

On The Specific Heat of the Liquid Helium

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Recebido em 14 de dezembro de 1979

We make an attempt to explain the A-singularity in the specific heat of the superfluid He^4 in terms of an order-disorder transition. We assume that the superfluid phase of Helium exhibit an off-diagonal long-range order.

Tenta-se explicar a singularidade A no calor específico do He^4 superfluido em termos de uma transição ordem-desordem. Supõe-se que a fase superfluida do Hélio exiba uma ordem de longo alcance fora da diagonal.

For low temperatures ($T < 0.6$ K) it is expected that all the thermal energy is associated with the excitation of longitudinal phonons¹. In this case, the de Broglie wavelength λ is bigger than the mean intermolecular separation a .

As the temperature rises, local atomic motions become relatively more important than the collective excitations: λ decreases so that $\lambda \leq a$. Let us call E the energy levels that an atom can assume in these local vibrations: $E_n = 0$ for $n = 0$ and $E_n = \epsilon_0 + \epsilon_n$ for $n = 1, 2, \dots$, with $\epsilon_1 = 0$. In our mean field approximation ϵ_0 is an adjustable parameter and is the minimum value of energy that a particle can have in local motions: for this value of energy $\lambda \sim a$. We have found that $a^* = \epsilon_0/kT = 2.6$, where $T_\lambda = 2.19$ K is the temperature of the A-point and k is the Boltzmann constant.

Due to the weak interaction of the Helium atoms, we must expect that the energy spectrum ϵ_n is close to the free particle spectrum.

Since we are not able, up to the present moment, to incorporate in a consistent scheme both phonon excitations and individual atomic motion, we take an additive superposition of the two contributions. The phonon energy can be easily obtained and is given, for instance, by London¹.

Let us now calculate the contribution of the individual atomic motion. If N is the total number of Helium atoms we have, using the Bose-Einstein statistics and the fact that the spectrum ϵ_n is quasi-continuum ($\epsilon_{n+1} - \epsilon_n \ll kT$):

$$N = N_0 + N_{\text{exc}} = \frac{1}{e^{\alpha} - 1} + kT \int_0^{\infty} \frac{1}{\psi(\epsilon, T)} \frac{d(\epsilon/kT)}{e^{\alpha + \epsilon/kT} - 1} \quad (1)$$

where $N_0 = 1/e^{\alpha} - 1$ is the number of particles in the ground state, $a' = a + \epsilon_0/kT$, and $1/\psi(\epsilon, T)$ is the number of states in the interval $d\epsilon$ of energy at a temperature T . We assume that the energy spectrum ϵ_n changes with the temperature. The function $1/\psi(\epsilon, T)$ increases, since the energy spectrum ϵ_n tends to an energy spectrum of free particles, when T increases.

If the particles were free, $1/\psi$ would be proportional to $\epsilon^{1/2}$ and if they were vibrating harmonically about a center of equilibrium, $1/\psi$ would be equal to $1/h\nu$, where ν is the fundamental frequency of vibration. Thus $1/\psi$ would be independent of ϵ .

It seems reasonable to expect that $1/\psi \propto \epsilon^{\delta}$ where δ is closer to $1/2$ than to 0 . As we will see in what follows this dependence of $1/\psi$ with ϵ^{δ} ($0 \leq \delta \leq 1/2$) is not important for $T < T_{\lambda}$. Thus we put simply $1/\psi(\epsilon, T) = 2/\sqrt{\pi} \epsilon^{1/2}/\psi(T)$. In this case, the number of excited particles N_{exc} given in equation (1) becomes¹:

$$N_{\text{exc}} = \frac{2}{\sqrt{\pi}} \frac{(kT)^{3/2}}{\psi(T)} \int_0^{\infty} \frac{x^{1/2} dx}{e^{\alpha + x} - 1} = \frac{(kT)^{3/2}}{\psi(T)} F_{3/2} \left(\alpha + \frac{\epsilon_0}{kT} \right) \quad (2)$$

As we will see, the choice of $\delta = 1/2$ is important for $T \geq T_\lambda$. For $T > T_\lambda$, the condition $N_{\text{exc}} = N$ must be satisfied.

The total energy U is given by:

$$U = \frac{2}{\sqrt{\pi}} \frac{1}{\psi(T)} \int_0^\infty \frac{(\epsilon + \epsilon_0) \epsilon^{1/2} d\epsilon}{e^{\alpha^2 + \epsilon/kT} - 1} = N_{\text{exc}} \left[\frac{3}{2} kT \frac{F_{5/2}(\alpha + \epsilon_0/kT)}{F_{3/2}(\alpha + \epsilon_0/kT)} + \epsilon_0 \right] \quad (3)$$

Our problem now is to obtain $\psi(T)$. As pointed out by Liu and Fisher² it is reasonably well established that the usual superfluid phase of Helium, liquid He II, possesses an off-diagonal long-range order (ODLRO), while the normally observed solid phase of Helium displays crystalline order³⁻⁹. To date no theory has been proposed to take into account both the long-range order in crystal and superfluid phases. In the absence of a complete theory Mullin⁷ and Liu and Fisher² have adopted the lattice gas model to describe a Bose system which has three possible phases: "normal" crystal with crystalline order, superfluid liquid with ODLRO and a third phase, having both crystalline order and ODLRO which was named "supersolid". In the phase diagram of He⁴ the supersolid phase may appear between the usual solid and superfluid phases².

The lattice gas was originally introduced by Matsubara and Matsuda¹⁰ as a model for studying the lambda transition in liquid Helium.

Despite the artificiality in many respects of the lattice model it seems to describe the essential features of the superfluid and crystal phases.

If the pressure is maintained constant the long range order of the superfluid decreases as the temperature increases. According to the lattice theory it seems reasonable to assume that the order parameter, that we call X , obeys the equation $X = \tanh(T_\lambda/T X)$.

We must expect that the energy spectrum ϵ_n is the free particle spectrum when the system is completely disordered ($X = 0$ at $T = T_\lambda$). So $\psi(T)$ must decrease when X decreases. Let us assume that $\psi(T) =$

$= \eta \cdot (1 + \xi X^\theta)$ where the parameter η is determined using the condition $N_{\text{exc}}(T_\lambda) = N$ and θ and ξ are adjustable parameters.

Now substituting $\psi(T) = \eta \cdot (1 + \xi X^\theta)$ in formulas (2) and (3) we obtain the specific heat per unit of mass C_v^- , for $T < T_\lambda$, which is given by:

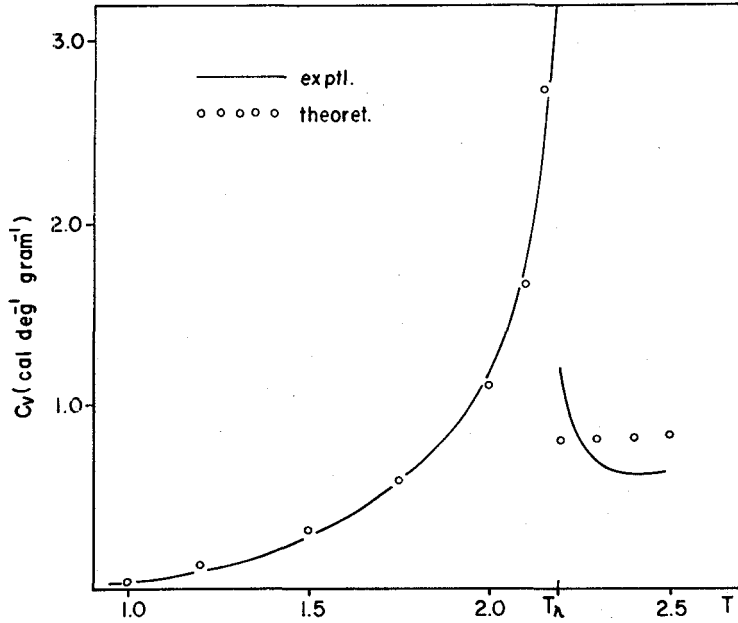
$$C_v^- = \frac{k}{m} \left(\frac{T}{T_\lambda}\right)^{3/2} \frac{1}{(1+\xi X^\theta)} \left[\frac{15}{4} + 3\alpha^* \frac{T_\lambda}{T} + \alpha^{*2} \left(\frac{T_\lambda}{T}\right)^2 \right] \exp\left[\alpha^* \left(1 - \frac{T_\lambda}{T}\right)\right] \\ + \frac{k}{m} \left(\frac{T}{T_\lambda}\right)^{1/2} \frac{\theta \xi X^\theta}{(1+\xi X^\theta)^2} \frac{\left[\frac{3}{2} + \alpha^* \frac{T_\lambda}{T}\right]}{\cosh^2\left[\frac{T_\lambda}{T} X\right] - \frac{T_\lambda}{T}} \exp\left[\alpha^* \left(1 - \frac{T_\lambda}{T}\right)\right]$$

For $T \geq T_A$ the specific heat per unit of mass C_v^+ is given by $C_v^+ = \frac{3}{2} \frac{k}{m}$, which is the specific heat of an ideal gas.

We see in the figure the theoretical results compared with the experimental results of Keesom and Clusius and Keesom and Keesom¹ (the phonons contribution, which is very small compared with C_v^- and C_v^+ , has been also taken into account). The best agreement with the experimental results was found putting $\theta = 0.22$, $\alpha^* = 2.60$ and $\xi = 8.0$. Our results for C_v^- diverges as $(T_\lambda - T)^{-0.89}$ at the λ -point and experimentally it seems to diverge as $\log|T - T_A|$. A better agreement at T_λ could be obtained by allowing $\psi(T)$ to be a more appropriate function of X .

To obtain C_v^- and C_v^+ , we have substituted $F_{3/2}(\epsilon_0/kT)$ and $F_{1/2}(\epsilon_0/kT)$ by $\exp(-\epsilon_0/kT)$ since $\epsilon_0/kT > 2.6$. The functions $F_\sigma(y)$ ($\sigma = 3/2$) can be approximate by $\exp(-y)$ for $y \geq 2.0$ with an error of only a few percent. For this reason, our result for C_v^- is only slightly modified if we put $\delta = 0$ in the function $1/\psi(\epsilon, T) \propto \epsilon^\delta$ and the agreement with the experimental results is the same as that obtained above with $\delta = 1/2$.

On the other hand for $\delta = 0$, $C_v^+ = k/m$ which is the specific heat of an isolated harmonic oscillator. In this case, the agreement with the experimental results is not as good as $3k/2m$, which we obtained above with $\delta = 1/2$.



The specific heat C as a function of T .

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