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Complex Energy Eigenstates in a Model with Two Equal Mass Particles

R. J. GLEISER, O.A. REULA and O. M. MORESCHI

Institudo de Matemática, Astronomia Y Física, Universidade Nacional de Córdoba, Laprida 854, 5 000 Córdoba, Argentina

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The properties of a simples quantum mechanical model for the decay of two equal mass particles are studied and related to some recent work on complex energy eigenvalues. It consists essentially in a generalization of the Lee-Friedrichs model for an unstable particle and gives a highly idealized version of the KO-KO system, including CP violations. The model is completely solvable, thus allowing a comparison with the well known Weisskopf-Wigner formalism for the decay amplitudes. A different model, describing the same system is also briefly outlined.

As propriedades de um modelo quântico simples para o decaimento de duas partículas de massas iguais são estudadas e relacionadas a alguns trabalhos recentes sobre autovalores complexos da energia. Consiste essencialmente em uma generalização do modelo de Lee-Friedrichs para uma partícula instável e dá uma versão altamente idealizada do sistema KO-F, incluindo violações CP. O modelo é completamento solúvel, permitindo uma comparação com o bem conhecido formalismo *de* Weisskopf-Wigner para as amplitudes de decaimento. Um modelo diferente, descrevendo o mesmo sistema, é também esboçado.

1. INTRODUCTION

A particularly interesting class of quantum rnechanical systems for which the time evolution may be studied are formulated in terms of a hamiltonian with a point spectrum and a continuous spectrum. The point spectrum corresponds to "particles" and the continuous spectrum to "decay products". The "particles" become unstable when a suitable perturbation, mixing the continuous and discrete portions of the spectrum, is applied. More specifically, the hamiltonian H is written as

$$H = H_0 + V \tag{1}$$

where H_0 is the unperturbed ("free") system and V is the perturbation ("interaction"). If $|E_0>$ is bound eigenstate of H_0, which the perturbation renders unstable, the unstable-state probability amplitude is

$$A(t) = <E_0| \exp(-iHt) |E_0> \tag{2}$$

It is well known that this expression does not lead to a purely exponential decay law if the spectrum of H is bounded from below [1]. This problem has been considered in some recent work on the quantum decay problem from the conceptual framework of complex energy eigenvalues[2]. It is shown there that the complex energy solutions of the Lee-Friedrichs model eigenvalue problem can be incorporated in a complete, biorthogonal set of generalized eigenfunctions with the continuum eigenvalues forming a contour deformed into the unphysical region of the complex energy plane. One obtains in this formulation, in rather natural way, pole contributions that would correspond to a purely exponential behaviour of the decay amplitude. A more general problemwas considered by Bailey and Schieve[3], who studied the construction of a complete orthonormal set of complex energy states for a general hamiltonian of the form given by Eq. (1). They also formulated a solvable model to further illustrate their discussion.

In this paper we consider another solvable model which can be studied along sililar lines. The spectrumof H_0 consists of two degenerate bound states and two continua. A discrete operation, which we call CP, commuting with H_0, is introduced. One bound state and one of the continua belong to the eigenvalue +1 of CP and the other bound state and continuum to the eigenvalue -1. This can be considered as a highly idealized version of the system consisting of the K^o and $\overline{K^o}$ (the two bound states) and their most important decay products, i.e., two pions or three pions (the two continua). H_0 would then represent the "strong interactions" and V the "weak interaction" leading to the decay of the kaons.

The experimentally verified small violation of CP invariance can be incorporated by including in V contributions that do not commute with CP.

2. A SOLVABLE MODEL

The present model is formulated as follows. We first consider a "free" hamiltonian H_0. Its spectral representation is of the form

$$H_0 = \sum_{i=1}^{2} \{M|K_i><K_i| + \int_{\mu_i}^{\infty} dE\, E|E_i><E_i|\} . \qquad (3)$$

The bound states $|K_i>$ are degenerate. The μ_i denote the corresponding thresholds of the continuum eigenvalues of H_0.

The eigenvectors of H_0 form a complete orthonormal set, satisfying

$$\begin{aligned} <K_i|K_j> &= \delta_{ij} \\ <E_i|E'_j> &= \delta_{ij}\, \delta(E_i - E'_j) \\ <K_i|E_j> &= 0 \end{aligned} \qquad (4.a)$$

The closure relation in the Hilbert space basis of H_0 is

$$I = \sum_{i=1}^{2} \{|K_i><K_i| + \int_{\mu_i}^{\infty} dE\, |E_i><E_i|\} \qquad (4.b)$$

The structure of H_0 has been chosen in such a way that we can define a hermitian operator in correspondence with the degeneracy index i. This operator has a discrete spectrum with only two eigenvalues. If we choose their values as +1 and -1, it could be considered as a sort of "parity" for the states. We choose the following representation,

$$CP = \sum_{i=1}^{2} (-1)^i \{|K_i><K_i| + \int_{\mu_i}^{\infty} dE\, |E_i><E_i|\} \qquad (5)$$

Notice that with these definitions we would have $|K^0> = (|K_1> + |K_2>)/\sqrt{2}$ and $|\overline{K^0}> = (|K_1> - |K_2>)/\sqrt{2}$.

We introduce next a hamiltonian $H_1 = H_0 + V_1$, where the (CP-conserving) perturbation potential V_1 is given by

$$V_1 = \int_{\mu_1}^{\infty} \{g^*(E)\,|K_1><E_1| + g(E)\,|E_1><K_1|\; dE \;, \tag{6}$$

so that the state $|K_1>$ is now unstable.

The CP = -1 sector of H_1 consists of the stable bound state $|K_2>$ plus the continuum $|E_2>$, i.e., we have

$$H_1|K_2> = M|K_2> \tag{7}$$

$$H_1|E_2> = E|E_2> \tag{8}$$

The CP = +1 sector, on the other hand, has the structure of the Lee-Friedrichs model[4,5]. The corresponding eigenvalue-eigenvector equation

$$H_1|\lambda;1> = \lambda|\lambda;1> \tag{9}$$

can be solved by a standard procedure. If we define the cornplex function

$$\alpha(z) = M - z + \int_{\mu_1}^{\infty} \frac{|g(\mu)^2}{z - \mu}\, d\mu \tag{10}$$

the eigenvectors $|\lambda;1>$ are given by

$$|\lambda;1> = \chi_\lambda\, |K_1> + \int_{\mu_1}^{\infty} \phi_\lambda\,(E)\,|E_1>\, dE \;, \tag{11}$$

where

$$\chi_\lambda = - \frac{g^*(\lambda + i\,\varepsilon)}{\alpha(\lambda + i\varepsilon)} \;, \tag{12}$$

and

$$\phi_\lambda(E) = \frac{\chi_\lambda\; g(E)}{\lambda - E + i\,\varepsilon} + \delta(\lambda - E) \tag{13}$$

and the limit $E \to 0^+$ is implied.

Up to this point, the model consists of a stable particle, the K_2, and a CP = -1 continuum, which we may call the "three-pion" sector, and an unstable particle, the K_1 with an associated CP = +1 continuum, which we may call the "two-pion" sector. It would therefore correspond, within our idealization, to making the long lived component K_L in K^0 decay stable, while considering only as unstable the short lived component K_S. Namely, the quantum mechanical survival amplitude for $|K_1>$ is given by

$$A_{11}(t) = <K_1| \exp(-iH_1 t) |K_1> = \int_{\mu_1}^{\infty} |\chi_\lambda|^2 \exp(-i\lambda t) d\lambda \qquad (14)$$

Therefore, for times not too long or too short, and if the perturbation is assumed "small",

$$A_{11}(t) \simeq \exp(-i\bar{P}_1 t) \qquad (15)$$

where $\bar{P_1} = M_1 - i\Gamma_1/2$ gives the position of the zero of $\alpha(z)$, reached by analytically continuing $\alpha(z)$ from above the cut into the unphysical region.

We complete now the formulation of our model by considering a total hamiltonian $H = H_1 + V_2$.

In the Hilbert space basis of H_1, H has the spectral representation

$$H = M|K_2><K_2| + \int_{\mu_2}^{\infty} E|E_2><E_2|dE + \int_{\mu_2}^{\infty} \lambda\ |\lambda;1><\lambda;1|d\lambda + V_2 , \qquad (16)$$

where

$$V_{/} = \int^{\infty} \{f(\lambda)|\lambda;1><K_2| + f^*(\lambda)|K_2><\Lambda;1|\}d\lambda . \qquad (17)$$

The new perturbation V_2, causes transitions between $|K_2>$ and $|\Lambda;1>$, that is, between the K_2 and the two-pion sector. It therefore includes a violation of CP invariance.

The structure of H in the H_1 Hilbert space basis is simply that of the Lee-Friedrichs model and its eigenvalue-eigenvector problem is solved accordingly. If we define

$$\beta(z) = M - z + \int_{\mu_1}^{\infty} \frac{|f(\lambda)|^2}{z - \lambda}\, d\lambda \quad , \tag{18}$$

it is easily shown that the vectors

$$|\omega\rangle = C_\omega |K_2\rangle + \int_{\mu_1}^{\infty} S_\omega(\lambda)\, |\lambda;1\rangle\, d\lambda \quad , \tag{19}$$

where

$$S_\omega(\lambda) = \frac{C_\omega\, f(\lambda)}{\omega - \lambda + i\varepsilon} + \delta(\omega - \lambda) \tag{20}$$

and

$$C_\omega = \frac{-f^*(\omega + i\varepsilon)}{(\omega + i\varepsilon)} \quad , \tag{21}$$

are eigenvectors of H and satisfy

$$H\,|\omega\rangle = \omega\,|\omega\rangle \tag{22.a}$$

while we also have

$$H|E_2\rangle = E\,|E_2\rangle \tag{22.b}$$

It follows from the known properties of the solutions of the Lee-Friedrichs model that the vectors $|\omega\rangle$ and $|E_2\rangle$ form a complete orthonormal basis, in which any vector in the Hilbert space can be expanded. In particular we have

$$|\lambda;1\rangle = \int_{\mu_1}^{\infty} S^*_\omega(\lambda)\, |\omega\rangle\, d\omega \tag{23}$$

To compute the survival probability amplitudes for the states $|K_1\rangle$ and $|K_2\rangle$, we first write these vectors in the $|\omega\rangle$ basis. Since

$$|K_1\rangle = \int_{\mu_1}^{\infty} \chi^*_\lambda\, |\lambda;1\rangle\, d\lambda \quad , \tag{24}$$

we have

$$|K_1\rangle = \iint_{\mu_1}^{\infty} S^*_\omega(\lambda)\, \chi^*_\lambda\, |\omega\rangle\, d\omega\, d\lambda \ . \tag{25}$$

Similarly

$$|K_2\rangle = \int_{\mu_1} C^*_\omega \, |\omega\rangle \, d\omega \ . \tag{26}$$

The matrix elements

$$A_{ij}(t) = \langle K_i| \exp(-iHt) \, |K_j\rangle \tag{27}$$

are of interest. Using Eqs. (25) and (26), the functions $A_{ij}(t)$ can be written as integrals along the real axis in the interval (μ_1,∞) in the complex plane. The contour deformation techniques of Refs. (2,3) can now be used to make explicit the contributions from the poles in the complex-ω plane that correspond to the unstable states.

The computations are sketched in the Appendix. If we neglect terms of higher order in f and g, the pole contributions to the $A_{ij}(t)$ are given by

$$A_{11}(t) = \exp(-iP_1^-t)+2\pi hf(P_1^-)g^*(P_1^-)(\exp(-iP_1^-t)-\exp(-iP_2^-t))/(P_1^+-P_2)(P_1^--P_2^-), \tag{28}$$

$$A_{12}(t) = 2\pi f(P_1^-)g^*(P_1^-) \, (\exp(-iP_1^-t) - \exp(-iP_2^-t))/(iP_1^- -iP_2^-) \ , \tag{29}$$

$$A_{21}(t) = ih \, \exp(-iP_2^-t)/(P_1^+ - P_2^-) \ , \tag{30}$$

$$A_{22}(t) = \exp(-iP_2^-t) \ , \tag{31}$$

where $P_1^- = M_1 - i\,\Gamma_1/2$ $(P_1^+ = M_2 + i\,\Gamma_1/2)$ gives the position of the zero of $\alpha(z)$ reached by analytically continuing $\alpha(z)$ into the unphysical sheet from above (below) the cut, and

$$\Gamma_1 \simeq 2\pi \, |g(M_1)|^2 \ , \tag{32}$$

$$\begin{aligned} M_1 &= M - \delta M_1 \\ &= M - P\int_\mu^\infty \frac{g(\mu)}{M_1 - \mu}\, d\mu \ , \end{aligned} \tag{33}$$

and similar definitions for $P_2^\pm$ replacing $\alpha(z)$ by $\beta(z)$ and $g(\mu)$ by $f(\mu)$. We have also set

$$h = 2\pi\{f^*(P_2^-)g(P_2^-) - f^*(P_1^+)\ g(P_1^+)\} \tag{34}$$

and

$$\Delta M = M_1 - M_2$$
$$= \delta M_1 - \delta M_2 \tag{35}$$

As it is discussed in many references, these contributions dominate the behaviour of the $A_{ij}(t)$ except for very short or very long times. The contributions of the contours, which we have not explicited here, are, of course, responsible for this departure from the exponential law.

3. COMPARISON WITH THE WEISSKOPF-WIGNER FORMALISM

A standard approximation for handling unstable particle problems, in which an exponential behaviour is assumed from the outset, is furnished by the Weisskopf-Wigner method[6].

This method was applied by Lee, Yang and Oheme[7] in their study of the K^0-$\overline{K^0}$ system. A comparison of their results with the present formalism is then of interest. There is however a point which should be indicated. Because of its very simple structure, the present model is invariant under time reversal. It is therefore not invariant under CPT, while this invariance is usually assumed in most discussions of the physical K^0-$\overline{K^0}$ system. The comparisons we can makewill then be relevant only as far as the approximations made in both treatments are concerned, since our highly idealized model does not allow direct comparison with experimental data.

If we use the treatment given e.g. by Gaillard[8], we obtain the following expressions for the decay eigenvectors:

$$|\Psi_1> = (a|K_1> - b|K_2|)/\sqrt{(|a|^2 + |b|^2)}\ , \tag{36}$$

and

$$|\Psi_2> = |K_2>\ , \tag{37}$$

where

$$\frac{a}{b} = \frac{-2\pi \; g(M_1) \; f(M_1)}{(\Gamma_1 - \Gamma_2)/2 - i\Delta M} \quad , \tag{38}$$

which, in this approximation, satisfy

$$H|\Psi_1> = (M_1 - i \; \Gamma_1/2) \; |\Psi_1> \quad , \tag{39}$$

and

$$H|\Psi_2> = (M_2 - i \; \Gamma_2/2) \; |\Psi_2> \quad . \tag{40}$$

We notice that this gives the same survival probability amplitudes as those in section 1 if we neglect there terms proportional to h. This is in fact consistent with the approximations involved in the derivation of Eqs. (34) and (35). We conclude that in this model the survival probability amplitudes obtained in the complex eigenstate formalism by just keeping the pole contributions are essentially the same as those in the Weisskopf-Wigner formalism.

4. COMMENTS

In the solvable model described in the previous sections one of the "particles" was stable at the "CP-conserving" level, becoming unstable through the "CP-violating" channel. A different solvable model, where both particles are unstable at the "CP-conserving" level can be formulated as follows. We start with H_0 given by

$$H_0 = H_0^{(1)} + H_0^{(2)} \quad , \tag{41}$$

where

$$H_0^{(i)} = M_i \, |K_i><K_i| + \int_{\mu_{0,i}}^{\infty} E|E_i><E_i|dE + \int_{\mu_{0,i}}^{\infty} \{g^{(i)}(E)\,|E_i><K_i| - g^{(i)*}(E)\,|K_i><E_i|\}dE . \tag{42}$$

As in the previous sections, the $|K_i>$ are the particles and the $|E_i>$ the corresponding decay channels. The normalization is

$$<K_i|E_j> = 0 \quad , \tag{43}$$

$$<K_i|K_j> = \delta_{ij} \quad , \tag{44}$$

$$<E_i|E'_j> = \delta(E - E')\delta_{ij} \quad . \tag{45}$$

The spectrum of the $|E_i>$ is continuous and bounded from below by $\mu_{0,i}$. In general $\mu_{0,1}$ may be assumed different from $\mu_{0,2}$ and $M_i > \mu_{0,i}$. The i index denotes the CP eigenvector:

$$CP\ |K_1> = |K_1> \quad , \tag{46}$$

$$CP\ |K_2> = -\ |K_2> \quad , \tag{47}$$

$$CP\ |E_1> = |E_1> \quad , \tag{48}$$

$$CP\ |E_2> = -\ |E_2> \quad . \tag{49}$$

Thus each CP sector is given by a Lee-Friedrichs model. They can be diagonalized to give the eigenvectors of $H_0^{(1)}$ and $H_0^{(2)}$ which we denote by $|\lambda,1>$ and $|\nu,2>$ respectively, where λ and ν indicate the corresponding eigenvalues.

We consider now an interaction term mixing states with different CP values given by

$$H_I = \int_{\mu_{0,1}}^{\infty} d\lambda \int_{\mu_{0,2}}^{\infty} d\nu\ \{L(\nu,\lambda)\ |\lambda,1><\lambda,2| + L^*(\nu,\lambda)\,|\nu,2><\lambda,1|\} \ . \tag{50}$$

The resulting system can be solved in closed form if

$$L(\lambda,\nu) = f(\lambda)\,h(\nu) \tag{51}$$

The computation are rather lengthy and we shall only give the main results. Introducing

$$F(\omega) = \int_{\mu_{0,1}}^{\infty} \frac{|f(\lambda)|^2}{\omega-\lambda+i\varepsilon}\, d\lambda \tag{52}$$

and

$$G(\omega) = \int_{\mu_{0,2}}^{\infty} \frac{|g(\nu)|^2}{\omega-\nu+i\varepsilon}\, d\nu ,$$

the eigenvectors of $H = H_0 + H_I$ are

$$|\omega,1\rangle = \int_{\mu_{0,1}}^{\infty} \left\{\delta(\omega-\lambda) + \frac{f^*(\omega)\; G(\omega)\; f(\lambda)}{(1 - F(\omega)\; G(\omega))(\omega - \lambda + i\varepsilon)}\right\} |\lambda, 1\rangle\, \lambda$$

$$+ \frac{f(\omega)}{1 - F(\omega)\; G(\omega)} \int_{\mu_{0,2}}^{\infty} \frac{h^*(\lambda)}{\omega - \lambda + i\varepsilon} |\nu,2\rangle\, d\nu \tag{53}$$

and

$$|\omega,2\rangle = \frac{h(\omega)}{1 - F(\omega)\; G(\omega)} \int_{\mu_{0,1}}^{\infty} \frac{f(\lambda)}{\omega - \lambda + i\varepsilon} |\nu,1\rangle\, d\nu$$

$$+ \int_{\mu_{0,2}}^{\infty} \left\{ \delta(\omega-\lambda) + \frac{h(\omega)\; F(\omega)\; h^*(\nu)}{(1 - F(\omega)\; G(\omega))(\omega - \nu + i\varepsilon)} \right\} |\nu,2\rangle\, d\nu. \tag{54}$$

It can be shown that they satisfy the orthonorrnalization and closure relations

$$\langle\omega,i|\omega',j\rangle = \delta_{ij}\, \delta(\omega - \omega') \tag{55}$$

and

$$\int_{\mu_{0,min}}^{\infty} \{|\omega,1\rangle\langle\omega,1| + |\omega,2\rangle\langle\omega,2|\}\, d\omega = I \tag{56}$$

From these relations one can compute explicit expressions for $|K_1\rangle$ and $|K_2\rangle$ in terms of the $|\omega,i\rangle$ and the functions $g^{(i)}(x)$, $f(x)$ and $h(x)$, which can be used to obtain the survival probability amplitudes.

Although in this model the "particles" are treated in a more symmetrical manner, both being unstable at the CP-conserving level, the same observations as in the previous section, regarding its relation to the physical kaon system, can be made. Clearly, more structure would be needed to improve this situation. The aim of our study has been mainly directed towards the construction of solvable models incorporating at least two important features experimentally encountered, i.e., a twice degenerate bound state and a discrete symmetry, which can be used to lavel degenerate states and which is eventually broken by the interactions. These models may eventually be useful in the construction of more elaborate treatrnents.

APPENDIX

In this Appendix we show how to compute the survíval probability amplitude matrix elements, sketching the'derivation for $A_{21}(t)$. The other matrix elernents are obtained in a similar way.

Frorn Eqs. (25), (26), (20) and (21) we have

$$A_{21}(t) = \langle K_2|\ \exp(-iHt)\ |K_1\rangle$$

$$= \iint_{\mu_1}^{\infty} C_\omega \chi_\lambda^* S_\omega^*(\lambda)\ \exp-(i\omega t)\ d\omega\ d\lambda$$

$$= - \iint_{\mu_1}^{\infty} \frac{\exp(-i\omega t)\ |C_\omega|^2\ f^*(\lambda)\ g(\lambda)}{\alpha(\lambda - i\varepsilon)(\omega - \lambda - i\eta)}\ d\lambda\ d\omega$$

$$= \int_{\mu_1}^{\infty} \frac{\exp(-i\omega t)\ g(\omega)\ f^*(\omega)}{\alpha(\omega - i\varepsilon)\ \beta(\omega + i\eta)}\ d\omega \quad .$$

Then using

$$\frac{1}{\omega - \lambda - i\eta} = i \int_0^\infty \exp((-i(\omega-\lambda) - \eta)x)\ dx.$$

we have

$$A_{21}(t) = -i\int_{\mu_1}^{\infty}\int_0^{\infty} dx\, \exp(-i(t-x))\ |C_\omega|^2\, d\omega\, \frac{\exp(i\lambda x)\ f^*(\lambda)\ g(\lambda)}{\alpha\ (\lambda - i\varepsilon)}\, d\lambda$$

$$+ \int_{\mu_2}^{\infty} \frac{g(\omega)\ f^*(\omega)\ \exp(-i\omega t)}{\alpha(\omega - i\varepsilon)\ \beta(\omega + i\eta)}\, d\omega$$

$$= -2\pi\int_{\mu_1}^{\infty} \exp(-iP_2^-(t-x) - iP_1^+x)\ f^*(P_1^+)\ g(P_1^+)\ dx$$

$$+ 2\pi\, i\, f(P_2^-)\ g(P_2^-)\ \exp(-iP_2^-t)/(P_1^+ - P_2^-)$$

$$A_{21}(t) = \frac{i\ h\ \exp(-iP_2^-t)}{P_1^+ - P_2^-}$$

$$= \frac{2\ h\ \exp((-iM_2 - \Gamma_2/2)t)}{-i2\ \Delta M - (\Gamma_1 - \Gamma_2)}$$

where we have made use of the following approximations

$$\int_{\mu_1}^{\infty} |C_\omega|^2\ \exp(-i\omega x)\ d\omega = \exp(-iP_2^-x)\ \Theta(x) - \exp(-iP_2^+x)\ \Theta(-x) + O(g^2/M),$$

$$\int_{\mu_1}^{\infty} \frac{\exp(-2\lambda x)\ h(\lambda)}{\beta(\lambda \mp i\varepsilon)}\, d\lambda = \mp\, 2\pi i\ h(P_2^{\mp})\ \exp(-iP_2^{\mp}x)\ \Theta(x) + O(g^2/M)\ ,$$

and

$$\alpha(P_2^-) \simeq P_1^+ - P_2^- \quad .$$

REFERENCES

1. L.A.Kalfin, Sov. Phys. JETP 6, 1053 (1958).
2. E.C.G.Sudarshan, C.B.Chiu and V.Gorini, Phys.Rev. D*18*, 2914(1978).
3. T.K.Bailey and W.C.Schieve, Nuovo Cimento *47*A, *2* (1978).

4. T.D.Lee, Phys. Rev. *95*, 1329 (1954).

5. K.O.Friedrichs, Comm. Pure and Applied Math. *1*, 361 (1948).

6. V.F.Weisskopf and E.P.Wigner, Z.Physik 63, 54(1930), 65, 18(1930).

7. T.D.Lee, R.Oehme and C.N.Yang, Phys.Rev.*106*, 340 (1957).

8. J.M.Gaillard, "The Neutral Kaon System", in *Weak Interactions* Ed. by M.K.Gaillard and M.Nikolic, Inst. National de Physique Nucleaire, Paris (1977).