{i}{k_I}\right) F_\ell(k_I r) \sum_{\ell' I'} \int_0^\infty dr'$$

$$\times\, F_\ell(k_I r')\, a_{\ell I \ell' I'}\, \frac{1}{r'^3}\, \psi_{(\ell' I')\ell_0}(r') \qquad (13)$$

Multiplying (12) by $\frac{1}{r}\,\frac{1}{k_{I''}}\,F_{\ell''}(k_{I''}r)$ and integrating over r we obtain the following relation

$$M_{\ell''\ell} = \frac{1}{k_{I''}}\, I_{\ell''\ell}(k_{I''},k_I)\,\delta_{II_0}\,\delta_{\ell\ell_0} + \left(-\frac{i}{k_I}\right) \sum_{\ell'} I_{\ell''\ell}(k_{I''}k_I)\, a_{\ell I \ell' I'}\, M_{\ell\ell'} \qquad (14)$$

where we have defined the coefficients

$$M_{\ell\ell'} \equiv \frac{1}{k_I} \int dr'\, F_\ell(k_I r')\, \frac{1}{r'^3}\, \psi_{(\ell' I')\ell_0}(r') \qquad (15)$$

and the in integrals $I_{\ell''\ell}(k_{I''},k_I)$ are the usual Coulomb excitation integrals[9]

$$I_{\ell''\ell}(k_{I''},k_I) = \int dr\, F_{\ell''}(k_{I''}r)\, \frac{1}{r^3}\, F_\ell(k_I r) \qquad (16)$$

Eq. (14) can easily be solved for $\Sigma_{\ell' I'}\, a_{\ell I \ell' I'}\, M_{\ell\ell'} \equiv Y_{\ell I}$

$$Y_{\ell I} = \sum_{\ell' I'} \{[1 + R\, i\, C]^{-1}\}_{R\blacksquare,2}\, \frac{1}{k_0}\, a_{\ell' 2 \ell_0 0}\, I_{\ell'\ell_0}(k_2,k_0) \qquad (17)$$

where we have introduced the coupling matrix C whose matrix elements are given by

$$C_{\ell''I'',\ell I} = \frac{1}{k_{I''}}\, a_{\ell''I''\ell I}\; I_{\ell''\ell}(k_{I''},k_I) \qquad (18)$$

The matrix $(\underline{1} + i\underline{C})^{-1}$ acts in the full channel space.

Since $a_{\ell' I' \ell_0 0}\, I_{\ell' \ell_0}$ corresponds to the coupling of the elastic channel to the 2+ channel (due to the quadrupole nature of the coupling only the 2+ can couple directly to the 0^+) the values ℓ' are completely determined by the 3-j symbols that appear in $a_{\ell' 2 \ell_0 0}$, i.e., $\begin{pmatrix} \ell_0 & \ell' & 2 \\ 0 & 0 & 0 \end{pmatrix}$. Therefore, ℓ' can be ℓ_0, $\ell_0 + 2$ or $\ell_0 - 2$. This sayr that Eq. (17) may be rewritten as (labeling the matrix elements of $[\underline{1} + \underline{C}]^{-1}$ by the intrinsic spin only)

$$Y_I = [[\underline{1} + i\,\underline{C}]^{-1}]_{I2}\, \underline{C}_{20} \tag{19}$$

The vector Y contains $I+1$ components (R and ■ must always add to $J=\ell_0$). Therefore, the matrix $[\underline{1} + \underline{C}]^{-1}$ is a $(I+1) \times 3$ matrix. Once $Y_{\ell I}$ is evaluated, the wave function $\psi_{\ell I}(r)$, which is also a vector with $I + 1$ components, is then obtained from Eq. (13) vis

$$\psi_{(\ell I)\ell_0}(r) = F_{\ell_0}(k_0 r)\, \delta_{I0}\, \delta_{\ell \ell_0} + (-i)\, F_\ell(k_I r) [\underline{1} + i\,\underline{C}]^{-1}_{I2}\, \underline{C}_{20} \tag{20}$$

Eq. (20) can be used to calculate the trivially equivalent local Coulomb polarization potential in the elastic channel as was done in Refs. 11 and 12. Inserting the second termof Eq. (20) in the r.h.s. of Eq. (4) for $\psi_{(\ell_0 0)\ell_0}(r)$we immediately obtain for the sum

$$\sum_{\ell' I'} V^{\ell_0}_{\ell_0 0 \ell' I'}(r)\, \psi^{\ell_0 0}_{(\ell' I')\ell_0}(r)$$

$$= a_{\ell_0 0 \ell 2}\, \frac{1}{r^3}\, (-i)\, F_\ell(k_2 r)\, [\underline{1} + i\underline{C}(\ell)]^{-1}_{22}\, C_{20}(\ell) \tag{21}$$

$$\equiv \frac{\mu}{\hbar^2}\, V_{\text{opt}}(r)\, F_{\ell_0}(k_0 r)$$

or

$$\frac{2\mu}{\hbar^2}\, V_{\text{opt}}(r) = -i(F(\ell;r))_{02}\, [\underline{1} + i\,\underline{C}(\ell)]^{-1}_{22}\, C_{20}(\ell) \tag{22}$$

where $F(\ell;r)$ is a 3-component vector given by

$$(F(\ell;r))_{02} = \left\{ \frac{F_{\ell_0+2}(k_2 r)}{F_{\ell_0}(k_0 r) r^3} a_{\ell_0 0, \ell_0+2, 2} \; , \; \frac{F_{\ell_0}(k_2 r)}{F_{\ell_0}(k_0 r) r^3} a_{\ell_0 0, \ell_0 2} \right.$$
$$\left. \frac{F_{\ell_0-2}(k_2 r)}{F_{\ell_0}(k_0 r) r^3} a_{\ell_0 0, \ell_0-2, 2} \right\} \qquad (23)$$

Eq. (22) describes the effect of the coupling of the elastic channel to all other inelastic channels. It is a simple realization of the Feshbach theory for the optical potential[13]. Denoting the projection operator that projects out the elastic channel of the full wave function by P and the complementary operator $Q \equiv 1-P$ which projects out all inelastic: channels, we can write Eq. (23) as

$$V_{\text{opt}} = V_{PQ}\left[\underline{1} + i\, \underline{C}_{Q}\right]^{-1} V_{QP} \qquad (24)$$

Therefore, the rnatrix $\left[\underline{1} + i\, \underline{C}_{QQ}\right]^{-1}$ is effectively the Q-space propagator.

In our particular case of multiple Coulomb excitation through the quadrupole coupling, V_{QP} is just V_{20} as only the 2^+ channel is coupled directly to the elastic, 0^+, channel. Within the Q-space the 2^+ channel couples directly only to the 4^+. The 4^+ then couples to the 6^+ state and so on. This suggests that the structure of the matrix $\left[\underline{1} + iC_{QQ}\right]^{-1}$ is

$$\left[\underline{1} + i\, \underline{C}_{QQ}\right]^{-1}_{22} = \left[\underline{1} + i\, \underline{C}_{22} + \underline{C}_{24}\left[\underline{1} + i\, \underline{C}_{Q-2\; Q-2}\right]^{-1} C_{44-42}\right]^{-1} \qquad (25)$$

where the symbol Q-2 refers to the subspace spanned by states in the Q-space except the 2^+. Eq. (25) can easily be derived by expanding the matrix propagator in $\underline{C}$ and inserting whenever calculating products of $\underline{C}$'s, i.e.,

$$
\begin{aligned}
\left[\underline{1} + i\,\underline{C}\right]^{-1}_{22} &= \underline{1} - i\,\underline{C}_{22} - (i)^2\,\underline{C}_{2Q}\,Q\,\underline{C}_{Q2} \\
&\quad - (i)^3\,\underline{C}_{2Q}\,Q\,\underline{C}_{QQ}\,Q\,\underline{C}_{Q2} \\
&= \underline{1} - i\,\underline{C}_{22} - (i)\,(\underline{C}_{22})^2 \\
&\quad - (i)^2\,\underline{C}_{24}\,Q\text{-}2\,\underline{C}_{Q2} \\
&\quad - (i)^3\,\underline{C}_{2Q}\,Q\text{-}2\,\underline{C}_{QQ}\,Q\text{-}2\,\underline{C}_{Q2} + \ldots
\end{aligned}
\tag{26}
$$

Rearranying and summing the series (26) we obtain (25). It should be emphasized that once the matrix propagators is decomposed into propagation in smaller subspace as given in (25) the calculation of V_{opt} becoaes quite simple. Realizing that C_{ij} is an $(i+1)\times(j+1)$ matrix (in magnetic quantum number space) such a calculation can then be easily made by inverting the matrix propagator as was done in Ref.14. In particular if all coupling except the C_{20} were made equal to zero we obtain the potential considered in Refs. 12 and 15

$$
V^{(2)}_{opt}(r) = -\frac{2}{r^3}\,(F(\ell;r)_{02}C_{20}(\ell) \tag{27}
$$

In the followinq we assume zero energy loss in the different excitation processes ($k_I = k_0$). We also utilize the simple relations, valid for large 1, among $F_\ell(kr)$, $F_{\ell+2}(kr)$ and $F_{\ell-2}(kr)$ obtained in Refs. 11 and 12,

$$
\frac{F_{\ell+2}(kr)}{F_\ell(kr)} = -1 + \frac{2}{\ell^2\,\eta^2}\left[\eta^2 + \frac{2\eta\,\ell^2}{kr} + \frac{\ell^4}{k^2r^2}\right] = \frac{F_{\ell-2}(kr)}{F_\ell(kr)} \tag{28}
$$

Finally, we shall use the large R values of the 3-j and 6-j symbols needed in the calculation ana given in Appendix 1. With the above approximations and assumptions the expression for $V^{(2)}_{opt}(r)$ given in (27) reduces to the form derived in Refs. 12 and 15.

$$V_{\ell}^{(2)}(r) = -i\frac{2}{5}\frac{E}{\eta} q_{0\to 2}^2 g_{0\to 2}(\xi_{0\to 2}) \left(\frac{a}{r}\right)^3 \left[\frac{3\bar{\ell}^2+1}{\bar{\ell}^2(\bar{\ell}^2+1)^2} - \frac{1}{\bar{\ell}^3}\arctan\bar{\ell} + \frac{4\bar{\ell}^2}{(\bar{\ell}^2+1)^2}\left(\frac{a}{r}\right) + \frac{2\bar{\ell}^4}{(\bar{\ell}^2+1)^2}\left(\frac{a}{r}\right)^2\right] \tag{29}$$

where we have introduced the quantity $\bar{\ell} \equiv \frac{\ell+1/2}{\eta}$ and have used the explicit forms of the Coulomb integrals $I_{\ell\ell'}(k_I=k_0)$ as given in Ref. 9,

$$I_{\ell\ell+2} = \frac{\eta^2}{[(\ell+1)^2+\eta^2][(\ell+2)^2+\eta^2]} \frac{1}{6a^2} \simeq \frac{1}{6a^2}\frac{1}{\bar{\ell}^2+1} \simeq I_{\ell\ell-2} \tag{30}$$

$$I_{\ell\ell} \simeq \frac{1}{\bar{\ell}^2}\left(1 - \frac{\arctan\bar{\ell}}{\bar{\ell}}\right)\frac{1}{2a^2} \tag{31}$$

We corrected for the $k_I = k_0$ approximation by inserting the semiclassical energy loss factors $g_{I\to I'}(\xi_{I\to I'})$, for the process $I\to I'$, [16] where $\xi_{I\to I'}$ is the adiabaticity parameter, which is given by

$$\xi_{I\to I'} \equiv \eta_{I'} - \eta_I = \eta_0\sqrt{E}\left[\frac{1}{(E-E_{I'})^{1/2}} - \frac{1}{(E-E_I)^{1/2}}\right] \simeq \frac{1}{2}\eta_0\frac{(E_{I'}-E_I)}{E} \tag{32}$$

where η_0 and E are, respectively, the Sommerfeld parameter and the center of mass energy in the elastic channel. E_I is the excitation energyof state I. Numerical values of $g_{I\to I'}(\xi_{I\to I'})$ are given in Ref. 16. It is easily seen that the simple r-dependence of the potential given in (29) also holds for ths more exact potential of Eq. (22) if the same assumptionsand approximations are made,[11] i.e.,

$$iV_{\text{opt}}(r) = \frac{a_\ell}{r^3} + \frac{b_\ell}{r^4} + \frac{c_\ell}{r^5} \tag{33}$$

This is clear since the r-dependence of $V_{\text{opt}}(r)$ is contained only in the vector $F(r;\ell)$. The propagator $[1 + iC(\ell)]^{-1}$, which is the quantity used to get approximate forms for $V_{\text{opt}}(r)$, does not depend on r. In Eq. (33) the complex coefficients, a_ℓ, b_ℓ and c_ℓ depend on the orbital angular momentum, ℓ, the Sommerfeld parameter, η, the center of mass energy, E,

and the quadrupole coupling strength q_{ij}. In Ref. 15 the potential $V_{opt}(r)$ was derived using the semiclassical theory of Coulomb excitation developed by Alder and Winther[9]. The R- and r-dependence of $V_{opt}(r)$ of Eq. (33) and that of Ref. 15 are quite different. One advantage of $V_{opt}(r)$ of Eq. (33) is its simple r-dependence as well as the explicit L-dependente which can be obtained by inverting the matrix $(\underline{1}+i\underline{C}(\ell))$ in Eq. (22). We shall give below the result for the case of the coupling of the 0^+ channel to the 2+ channel including the reorientation of the 2+ to all orders. We shall also present the result when including the coupling to the 4+ channel. In order to perform these calculations we first rewrite Eq. (25) in the equivalent, but more transparent form

$$\left[\underline{1} + i\underline{C}(\ell)\right]^{-1}_{22} = \left[1 + i\underline{C}_{22}(\ell) + \underline{C}_{24}(\ell)\left[1 + i\underline{C}_{44}(\ell) + \underline{C}_{46}(\ell)\left[..\right]^{-1}\underline{C}_{42}(\ell)\right]^{-1}\right. \tag{34}$$

In the above each coupling matrix $\underline{C}_{ij\neq i}(\ell)$ contains an appropriate energy loss factor $\sqrt{g_{ij}(\xi_{ij})}$ whereas the reorientation matrices $\underline{C}_{ii}(\ell)$ do not. Therefore, the Coulomb polarization potential for the two channel case with reorientation is given by

$$V^{(2)\text{Reor.}}_{opt}(r) = -\frac{i}{r^3}\left(\frac{\hbar^2}{2\mu}\right)\left(F(r;\ell)\right)_{02}\left[\underline{1} + i\,\underline{C}_{22}(\ell)\right]^{-1}\underline{C}_{20}(\ell) \tag{35}$$

The matrix $\underline{C}_{22}(\ell)$ has the form

$$C_{22}(\ell) = \frac{2}{7}\,q_{2\to 2}\begin{pmatrix} \alpha & \beta & 0 \\ \beta & -\alpha & \beta \\ 0 & \beta & \alpha \end{pmatrix} \tag{36}$$

where we have defined

$$\alpha \equiv -\frac{1}{\bar{\ell}^2}\left[1 - \frac{\tan^{-1}\bar{\ell}}{\bar{\ell}}\right]$$
$$\beta \equiv \frac{1}{\sqrt{6}}\,\frac{1}{1+\bar{\ell}^2} \tag{37}$$

Since $\underline{C}_{22}(\ell)$ is symmetric and real we could immediately invert the propagator and write down the expressions for the real and imaginary part of $V_{opt}^{(2)Reor}$ utilizing the form for $\underline{C}_{20}(E)$

$$\underline{C}_{-20}(E) = \frac{1}{\sqrt{5}} q_{0\to 2}\, g_{02}(\xi_{0\to 2}) \begin{pmatrix} \beta \\ -\alpha \\ \beta \end{pmatrix} \tag{38}$$

$$\begin{aligned} \mathrm{Re}\, V_{opt}^{(2)Reor}(r) = & -\frac{4}{35}\left(\frac{E}{\eta}\right) q_{0\to 2}^2\, q_{2\to 2} \left(\frac{a}{r}\right)^3 g_{02}(\xi_{0\to 2}) \\ & \times \left[1 + \frac{4}{49} q_{2\to 2}^2 \left(\frac{1}{\bar{\ell}^4}\left(1 - \frac{\arctan\bar{\ell}}{\bar{\ell}}\right)^2 + \frac{1}{3}\,\frac{1}{(1+\bar{\ell}^2)^2}\right)\right]^{-1} \\ & \times \left[\frac{(1-\arctan\bar{\ell}/\bar{\ell})^2}{\bar{\ell}^4} + 2\,\frac{(1-\arctan\bar{\ell}/\bar{\ell})}{\bar{\ell}^2}\left(\frac{\bar{\ell}^2-1}{(\bar{\ell}^2+1)^2}\right)\right. \\ & - \frac{1}{3(\bar{\ell}^2+1)^2} - 8\,\frac{(1-\arctan\bar{\ell}/\bar{\ell})}{(\bar{\ell}^2+1)^2}\left(\frac{a}{r}\right) \\ & \left. - 4\,\frac{\bar{\ell}^2(1-\arctan\bar{\ell}/\bar{\ell})}{(\bar{\ell}^2+1)^2}\left(\frac{a}{r}\right)^2\right] \end{aligned} \tag{39}$$

and

$$\begin{aligned} \mathrm{Im}\, V_{opt}^{(2)Reor}(r) = & \left[1 + \frac{4}{49} q_{2\to 2}^2 \left(\frac{1}{\ell^4}(1-\arctan\bar{\ell}/\bar{\ell})^2\right.\right. \\ & \left.\left. + \frac{1}{3}\,\frac{1}{(\bar{\ell}^2+1)^2}\right)\right]^{-1} V_{\ell}^{(2)}(r) \end{aligned} \tag{40}$$

where $V_{\ell}^{(2)}(r)$ is given in Eq. (29). Eq. (40) is the potential given in Refs. 11 and 14. It is interesting to notice that $\mathrm{Re}\, V_{opt}^{(2)Reor}(r)$ becomes indentically zero when $q_{2\to 2} = 0$, i.e., no reorientation effect. As a matter of fact this result is more general, $\mathrm{Re}\, V_{opt}(r) = 0$ when all reorientation couplings vanish, $q_{ii} = 0$, as can be seen from Eq. (34) and (22). The above observation sheds some light on the results of Ref. 17 where $V_{opt}(r)$ was calculated assuming a harmonic vibrational spectrum for the

target nucleus (all $q_{ii} = 0$) and it was found that $V_{opt}(r)$ is purely imaginary.

Including the coupling to the 4+ state to all orders results in a potential $V_\ell^{(4)}(r)$ which has the form[14]

$$V_\ell^{(4)}(r) = -i\,\underline{F}(r,\ell)\;\frac{1}{1+i\underline{C}_{22}(\ell)+\underline{C}_{24}(\ell)\underline{C}_{42}(\ell)}\;\underline{C}_{20}(\ell)$$

$$= -i\frac{2}{5}\frac{E}{\eta}\,q_{0\to2}^2\,\left(\frac{a}{r}\right)^3 g_{02}(\xi_{02})\{(-1+f(r))\times$$

$$\times\left[1+\frac{3}{49}\,q_{2\to4}^2\,g_{24}(\xi_{24})\left(\alpha^2+\frac{\gamma^2}{3}\right)-i\,\frac{4}{7}\,q_{2\to2}\,\alpha\right]\gamma$$

$$-\left(1+\frac{6}{49}\,q_{2\to4}^2\,g_{24}(\xi_{24})\left(\alpha^2+\frac{\gamma^2}{3}\right)\right)\alpha$$

$$+i\,\frac{2}{7}\,q_{2\to2}\left(\frac{\gamma^2}{3}-\alpha^2\right)\}$$

$$\times\left\{\left(1+\frac{3}{49}\,q_{2\to4}^2\,g_{24}(\xi_{24})\left(\alpha^2+\frac{\gamma^2}{3}\right)\right)\times\right.$$

$$\left.\times\left(1+\frac{36}{245}\,q_{2\to4}^2\,g_{24}(\xi_{24})\left(\alpha^2+\frac{\gamma^2}{3}\right)\right)\right.$$

$$\left.+\frac{4}{49}\,q_{2\to2}^2\left(\alpha^2+\frac{\gamma^2}{3}\right)-i\,\frac{18}{15\times49}\,q_{2\to4}^2 g_{2\to4}(\xi_{24})q_{2\to2}(\gamma^2-\alpha^2)\alpha\right\}^{-1} \tag{41}$$

where $f(r)$ is given by

$$f(r) = \frac{2}{1+\bar{\ell}^2}\left[1+2\bar{\ell}^2\left(\frac{a}{r}\right)+\bar{\ell}^4\left(\frac{a}{r}\right)^2\right]$$

and $\gamma = \beta\sqrt{6}$. Eq. (41) reduces to (39) and (40) in the limit $q_{2\to4} = 0$. It is clear that one could calculate explicitly $V_{opt}(r)$ to whatever order desired. However, such expressions become more and more complicated as already indicated in Eq. (41). Instead one could simply invert the matrix

propagator numerically in conjunction with optical model calculations. In Fig. 1 we show the coefficients a_ℓ, b_ℓ and c_ℓ as function of $\bar{\ell}$ for the different cases studied above as well as for that including the excitation of up to the $I = 16^+$ state. It is clear from Fig. 1 that the imaginary part of $V_{opt}(r)$, determined by Re a_ℓ, Re b_ℓ and Re c_ℓ, behaves basically like r^{-3} for small values of $\bar{\ell}$ whereaç for large values of $\bar{\ell}$ it goes as r^{-5}. This fact seems to hold irrespective of the value of the quadrupole coupling strength q. The above result continues to hold when the energy loss is properly included by use of the semiclassical energy loss factors

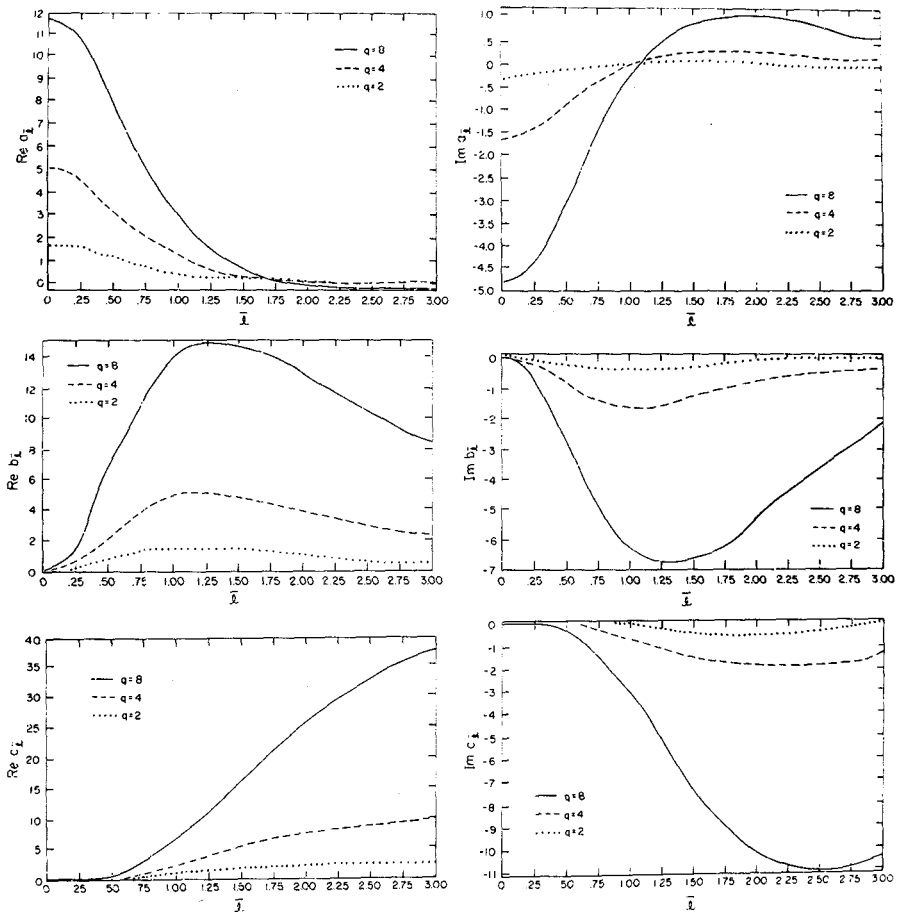


Fig.1 - The coefficienr a_ℓ, b_ℓ and c_ℓ plotted as functions of $\bar{\ell} \equiv \frac{\ell+1/2}{\eta}$ for several values of the quadrupole coupling parameter q. A factor $\frac{\eta}{E}$ was taken out of the coefficients in order to present the result in as general a form as possible.

as discussed before. The real part of $V_{opt}(r)$ shows a similar behavior to Im $V_{opt}(r)$ namely it goes as r^{-3} for small ℓ and as r^{-5} for large $\bar{\ell}$. For intermediate values of $\bar{\ell}$ both the real and the imaginary parts of $V_{opt}(r)$ show the general r-dependence of Eq. (33). The q-dependence of V_{opt} is shown in Fig.2 . The coefficient a , b_ℓ and c_R have a rather smooth dependence on q. Notice that for $\bar{\ell} = 0$ the coefficients $b_{\bar{\ell}=0}$ and $c_{\bar{\ell}=0}$ are identically zero for all q. This coordinates the discussion above about the r-dependence of $V_{opt}(r)$. It is instructive to compare our results for $V_{opt}(r)$ with the potential derived in Ref.15 where the semiclassical theory of multiple Coulomb excitation has been used. From the closed expres-

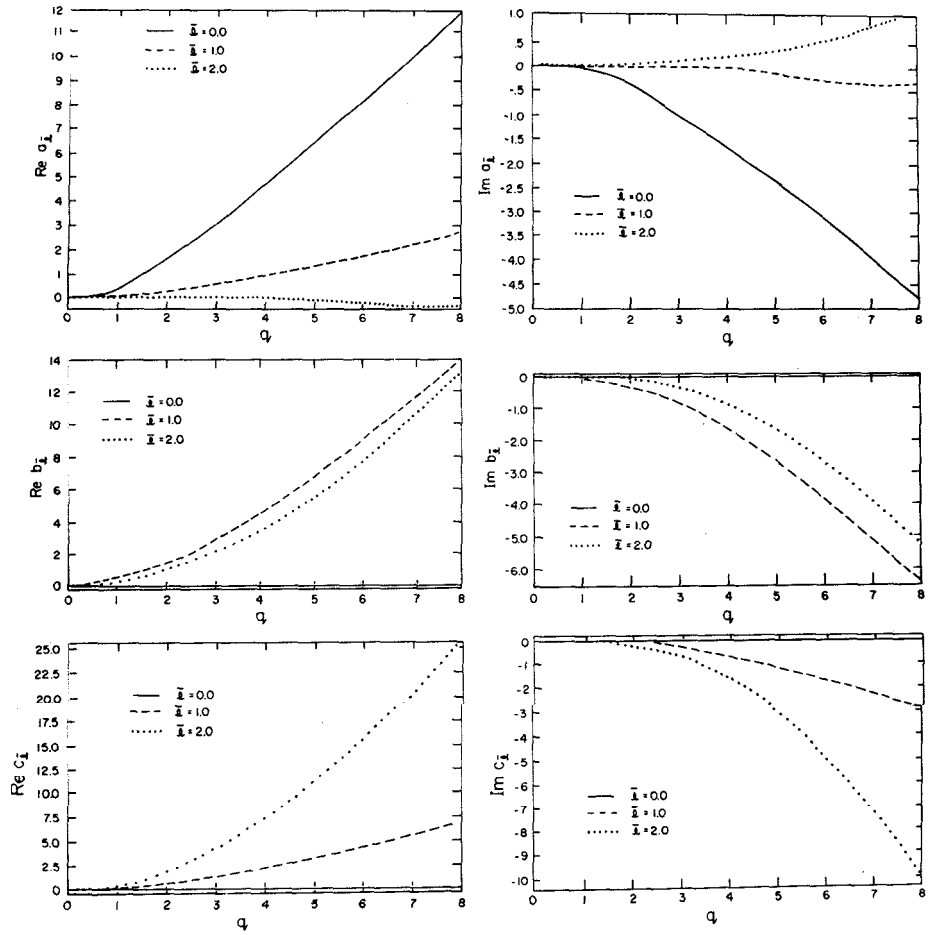


Fig.2 - The coefficient a_ℓ, b_ℓ and c_ℓ plotted as functions of the quadrupole coupling parameter q for several values of $\bar{\ell}$ r $\frac{\ell+1/2}{\eta}$. A factor $\frac{\eta}{E}$ was taken out of the coefficients (see caption to Fig.1).

sion for the elastic amplitude for $\bar{\ell}=0$ and $\xi=0$ (assuming a pure rotational band) obtained by Alder and Winther[9], one can extract the $V_{opt}(r)$ following the method developed in Ref.15.

$$\mathrm{Re}\, V^{AW}_{\ell=0}(r) = -3\frac{E}{\eta}\left[\frac{2}{3}q - \arctan\left[\frac{S_2(2q)}{C_2(2q)}\right]\right]\left(\frac{a}{r}\right)^3$$

$$\mathrm{Im}\, V^{AW}_{\ell=0}(r) = -3\frac{E}{\eta}\ln\left[\sqrt{\frac{\pi}{4q}}\left\{\left[C_2(2q)\right]^2 + \left[S_2(2q)\right]^2\right\}\right]\left(\frac{a}{r}\right)^3 \qquad (42)$$

where C and S are the Fresnel integrals[9].

We have compared our potentials of Eq. (22) with that of Alder and Winther given in (42) and the results are shown in Fig. 2 where we have plotted $V_{\ell=0}$ as a function of the quadrupole strength parameter q_{02}. It is clear that our potential is ~50% smaller than V^{AW}. More importantly we do not reproduce the broad oscillations seen in $\mathrm{Im}\, V^{AW}_{\bar{\ell}=0}(r)$. One possible reason for the disagreement can certainly be tracked down to a shortcoming of the on-energy-shell approximation. Another possibility is the fact the potential given in (42) is extracted from the elastic amplitude, an asymptotic quantity, and therefore is a phase- shift equivalent potential whereas our potential is a wave function equivalent potential being obtained directly from the wave function. We shall address ourselves here to the question of the on-shell approximation and itslimitations. Implicit in our calculation is the neglect of all terms involving the irregular Coulomb solution $G_\ell(kr)$. If we were to calculate the potential to first order, as was done in Ref. 12 such an approximation is all right since it amounts to neglecting terms of the type (assuming zero energy loss)

$$\int F_\ell(kr)\,\frac{1}{r^3}\,G_{\ell'}(kr)\,dr \qquad (43)$$

with $R' = R + 2, R-2, 9..$ It is clear that for large ri, the Sommerfeld parameters, F_ℓ and G_ℓ oscillate out of phase and therefore the above term may be neglected as compared to the dominant term $\int F_\ell(kr)\,\frac{1}{r^3}\,F_{\ell'}(kr)\,dr$. It would seem reasonable to adopt the above approximation, i.e., neglect all integrals involving F_ℓ and G_ℓ even in the calculation of higher order terms in the potential. However, upon an inspection of our series (12),

one notices immediately that terms like $\int G_\ell \frac{1}{r^3} G_\ell$ start appearing andone certainiy cannot neglect them as they could be as important as the terrns involving pairs of F_ℓ's. Short of actually taking full account of these terms we, instead, attached one parameter, a, to all the reorientation matrices $\underline{C}_{ii}(\ell)$ and another one, to all the coupling matrices $C_{ij}(\ell)$ in our expression for $V_{opt}(r)$ and adjusted these two parameters to obtain a best fit to the Alder-Winther potential of Eq. (42). Our results are shown in Fig. 3. The final adjusted values of a and β are 2.23 and 4.46, respectiveiy. It is clearly seen that the real part can be nicely fitted, the imaginary part, however, fits only on the average without exhibiting any oscillations. The fact that the parameters a and β which fit the average behavior of V^{AW} come out to be greater than unity is an indication that the neglected terms if included, would tend to enhance the matrix elements $C_{ii}(\ell)$ and $C_{ij}(\ell)$.

4. THE SUB-BARRIER ELASTIC CROSS SECTION

Once the optical potential is obtained one may then calculate the sub-barrier elastic scattering cross section either from an exact, one-channel, optical model calculation or simply by using the WKB approximation. Due to the fact that the number of partial waves involved in the heavy ion reactions we are considering is quite iarge, one may use the usual argurnents of replacing the partial wave sum in the elastic amplitude by an integral and perforrn the integration using the stationary phase method. Insofar as the real part of $V_{opt}(r)$ of *Eq.* (22) is quite small as compared with the dominant monopole-monopole Coulornb potential one may then use the Coulomb deflection function to relate $\bar{\ell} = \frac{\ell+1/2}{n}$ to the c.m. scattering angle θ,

$$\bar{\ell} = \cot \frac{\theta}{2} \tag{44}$$

A numerical calculation was reported in Ref. 14 where the $V_{opt}^{(2)\,Reor}(r)$ corresponding to two channel coupling with reorientation was incorporated into an optical model code. The deflection function extracted from the resulting phase shifts was found tc be very clcse to a pure Rutherford deflection function, Eq. (44). This gives us confidence in

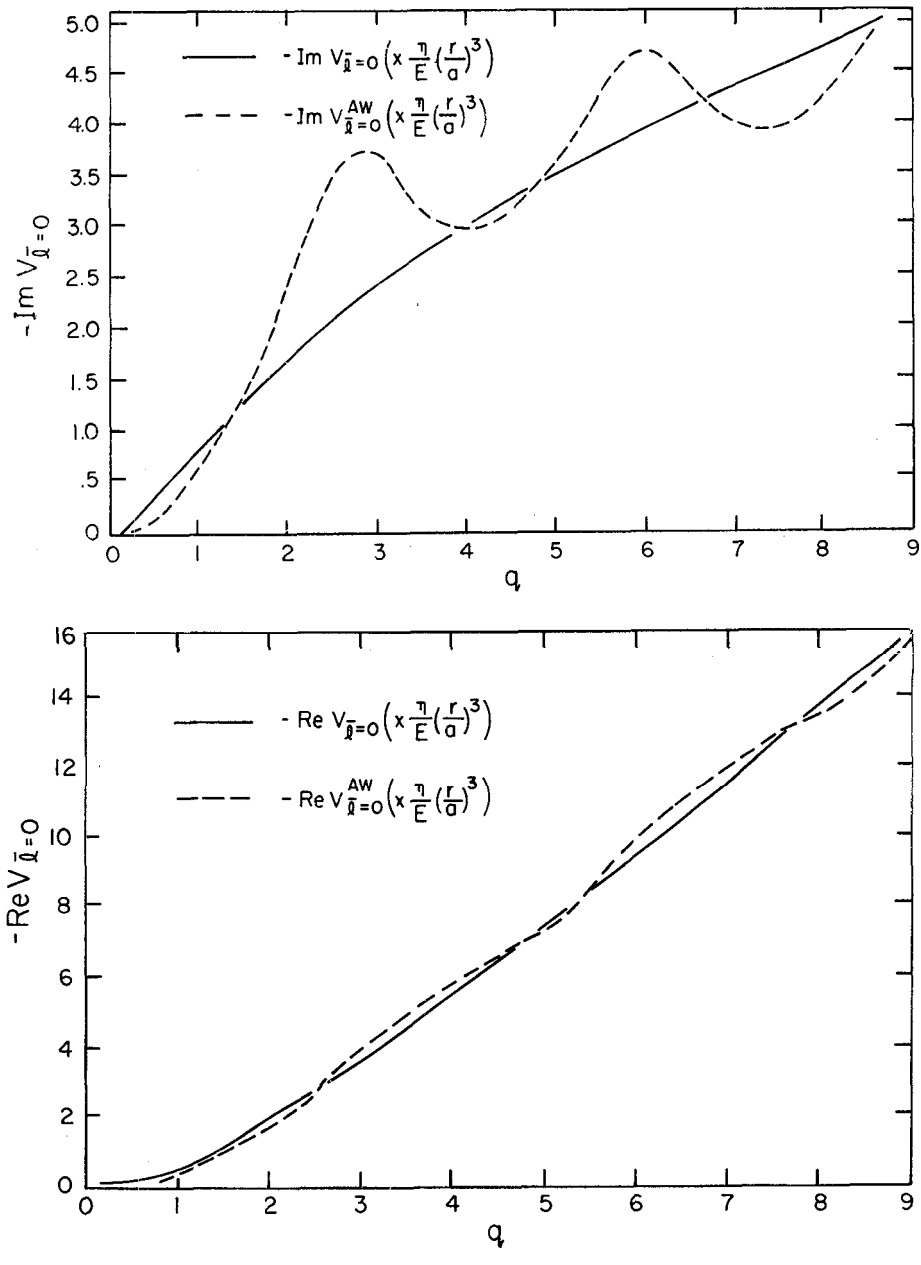


Fig.3 - The potential $V_{\bar{\ell}=0}$ plotted as a function of q and adjusted to the Alder-Winther potential (dashed line). The adjusted values of the parameters a and β used to fit the potential to the A-W potential wave 2.3 and 4.6, respectively (see text).

using the WKB formula for σ_{eR} given in Refs.2 and 18, which adapted to our case, becomes

$$\frac{\sigma_{e\ell}}{\sigma_R}(\theta) = \exp\left[-2\,\frac{2\mu}{\hbar^2}\int_{r_t(\bar{\ell}(\theta))}^{\infty}\frac{\mathrm{Im}\,V_{opt}(r)}{k_{\bar{\ell}(\theta)}}\,dr\right] \tag{45}$$

where $\bar{\ell}$ is given by Eq. (44), and

$$k_{\bar{\ell}}(r) = \left[\frac{2\mu}{\hbar^2}E\left(1 - \bar{\ell}^2\,\frac{a^2}{r^2} - 2\,\frac{a}{r}\right)\right]^{1/2} \tag{46}$$

and $r_t(\bar{\ell})$ is the classical turning point determined by the larger root of $k^2_{\bar{\ell}}(r) = 0$. From the general r-dependence of our potential given in Eq. (33), Eq. (45) may be rewritten in the form

$$\frac{\sigma_e}{\sigma_R}(\theta) = \exp\left\{-\left[\mathrm{Re}\,a_{\bar{\ell}(\theta)}\,I_3(8) + \mathrm{Re}\,b_{\bar{\ell}(\theta)}\,I_4(\theta) + \mathrm{Re}\,c_{\bar{\ell}(\theta)}\,I_5(\theta)\right]\right\} \tag{47}$$

where

$$I_i(\theta) \equiv 2\left(\frac{2\mu}{\hbar^2}\right)\int_{r_t(\bar{\ell})}^{\infty}\frac{1}{(r)^i k_{\bar{\ell}(\theta)}(r)}\,dr \tag{48}$$

and $\mathrm{Re}\,a_{\bar{\ell}(\theta)}$, etc. are functions only of the center of mass angle, the quadrupole strength parameters q_{ij} and the energy loss factors $g_{ij}(\xi_{ij})$. The integrals (48) can be evaluated in closed form and the final expressions for $i = 3$, 4 and 5 are collected in Appendix 2.

The above formula for $\sigma_{e\ell}(\theta)/\sigma_R$ was used in Refs. 11 and 12 to calculate the Coulomb damping in the elastic channel. In Ref. 12 the Coulomb orbit integrals of Eq. (48) were evaluated in a slightly different way from our expressions given in Appendix 2 but comparison of our expression for $\frac{\sigma_{e\ell}}{\sigma_R}(\theta)$ with the $V_{opt}(r)$ given in Eq. (29), i.e., $V^{(2)}_{\ell}(r)$ and with the $I_i(\theta)$ given in Appendix 2 and their expression (Eq. (9), Ref.12) indicate that they are practically identical. In Ref.11 the potential $\mathrm{Im}\,V^{(2)\mathrm{Reor}}_{opt}(r)$ given in Eq.(40), was used in

$\sigma_{el}(\theta)/\sigma_R$ and the expression found was (assuming only target excitation)

$$\frac{\sigma_{el}}{\sigma_R}(\theta) = \exp\left[-\frac{16}{45}\, q^2_{0\to 2}\, g_{02}(\xi_{02})\left(1 + \frac{4}{49}\, q^2_{2\to 2}\, f_1(\theta)\right)^{-1} f(\theta)\right] \qquad (49)$$

where

$$f_1(\theta) = \left[\left(1 - \left(\frac{\pi-\theta}{2}\right)\tan\frac{\theta}{2}\right)^2 \tan^4\frac{\theta}{2} + \frac{1}{3}\sin^4\frac{\theta}{2}\right]$$

and $f(\theta)$ is the universal angle function given in Ref. 11. Again the comparison of Eq. (49) with the corresponding expression obtained from Eq. (47) and using the I_i's given in Appendix 2 indicate that they give almost identical results. When performing calculations with the general potential of Eq. (35), Eq. (47) for $\frac{\sigma_{el}}{\sigma_R}(\theta)$ ismoreappropriate. In Fig. 4 we present the results for the system $^{20}Ne + {}^{152}Sm$ at $E_{lab} = 70$ MeV. The agreement with the data (from Ref. 19) is quite good. In Fig. 5 we present the results for $\frac{\sigma_{el}}{\sigma_R}(\theta)$ for two values of q (assuming a pure quaclrupole rotational band) fixing the potential V_{opt} so that the maximum I included satisfies the inequality

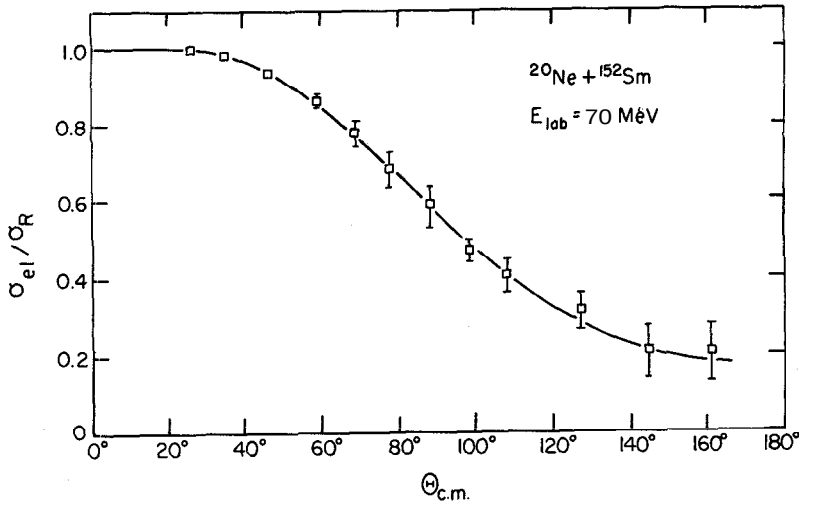
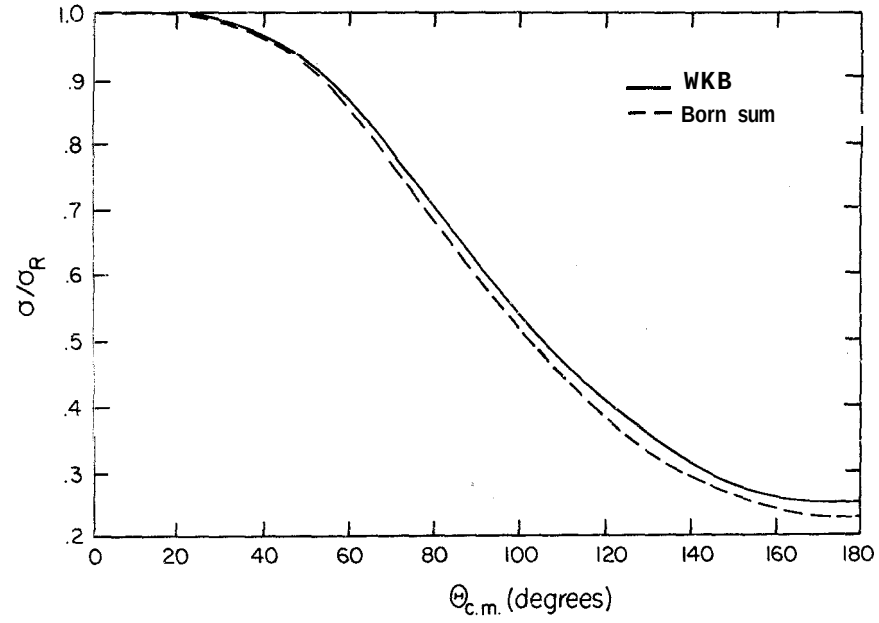


Fig. 4 - The sub-barrier elastic cross section normalized to the Rutherford cross section, plotted vs. the center of mass angle for the system $^{20}Ne + {}^{152}Sm$ ($E_{lab} = 70$ MeV). included in the calculation is the coupling to the 2^+ state as well as the reorientation of the 2^+ to all orders in both target and projectile. The data are from Ref. 19.



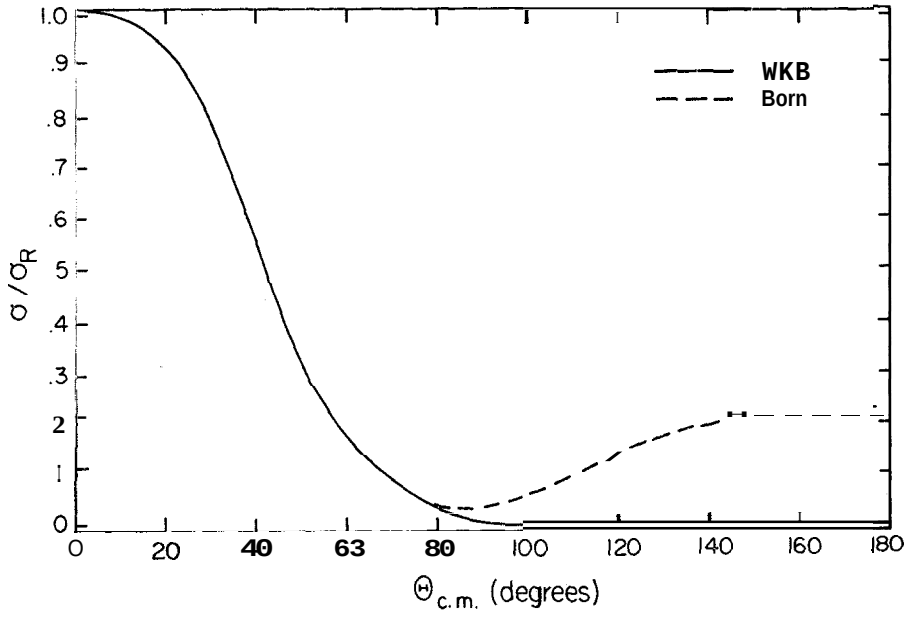


Fig.5 - The ratio $\frac{\sigma_{el}}{\sigma_R}$ calculated both from the WKB expression Eq. (47) (solid lines) as well as from the summed Born series Eq. (52) (dashed lines). a) q = 2,36; b) q = 9.56. Energy loss was accounted for by inserting the appropriate semiclassical energy loss factors (see text).

$$I_{max} \geq 2q \tag{50}$$

The above inequality guarantees, for a given value of q, the presence of the effects of the coupling to all channeis included in the construction of $V_{opt}(r)$. This follows from the classical relation between the value of q and the maximum angular momentum transferred namely[9],

$$2q \tilde{=} \langle I \rangle \tag{51}$$

Another way of calculating the elastic scattering cross section is by summing of the Born series for the elastic scattering amplitude obtained directly from Eq. (4) upon replacing the right hsnd side by $V_{opt}(r)\ \psi(r)$. This results in the following simple expression for $\frac{\sigma_{e\ell}}{\sigma_R}(\theta)$

$$\frac{\sigma_{e\ell}}{\sigma_R}(\theta) = \left| \frac{1 - \frac{i}{k}\int_0^\infty F_{\bar{\ell}(\theta)}(kr) V_{opt}(r)\ \ F_{\bar{\ell}(\theta)}(kr)\ dr}{1 + \frac{i}{k}\int_0^\infty F_{\bar{\ell}(\theta)}(kr) V_{opt}(r)\ \ F_{\bar{\ell}(\theta)}(kr)\ dr} \right|^2 \equiv |A_{00}(\theta)|^2 \tag{52}$$

Here we have adopted the on-energy-shell approximation for the elastic channel Green's function. The amplitude $A_{00}(\bar{\ell})$ may be written in terms of the coupling matrices $\underline{C}_{ij}(\ell)$ as can be seen from the structure of V_{opt} given in Eq. (22) and the definition of $\underline{C}_{ij}(\ell)$ given in Eq. (18)

$$A_{00}(\bar{\ell}) = \frac{1 - i\,\underline{C}_{02}(\bar{\ell}) \left\langle 2 \left| \frac{1}{\underline{1} + i\,\underline{C}(\ell)} \right| 2 \right\rangle \underline{C}_{20}(\ell)}{1 + i\,\underline{C}_{02}(\ell) \left\langle 2 \left| \frac{1}{\underline{1} + i\,\underline{C}(\ell)} \right| 2 \right\rangle \underline{C}_{20}(\ell)} \tag{53}$$

where the Q-space "propagator" of the 2+ state $\left\langle 2 \left| \frac{1}{\underline{1} + i\,\underline{C}(\ell)} \right| 2 \right\rangle$ is given in Eq. (25).

We have evaluated $\frac{\sigma_{e\ell}}{\sigma_R}(\theta)$ as given in Eq. (52) including in the construction of the 2+ propagator states up to $I = 18$, i.e., in-

cluding up to 8 members of the rotational band and taking q_{02} to be 2.36 and 9.56. The result is shown as the dashed lines in Fig. 5. It is clear that the WKB expression for σ_{el}/σ_R given in Eq. (47) and the summed Born series of Eq. (52) give almost identical resulf for small values of q. However, for large q the two expressions, although give close results at small angles, presents qualitatively different behavior as can be seen in Fig. 5b. We have accounted for the energy loss by inserting the factors $g_{ij}(\xi_{ij})$ in the coupling matrices $C_{ij\neq i}$. The semiclassical coupled channels-calculation of σ/σ_R for the above system has been reported recently and it shows that σ/σ_R oscillates slightly at angles larger than 60°.

Finally, we give below the elastic scattering cross section evaluated at $\theta = \pi$ (included in this calculation are the couplings to the 2+ and 4+ states as well as the reorientation of the 2+ state)

$$\frac{\sigma_{el}(\pi)}{\sigma_R(\pi)} = \exp\left[-\frac{16}{45} q_{0\to 2}^2 \, g_{0\to 2}(\xi_{0\to 2}) \times \frac{1 + 0.065\, q_{2\to 4}^2 \, g_{2\to 4}(\xi_{2\to 4})}{0.036\, q_{2\to 2}^2 + \left[1 + 0.065\, q_{2\to 4}^2 \, g_{2\to 4}(\xi_{2\to 4})\right]^2} \right] \tag{54}$$

It is interesting to note that at back angles both the WKB and the summed Born series expressions for $\frac{\sigma_{el}}{\sigma_R}$ (8) become, for a pure rotational band and in the zero-energy loss limit, a function of only one variable, the quadrupole strength parameter $q_{0\to 2}$. This is also the casewith the elastic scattering probability calculated by Alder and Winther[9] without resorting to the Coulomb polarization potential.

From the elastic scattering cross section one may calculate the total inelastic scattering cross section using the unitariry condition which the matrix must satisfy.

$$\frac{\sigma_{inel}(\theta)}{\sigma_R(\theta)} \equiv \sum_{i=2,4,\ldots} \frac{\sigma_{i^+}(\theta)}{\sigma_R(\theta)} = 1 - \frac{\sigma_{el}(\theta)}{\sigma_R(\theta)} \tag{55}$$

In order to exhibit this property of the S-matrix we notice that a manifestly unitary S-matrix may be constructed from our equation (20) through[21]

$$S = (k)^{+1/2} \; \hat{S}(k)^{-1/2} \tag{56}$$

where $\hat{S}$ is obtained from Eq. (20),

$$\hat{S} = 1 - 2ik^{-1} \int_0^\infty dr \; F \, V\psi^{(+)} \tag{57a}$$

where F and ψ^+ are matrices in channel space and V is the coupling matrix. From the unitary S-matrix above we can determine the T-matrix which is related to S by

$$1 - 2iT = S$$

or

$$\begin{aligned} T = k^{1/2} \; k^{-1} \int FV\psi \; k^{-1/2} \\ k^{-1/2} \int FV\psi \; k^{-1/2} \end{aligned} \tag{57b}$$

Defining a new coupling matrix $\underline{C}'$ to be

$$\underline{C}' = k^{1/2} \; k^{-1} C \; k^{-1/2} = k^{-1/2} \underline{C} \; k^{-1/2}$$

we finally obtain

$$T = \frac{k^{-1/2} \underline{C} \; k^{-1/2}}{1 + ik^{-1/2} \; \underline{C} \; k^{-1/2}} = \frac{\underline{C}'}{1 + iC'} \tag{58}$$

and

$$S = \frac{1 - ik^{-1/2} \; \underline{C} \; k^{-1/2}}{1 + i \; k^{-1/2} \; \underline{C} \; k^{-1/2}} = \frac{1 - i\underline{C}'}{1 + i\underline{C}'}$$

In the above symmetrized form, T may be used to calculate the different inelastic cross sections. This calculation is reported in the following paper[22].

5. DISCUSSION AND CONCLUSIBMS

In this paper we have explored the consequences of the on-energy-shell approximation for the channel Green's function on the multiple Coulomb excitation coupled channel problem. We have, among other things, derived a closed expression for the Coulomb polarization potential in the limit of zero energy loss accounting partially for the last through the semiclassical energy ioss factors. Our general expression for the Coulomb polarization potential reduces to the more approximate expressions given in Refs.11 and 12 as limiting cases. As a check on our results we have compared our potential for R=0 with the potential based on the Alder-Winther theory, obtained in Ref.15. The discrepancy found is accounted for partly by the terrns in the Born series neglected in the on-energy-shell approximation. We have been able to fit, on the average, our potential to the Alder-Winther one by adjusting two parameters--one attached to the reorientations matrices and the other to the coupling matrices, It is argued that the failure of our adjusted imaginary potential to reproduce the oscillations found in the Alder-Winther potential could be a consequence of both the fact that the AW potential was extracted from the asymptotic elastic amplitude whereas ours directly from the wave function, as well as the on-energy-shell approximation. Although the rnethods develcped in this paper are not meant to substitute for more exact cuupled channels calculations, they do, nevertheless, supply us with a very simple closed form for the sub- barrier elastic cross section appropriate for the case of strong coupling. Furthermore, having obtained the elastic channel Coulomb polarization potential, which contains multiple Coulomb excitation effects, one could use it to simplify an otherwise more complicated coupled channels calculation. As an example, we cite the case of sub-barrier fusion of deformed nuclei [23]. As is known, deformation of the target induces both static as well as dynamic effects on the fusion cross section [24]. If the target is so deformed that multiple Coulomb excitations is important, one needs to perform a calculation involving several strongly coupled channels to account correctly for these effects. Our potential $V_{opt}(r)$, makes it possible to account almost completely for the dynamic effects in a one-channel opticai model description. The study cf static effects is thus made easier and could be handled following, for example, the ideas in

Ref.23. Aside from the above practical aspects of our results we consider our findings as a first step in the direction of exploring simplifying schemes for the coupled channels calculation in tne more general case where the nuclear force is included. In such cases the on-energy-shell approximation arnounts to replacing the channel Green's function by (Ref.7)

$$G_{\ell,c}(r,r') \propto \psi_{\ell,c}(r)\ \psi_{\ell,c}(r')/S^0_{\ell,c}(k) \qquad (59)$$

Here $\psi_\ell(r)$ is the regular solution of the optical rnodel Schrödinger equation describing the elastic scattering in channel c and $S^0_{\ell,c}(k)$ is the elastic S-matrix element in channel c. Due to strong absorption for low ℓ partial waves the factor $(S^0_{\ell,c}(k))^{-1}$ could become quite large for sniali L, resulting in a pnssible overestimation of the coupled channel effects[6]. However, as demonstrated in Ref.7, this needs not be a worrying point if the amplitudes were to be evaluated in closed form as anotheir factor of $S^0_\ell(k)$, which comes from the distorted waves, approximately cancels the dangerous $[S^0_{\ell,c}(k)]^{-1}$ factor in the channel Green's function. Finally, the inclusion of the nuclear excitation into our expressions for the inelastic amplitudes, which are calculated withoutthe nuclear effects in the following paper, would make the investigation of the dependence of the nuclear-Coulomb interference effects on the spins of the excited states[25] simple and transparent. These ideas will be explored and developed fully in the third part of this. series[26].

We were fortunate to have A.J. Baltz collaborate with us on some aspects of the present work.

APPENDIX 1

In this appendix are given the approximate, large ℓ limit, values of the 3-j and 6-j symbols needed in the calculation. Throughout we shall use the definitions and convention of Edmonds[27].

a) 3-j syrnbols

$$\begin{pmatrix} \ell & \ell & 2 \\ 0 & 0 & 0 \end{pmatrix} = (-)^{\ell+1} \left[\frac{\ell(\ell+1)}{(2\ell+3)(2\ell+1)(2\ell-1)}\right]^{1/2}$$

$$\simeq \frac{(-)^{\ell+1}}{\sqrt{8\ell}} \qquad \text{(A1-1)}$$

$$\begin{pmatrix} \ell & \ell+2 & 2 \\ 0 & 0 & 0 \end{pmatrix} = (-)^{\ell} \left[\frac{3}{2} \frac{(\ell+2)(\ell+1)}{(2\ell+5)(2\ell+3)(2\ell+1)}\right]^{1/2}$$

$$\simeq (-)^{\ell} \sqrt{\frac{3}{16\ell}} \qquad \text{(A1-2)}$$

b) 6-j symbols

In the **following** $\ell >> \mathbf{I}$, $R >> m$.

$$\begin{Bmatrix} \ell & \ell+m & I \\ 2 & I-2 & \ell+m-2 \end{Bmatrix} \simeq \left[\frac{1}{2\ell} \frac{(m+I-3)(m+I-2)(m+I-1)(m+I)}{(2I-3)(2I-2)(2I-1)(2I)(2I+1)}\right]^{1/2}$$

(A1-3)

$$\begin{Bmatrix} \ell & \ell+m & I \\ 2 & I-2 & \ell+m \end{Bmatrix} \simeq \left[\frac{3}{\ell} \frac{(m+I-1)(m+I)(I-m-1)(I-m)}{(2I-3)(2I-2)(2I-1)(2I+1)(2I)}\right]^{1/2}$$

(A1-4)

$$\begin{Bmatrix} \ell & \ell+m & I \\ 2 & I-2 & \ell+m+2 \end{Bmatrix} \simeq \left[\frac{1}{2\ell} \frac{(I-m-3)(I-m-2)(I-m-1)(I-m)}{(2I-3)(2I-2)(2I-1)(2I)(2I+1)}\right]^{1/2} \qquad \text{(A1-5)}$$

$$\begin{Bmatrix} \ell & \ell+m & I \\ 2 & I & \ell+m+2 \end{Bmatrix} \simeq \left[\frac{3}{\ell} \frac{(m+I-1)(m+I)(I-m+1)(I-m+2)}{(2I-1)(2I)(2I+1)(2I+2)(2I+3)}\right]^{1/2} \qquad \text{(A1-6)}$$

$$\begin{Bmatrix} \ell & \ell+m & I \\ 2 & I & \ell+m \end{Bmatrix} \simeq \frac{(3m^2 - I(I+1))}{\left[\ell I(2I-1)(2I+1)(2I+2)(2I+3)\right]^{1/2}} \qquad \text{(A1-7)}$$

APPENDIX 2

The Coulomb orbit integrals defined in Eq. (45) are easily evaluated following the methods described in Ref. 28. We obtain for $I_3(\theta)$, $I_4(\theta)$ and $I_5(\theta)$, the following expressions

$$I_3(\theta) = \frac{1}{\bar{\ell}^2}\left[1 - \frac{\arctan\bar{\ell}}{\bar{\ell}}\right]\frac{2\eta}{Ea^3}$$

$$= \tan^2\frac{\theta}{2}\left[1 - \left(\frac{a-8}{21}\right)\tan\frac{\theta}{2}\right]\frac{2\eta}{Ea^3} \qquad \text{(A2-1)}$$

$$I_4(\theta) = \frac{}{2\bar{\ell}^4}\left[(\bar{\ell}^2 + 3)\frac{1}{\bar{a}}\arctan\bar{\ell} - 3\right]\frac{2\eta}{Ea^4}$$

$$= 2\tan^4\frac{\theta}{2}\left[\left(\cot^2\frac{\theta}{2} + 3\right)\left(\frac{\pi-\theta}{2}\right)\tan\frac{\theta}{2} - 3\right]\frac{2\eta}{Ea^4} \qquad \text{(A2-2)}$$

$$I_5(\theta) = \frac{1}{2\bar{\ell}^6}\left[\left(\frac{4\bar{\ell}^2}{3} + 5\right) - (3\bar{\ell}^2 + 5)\frac{1}{\bar{\ell}}\arctan\bar{\ell}\right]\frac{2\eta}{}$$

$$= 2\tan^6\frac{\theta}{2}\left[\left(\frac{4}{3}\cot^2\frac{\theta}{2} + 5\right) - (3\cot^2\frac{\theta}{2} + 5)\left(\frac{\pi-\theta}{2}\right)\tan\frac{\theta}{2}\right]\frac{2\eta}{Ea^5} \qquad \text{(A2-3)}$$

In obtaining the final angular dependence of $I_{\eta}(\theta)$ we have used the Rutherford $\bar{\ell} = \cot\frac{\theta}{2}$ (Eq. (41)).

REFERENCES

1. U.Smilansky, Invited talk at the International Conference on Dynamical Properties of Heavy-Ion Reactions, Johannesburg (1979).

2. S.Landowne, C.H.Dasso, B.S.Nilsson, R.A.Broglia and A.Winther, Nucl. Phys. A259 , 99 (1976).

3. W.H.Miller, Adv.Phys.Chem. 25, 67 (1974).

4. S.Levit, U.Smilansky and D.Pelte, Phys.Lett. *53*B, 39 (1974); H. Massmann and J.O.Rasmussen, Nucl.Phys. A*243*, 155 (1975); R. Oonangelc, M.W.Guidry, J.P.Boisson and J.O.Rasmussen, Phys.Lett. *46*B, 377 (1977).

5. P.Frobrich, K.Mohring and Q.Lin, Comm.To European Conf. on Nuclear Physics with Heavy Ions, p.96, Caen, France (1976).

6. S.R.Cotanch and C.M.Vincent, Phys. Rev. C*14*, 1739 (1976).

7. W.E.Frahn and M.S.Hussein, Proceedings of the Symposium on Heavy Ion Physics from 10 to 200 MeV/AMU, Brookhaven National Laboratory, July 16-20 (1979), and to be pubiished.

8. W.E.Frahn, in Heavy-Ion, High-Spin States and Nuclear Structure (IAEA, Vienna, 1975), p.157; W.E.Frahn, Nucl.Phys. A*272*, 413 (1976); W.E.Frahn and K.E.Rehm, Phys. Reports 37 C, 18 (1978).

9. K.Alder and A.Winther, *Electromagnetic Excitation* (North Holland Publishing Co., Amsterdam (1975)).

10. J.de Boer, G.Dannhauser, H.Massmann, F.Roesel and A.Winther, J. of Phys. G: Nucl. Phys. 3, 889 (1977).

11. M.S.Hussein, to appear in Phys.Lett. B (1979).

12. A.J.Baltz, S.K.Kauffmann, N.K.Glendenning and K.Pruess, Phys. Rev. Lett. 40, 20 (1978); Nucl. Phys. A; to appear.

13. H.Feshbach, Ann. of Phys. 19, 287 (1962).

14. B.V.Carlson, M.S.Hussein and A.J.Baltz, submitted for publication (1979).

15. R.Donangelo, L.F.Canto and M.S.Hussein, Nucl.Phys. A*320*, 422 (1979).

16. K.Alder, A.Bohr, T.Huus, B.Mottelson and A.Winther, Rev. Mod. Phys. *28*, 1432 (1958).

17. R.Donangelo, L.F.Canto and M.S.Hussein, Phys.Rev. C*19*, 1801 (1979).

18. W.E.Frahn and T.F.Hill, Zeitschrift fur Physik, A*285*, 315 (1978).

19. P.Doll, M.Bini, D.L.Hendrie, S.K.Kauffmann, J.Mahoney, A. Menchaca-Rocha, D.K.Scott, T.J.M.Symons, K.Van Bibber, M.P.Viyogi, H.Wieman and A.J.Baltz, Phys.Lett. *76*B, 566 (1978).

20. P.Frobrich, R.Lipperheide and H.Fiedeldey, Phys.Rev.Lett. 43, 114 (1979).
21. R. G.Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill, 1968).
22. B.V.Carlson and M.S.Hussein, Nucl.Phys. A, following paper.
23. R.G.Stokstad, Y.Eisen, S.Kaplanis, D.Pelte, U.Smilansky, and I. Tserruya, Phys.Rev.Lett *41*, 465 (1978).
24. M.S.Hussein, L.F.Canto and R.Donangelo, to appear in Phys. Rev. C 1979).
25. M.W.Guidry, P.A.Butler, R.Donangelo, E.Grosse, Y.El-Masri, I.Y.Lee, F.S.Stephens and R.M.Diamond, L.L.Rideinger, C.R.Bingham, A.C. Kahler, J.A.Varba, R.Edmonds, E.L.Robinson and N.R.Johnson, Phys. Rev. Lett.40, 1016 (1978).
26. B.V.Carlson and M.S.Hussein, to be published.
27. A Edmond, *Angular Momentum* in *Quantum Mechanics* (2nd ed.)(Princeton University Press, 1960).
28. I.S.Gragshteyn and I.M.Ryzhik; *Tables of Integrals, Series and &O-ducts* (Academic Press, 1965).