Revista Brasileira de Física, Vol. 10, Nº 3, 1980

Macroscopic Quantum Waves in Boson Systems II

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Recebido em 10 de Junho de 1980

The existence of macroscopic quantum waves in the theory $-\lambda|\phi|^4 + g|\phi|^6$ is verified.

Verifica-se a existência de ondas quânticas macroscópicas na teoria $-\lambda|\phi|^4 + g|\phi|^6$.

Last year[1] a new scheme was proposed to uriderstand the interesting properties of liqiiid ^{4}He (superfluidity, the λ-transition, the nature of "rotons", etc.) from a microscopic view point. It has been observed that Bogoliubov superfluidity theory[2] describes som macroscopic quantum waves (MQWs), which behave as moving Bloch walls, cutting the condensate into sectors of phase, analogous to magnetic domains[1]. In addition, the MQWs bind a new set of quasi-particles, that look much like the liquid Helium elementary excitations observed in neutron scattering experiments[3]. We believe, therefore, that these waves provide the link between the microscopic world and the phenomenology of superfluidity.

Macroscopic quantum waves appear also in a large class of non-local theories[4], as well as in the $\lambda|\phi|^n$ theories, with $\lambda>2$ [5].

Here we study the MQWs of the model $-\lambda|\phi|^4+g|\phi|^n$ (λ and g positive constants). We notice that the attractive term $-\lambda|\phi|^4$ does not exist in any other theory, where the existence of MQWs has been establi-

* Supported by CNPq.

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shed. Hence, the results presented in this paper reinforce the idea[1] that MQWs are common to every stable boson theory, and, as a consequence, that they should occur in the microscopic theory of the real helium atoms.

According to Burt[6], the terms $-\lambda|\phi|^4$ and $g|\phi|^6$ correspond respectively to attractive two-body and repulsive three-body interactions, generated by delta-like potentials.

ϕ being a nonrelativistic boson field, consider the Hamiltonian

$$H = \int d\vec{x} \left\{ \frac{1}{2m} \vec{\nabla}\phi^*\vec{\nabla}\phi - \frac{\lambda}{2} (\phi^*\phi)^2 + \frac{g}{3} (\phi^*\phi)^3 \right\} \tag{1}$$

and the associated classical equation of motion,

$$i\partial_t\phi = -\frac{1}{2m} \nabla^2\phi - \lambda(\phi^*\phi)\phi + g(\phi^*\phi)^2\phi \tag{2}$$

This is the Gross-Pitaevskii equation[7]. The x-independent solutions,

$$\Omega = \sqrt{\rho}\, \exp\{-i(g\rho^2 - \lambda\rho)t\} \tag{3}$$

describe the fluid condensate[1,7,8], and ρ is the condensate density.

As this theory has an attractive term, the condensate is unstable at low densities. The stable configuration for these low densities is a sort of vapor state. We can determine the conditions under which the condensate is stable by studying the motion of the phonons.

Let us write ϕ in the form

$$\phi = (\sqrt{\rho} + \eta)\exp\left[-i(g\rho^2 - \lambda\rho)t\right] \tag{4}$$

where the small fluctuation $\eta(x,t)$ is a sound wave. Inserting ϕ into the equation of motion and keeping only terms up to first order in η, we get:

$$i\partial_t\eta = -\frac{1}{2m} \nabla^2\eta + (2g\rho^2 - \lambda\rho)(\eta^* + \eta) \tag{5}$$

This equation is the same as Eq.(15) of ref.(1). Then, following the steps of that reference, we obtain $mc^2 = 2g\rho^2 - \lambda\rho$, where c is the velocity of large wave-length phonons:

$$c = \left[(2g\rho^2 - \lambda\rho)/m\right]^{1/2} \tag{6}$$

c must be real, otherwise we would have imaginary frequencies, and the condensate stability would no longer be ensured. The stability condition is, therefore,

$$\rho > \rho_c = \frac{\lambda}{2g} \tag{7}$$

To construct the MQWs solutions, we represent the field as follows:

$$W(\xi) = \sqrt{s(\xi)}\ \exp\{i\theta(\xi)\}\sqrt{\rho}\ \exp\{-i(g\rho^2 - \lambda\rho)t\} \tag{8}$$

where $\xi = mc(x-cVt)$, x is a particular coordinate, and $|V|<1$, V being a real number. It is clear that cV is the MQW's velocity.

In the limit $|\xi|\to\infty$, we impose $s(\xi)\to 1$, and $\frac{d\theta}{d\xi}\to 0$. Thus, asymptotically, the solution we seek will tend towards condensates of density ρ.

The substitution of $W(\xi)$ into the equation of motion leads to a pair of coupled equations for $s(\xi)$ and $\theta(\xi)$:

$$mc^2 V\sqrt{s}\ \partial_\xi\theta + \frac{mc^2}{2}\{\partial_\xi^2\sqrt{s} + \sqrt{s}\ (\partial_\xi\theta)^2\} = (\lambda\rho - g\rho^2)\sqrt{s} +$$

$$- \lambda\rho(\sqrt{s})^3 + g\rho^2(\sqrt{s})^5 \tag{9a}$$

and

$$2V\partial_\xi\sqrt{s} = 2\partial_\xi\sqrt{s}\ \partial_\xi\theta + \sqrt{s}\,(\partial_\xi\theta)^2 \tag{9b}$$

If now we integrate (9b), taking into account the boundary conditions, we get

$$\partial_\xi\theta = \sqrt{(1 - \frac{1}{s})} \tag{10}$$

Inserting, then, (10) in (9a) leads to an equation for $s(\xi)$

$$\frac{1}{2}\partial_\xi^2 \sqrt{s} + \frac{V^2}{2}\sqrt{s}\left(1 - \frac{1}{s^2}\right) + \sqrt{s} - Q\sqrt{s} + (2Q-1)(\sqrt{s})^3 - Q(\sqrt{s})^5 = 0 \tag{11}$$

The next step is to multiply Eq.(11) by $2\partial_5\sqrt{s}$. After integration it follows:

$$\frac{1}{2}\left(\partial_\xi \sqrt{s}\right)^2 + \frac{V}{2}\left(s + \frac{1}{s}\right) + s - Q\left(s + \frac{s^3}{3}\right) + (2Q-1)\frac{s^2}{2} = \text{const.} \tag{12}$$

This constant is determined by imposing again the boundary conditions: const $= V^2 + 1/2 - Q/3$.

If we still multiply this last equation by $8s$, we finally obtain

$$(\partial_\xi s)^2 + U(s) = 0 \tag{13a}$$

where

$$U(s) = (s-1)^2\left\{-\frac{8}{3}Qs^2 + \left(\frac{8}{3}Q - 4\right)s + 4V^2\right\} \tag{13b}$$

integrating Eq.(13) is the same as solving a problem of classical mechanics: the motion of a particle of mass 1/2 and energy zero, under the action of the potential $U(s)$.

Properties of the potential $U(s)$:

(i) Whenever $|V|>1$, s=l is a point of minimum, and, therefore, the only solution of Eq.(13), which obeys the boundary condition $s(-\infty) = 1 = s(+\infty)$ is $s(\xi)=1$. This implies that $\theta(\xi)$ = constant (see Eq. (10)). Thus, under these circumstances, we are led, through Eq. (8), to a solution like (3), and no MQWs exist at all.

(ii) If $|V|<1$, the $U(1)=1$, and the point s=l is a point of maximum, as is shown in fig.1. In this second case it is clear that solutions of Eq.(13) will exist, which, even obeying the boundary conditions, are different from the trivial condensate-like solution. These are the MQWs.

By chance, Eq.(13) is amenable to exact integration. When the boundary conditions are imposed, we get:

$$s(\xi) = 1 - \frac{2}{A + B \cosh 2\gamma\xi} \tag{14a}$$

$$\gamma^2 = 1 - V^2 \tag{14b}$$

$$A = \frac{1}{2} + \frac{Q}{3} \tag{14c}$$

and

$$B = \left\{ \left(\frac{1}{2} + \frac{Q}{3} \right)^2 - \frac{2}{3} Q\gamma^2 \right\}^{1/2} \tag{14d}$$

Using Eq.(10) and choosing $\theta(0)=0$ (this freedom of choice comes from the gauge invariance of the theory), it follows

$$\theta(\xi) = - \arctan \left\{ \frac{B\, e^{2\gamma\xi} + A - \gamma^2}{\gamma V} \right\} \tag{15}$$

Inserting $s(\xi)$ and $\theta(\xi)$ in Eq.(8) and simplifying, we find the function $W_V(\xi)$, which describes a macroscopic quantum wave of velocity cV.

$$W_V(\xi) = \frac{D \cosh\gamma\rho - (\frac{\gamma V}{D}) \sinh \gamma\xi}{(A + B \cosh 2\gamma\xi)^{1/2}} \sqrt{\rho} \exp \{- i(g\rho^2 - \lambda\rho)\, t\} \tag{16}$$

The momentum per unit area carried is

$$p = - i \int dx\, W^* \partial_x W = - (\frac{\rho V}{2}) (\frac{2Q}{3})^{-1/2} \ln \left\{ \frac{A + \gamma(\frac{2Q}{3})^{1/2}}{A - \gamma(\frac{2Q}{2})^{1/2}} \right\}$$

If $H(W)$ is the energy associated with the MQW system and $H(\Omega)$ is the ground state energy, the energy carried by the MQW shall then be $H(W)-H(\Omega)$. This subtraction is somewhat subtle, because we must be careful when seeking the ground state of the problem. An analogous calculation is carried out in ref.(1). In the present case, the energy per unit of area obtained is

$$\sigma(V) = \frac{c\rho}{4}\left\{3\gamma\left(1 - \frac{1}{2Q}\right) - (2A^2+B^2)\left(\frac{2Q}{3}\right)^{-1/2} \ln\left\{\frac{A + \gamma\left(\frac{2Q}{2}\right)^{1/2}}{A - \gamma\left(\frac{2Q}{3}\right)^{1/2}}\right\}\right\} \qquad (17)$$

When $\lambda=0$, the theory studied here becomes the pure $|\phi|^6$ theory. In that case we have

$$\tilde{W}_V = \sqrt{2}\,\frac{\cos\delta\,\cosh\gamma\rho - i\,\sin\delta\,\sinh\gamma\rho}{\left\{\frac{2}{\sqrt{4-3\gamma^2}} + \cosh 2\gamma\xi\right\}^{1/2}}\sqrt{\rho}\,\exp\left(-i\,\frac{mc^2}{2}\,t\right) \qquad (18a)$$

where

$$6 = \arctan\left\{\frac{\gamma}{\sqrt{4-3\gamma^2} + 2 - 3\gamma^2}\right\} \qquad (18b)$$

(the topological charge[1-5] of $\tilde{W}_V$ is just -2δ), and

$$\tilde{p}(V) = -\frac{\sqrt{3}}{2}\,\rho V \ln\left(\frac{2 + \sqrt{3}\,\gamma}{2 - \sqrt{3}\,\gamma}\right) \qquad (19a)$$

and

$$\sigma(V) = c\rho\left\{\gamma + \frac{\sqrt{3}}{4}\gamma^2 \ln\left(\frac{2 + \sqrt{3}\,\gamma}{2 - \sqrt{3}\,\gamma}\right)\right\} \qquad (19b)$$

Eqs. (18) and (19) agree with the results of ref. (5).

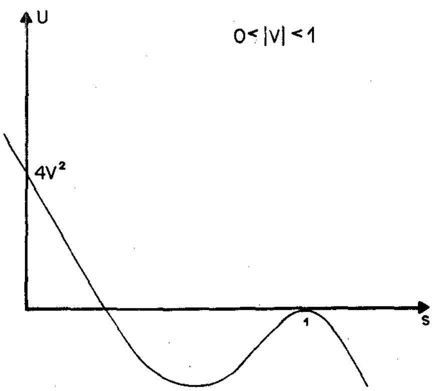

FIG. 1: Shape of U(s) in the subsonic case

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