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The Mass-Velocity Relation in Special Relativity

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The relativistic time-dilation is taken as a starting point for deriving the mass-increase formula of special relativity. The analysis of two examples of mechanical systems shows that the slowing down of their internal processes is incompatible with newtonian mechanics, and leads naturally to the assumption of a variable mass. The relevant physical presupposition used in these two new derivations and in old ones is momentum conservation. Although supported by analogy to classical conservation laws, this supposition as commonly presented shows an arbitrariness which can be removed by operational elucidation of thedynamical concepts.

Utiliza-se o efeito relativístico de "dilatação do tempo" como ponto de partida para a dedução da fórmula de aumento de massa da relatividade especial. A análise de dois exemplos de sistemas mecânicos mostra que o retardamento de seus processos internos é incompatível com a mecânica newtoniana, e leva naturalmente à suposição de uma massa variável. O pressuposto físico relevante utilizado nessas duas novas deduções e em outras antigas é a conservação do momento. Embora apoiada por analogia com as leis clássicas de conservação, esta suposição, do modo como é comumente apresentada, exibe uma arbitrariedade que pode ser removida pelo esclarecimento operacional dos conceitos dinâmicos.

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1. INTRODUCTION

The relativistic relation between (transverse) mass and speed of a material body,

$$m = m_0/(1-v^2/c^2)^{1/2} \tag{1}$$

has historically ariseri from the study of the electromagnetic mass of the electron, iniciated by Abraham[1]. It was correctly derived for the first time by Lorentz[2] and appears on Einstein's first paper on relativity[3], where it is derived from eiectrodynamical considerations. These pioneer approaches were not intended to be general: the obtained results dependend on the adopted definition of force and acceleration, as pointed by Einstein[3], who has also produced a different formula, by considering another definition of transverse electromagnetic force[4]. There, he arrives at:

$$m = m_0/(1-v^2/c^2) \quad . \tag{2}$$

Besides its historical relevance, the electrodynamical approach in the form first presented in 1911 by Laue[5], where particle dynamics is derived from continuum mechanics, is considered by Costa de Beauregard[6] to be the best way of forrnulating special relativity, because this allows the total number of independent postulates to be reduced to a minimurn.

Purely mechanical formulations of relativistic dynamics have been produced from time to time, and an excellent survey of these is presented by Arzeliès[7], where relevant references can be found. Some modern advanced textbooks[8,9] use four-vectors and covariance properties in order to obtain the relativistic momentum-energy, and then define a relativistic mass in accordance with these formulas. This approach was presented earlier by Einstein[10] in a simplified way; its physical meaning was implicit in a still earlier paper by Lalan[11].

Many authors of introductory textbooks[12-15] and classical treatises[16,17] prefer the classical derivation by Lewis and Tolman[18] where an example of collision tliough-experiment is analysed, and momentum transformation is derived from relativistic kinematics and momentum conservation. Then, by analogy to newtonian definition of momentum, a relativistic (transverse) mass is defined. Although somewhat lengthy, this seems to be hitherto the best didactic way of introducing relativistic dynamics, because it employs only elementary mathematics, and makes use of enlightening physical analysis of the collision situation.

In our teaching practice, however, we have felt that physics students do not usually understand the important physical assumptions at issue, and do never perceive any conceptual relation between formula (1) and similar relations from relativistic kinematics. such as the "time dilation" formula:

$$T = T_0/(1-v^2/c^2)^{1/2} \quad (3)$$

We have developed two new derivations of (1) that show a simple and clear relation between formulas (1) and (3), and that have not been published previously[19]. In this paper, after presenting the derivations in their simplest form, some conceptual problems are discussed. A methodological imperfection is found, which is also present in Lewis and Tolman's method, and that can only be surmounted by acoherent presentation of relativistic dynamics.

2. TIME DILATION AND MECHANICAL CLOCKS

Let us suppose that M is a material system where a periodic process with proper period T_0 occurs. If M is initially at rest relative to an inertial referential S, and is then slowly accelerated (so that its mechanism is neither damaged nor suffers any sensible elastic deformation) until it attains a speed v relative to S, then it will be observed to have a new period T relative to S, as given by (3). This is a consequence of relativistic kinematics, and must apply to any kind of periodical process, independently of its interna1 working. Now, this result it incompatible with classical dynamics, because tnere are very simple kinds of mechanical periodic motions which could not undergo any change of period when the system is accelerated, if newtonian mechanics were valid. Two such periodic processes will be discussed in this paper (Sections 3 and 4): a frictionless gyroscope and a bouncing material body moving along a frictionless beam kept perpendicular to the direction of acceleration. In these simples cases to be analysed, the constancy of the period predicted by classical mechanics is a consequence of four assumptions:

P1) geometry conservation;

P2) definition of momentum;

P3) parallelism between force and momentum change;

P4) mass constancy.

When the systems are analysed according to special relativity, it is seen that relativistic kinematics allows us to retain assumption (P1) when relevant geometrical factors are perpendicularto the accelerotion of M. Then, at least one of the remaining classical assumptions must be changed in order to allow a period change of these processes. It will be shown that if we drop (P4), (P2) and (P3) may be retained, and relation (1) follows as a necessary consequence of these assurnptions and relativistic kinematics. The scientific status of (P2) and (P3) will be discussed afterwards (Section 5).

3. THE BOUNCING CYLINDER

Let us suppose that a material body such as a drilled cylinder C moves up and down[20] along a frictionless beam B inside a box which is at rest relative to referential system S (Fig.1).

The speed u_z of the rnoving cylinder C is constant, relative to 3, except when it interacts with small springs at the ends of B. The inertial mass of the box is supposed to be so much larger than that of C, that the box recoil may be neglected, in order that it can be supposed to be at rest.

The springs produce a velocity inversion of C. The interaction time is supposed to be negligibly small, as compared to the period of the oscillations. If the vertical[20] distance travelled by the cylinder inside the box is h, the period of its oscillations, relative to S, will be:

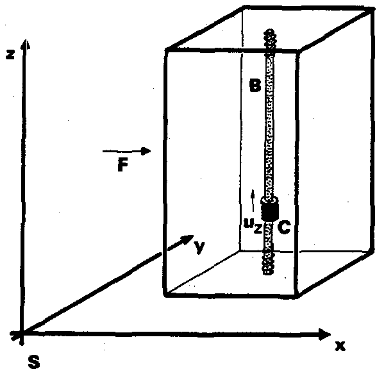

Fig.1 - A cylinder C slides up and down aiong a frictionless beam 8. When the box is accelerated by a horizontal force F, the cylinder moves slower. This could not happen if newtonian mechanics were valid. The speed change may be explained by a mass increase.

$$T_0 = 2.h_0/u_z \tag{4}$$

If the box is now submited to a force F directed parallel to the x-axis of system S, and reaches any horizontal speed relative to S, and then its velocity remains constant, the period of the cylinder oscilations would not change, if classical rnechanics were valid: the frictionless beam B can only communicate to C a force perpendicular to the beam; so, p_z (the z-component of the cylinder's momentum p) cannot change, by (P3). Now, by the classical definition of momentum (P2), we have:

$$p_z = m.u_z \tag{5}$$

If p_z is constant, and if we assume that m is constant (P4), relation (5) implies that u_z must remain constant. Now, the period depends only on h and u . But there is no classical reason for súpposing that the geometry of the box will change (P1); so, the period should be constant.

But, by relativistic kinematics, we know that this period must chanqe, and that it will generally have different values relative to non-equivalent inertial referential frames. After attaininq a speed v, its new period T' will become

$$T' = T_0/(1-v^2/c^2)^{1/2} \tag{6}$$

Kinematically, it is easily seen that this period must be equal to:

$$T' = 2.h'/u'_z \qquad (7)$$

where h' and u'_z are respectively the new vertical distance travelled by the cylinder and its new vertical speed relative to S. Now, as in this case the height of the box is perpendicular to its motion, it does not change, because Lorentz contraction is always parallel to the velocity. So, $h'=h_0$, and by (4) and (7) we have:

$$T'/T_0 = u_z/u'_z \qquad (8)$$

Applying (6) we deduce that the vertical component of the velocity of the cylinder has been reduced:

$$u'_z = u_z(1-v^2/c^2)^{1/2} \qquad (9)$$

Why should the speed change, if there is no vertical force applied to the cylinder? The answer must be a dynamical one. If the z-component of the momentum has not change (P3) and if we addopt the classical definition of momentum (P3), we must have

$$p'_z = p_z \qquad (10)$$

$$p_z = m.u_z \qquad (11)$$

$$p'_z = m'.u'_z \qquad (12)$$

and from these formulas we deduce

$$m.u_z = m'.u'_z \qquad (13)$$

Using (6) and (8) we obtain:

$$m'/m = u_z/u'_z = T'/T_0 = (1-v^2/c^2)^{1/2} \qquad (14)$$

This shows the whole of the argument is a contracted form : mass must change because vertical speeds change, and these change because the period must change.

4. THE GYHOSCOPE

A similar deduction may be taken from the analysis of a "gyroscopic clock", which for simplicity will be supposed to be similar to a bicycle wheel W which spins along an axis A fastened by frictionless bearings HH to the walls of a rigid box. The box is at rest relative to a referential system S, and the axis of the wheel is parallel to its X-axis (Fig.2). The radius of the spinning wheel is r, and its whole mass m is supposed to be at this distance of the axis. Its moment of inertia I will be:

$$I = m.r^2 \tag{15}$$

and its angular momentum M is:

$$M = I.\, 2\pi/T = 2\pi m.r^2/T \ . \tag{16}$$

Now, if the box acquires some speed relative to S, by the application of a force parallel to the x-axis, no torque will be applied to the wheel, and so, by classical mechanics, its angular momentum must remain constant. As the radius and mass of the wheel could not classically change, the constancy of the angular momentum implies the constancy of angular speed and period.

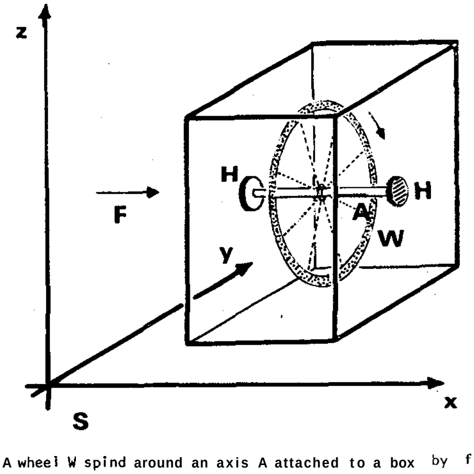

Fig.2 - A wheel W spind around an axis A attached to a box by frictionless bearings HH. When the external force F transmits to this system a speed v, the wheel must turn slower, by relativistic time-dilation. Asthe angular momentum could not vary, its mass must have changed.

In this analysis, two of the relevant suppositions are slightly different from those used in Section 3. They could be thus expressed:

P2') definition of angular momentum;

p3') constancy of angular momentum in the absence of an applied torque.

These assumptions, in classical mechanics, are intimately related to the previous ones, as is well known.

The analysis from the point of view of special relativity shows that r remains constant, in this case. So, if the period changes, we must accept either that angular momentum may change without any externally applied torque, or that m is not constant, in order to explain the period change. If we retain (P2') and (P3') in relativistic dynamics, then, from time-dilation formula (5) and from the constancy of M we deduce:

$$2\pi m'.r^2/T' = 2\pi m.r^2/T \tag{17}$$

$$m'/m = T'/T \tag{18}$$

$$m' = m/(1-v^2/c^2)^{1/2} \quad . \tag{19}$$

5. CONCEPTUAL DISCUSSION

There are three central points of these derivations which deserve discussion. Two of them are completely answerable problems.

5.1 - In the two examples presented at sections 3 and 4, we compare the behaviour of the same physical system, before and after acceleration, relative to S, in order to derive the mass-speed relation. But the presented relations of special relativity (particularly the time-dilation formula) are not derived as consequences of the acceleration of a material system: they are deduced as the necessary relations between measures of a system which is always at rest relative to a given referential system, but which is observed and measured also relative to another referential system which moves relative to the first one, at a constant velocity. So, the question arises: is it valid to use the time-dilation formula in the studied examples? If it is valid, may the deduced consequence (mass - velocity relation) be applied to situations where no acceleration is applied to the material system whose mass is considered?

We may answer affirmatively to both question. This is a consequence of the supposition that, if a physical system acquires in any non-destructive way any constant velocity relative to a given inertial referential system I, the final behaviour of this physical system, relative to its rest frame, is independent of its speed relative to I, and is also independent of the accelerative process. This is one of the ways of expressing the physical content of the principle of special relativity.

5.2 - In its usual form, the mass-velocity relation (1) compares the rest mass m_0 of a material system with its mass m relative to another referential frame. But in the two cases presented in this paper, the material body whose mass change is cornputed is never at rest relative to the used referential systems: it has a vertical or circular motion, even before acceleration of its enclosing bo~.May we assume that relations (14) and (19) do also apply to any object initially at rest relative to S ? Is the result mathematically consistent with (1) ? Let us see.

If any part of a material system suffers a relativistic mass change, all of its other parts must undergo a proportional change, because if this did not happen, the moving system would not behave in the same way as when it was at rest. Besides, the initial motion of the cilinder and of the spinning wheel may be as slow as one wishes, and so there is no reason for supposing that the mass change would be different if they were initially at rest.

As to the mathematical consistency, some simple calculations show that (1), (14) and (19) are compatible. We will study only the cylinder example:

Let us suppose that the rest mass of the cylinder C (Fig. 1) is m_0. When the box which contains C is at rest relative to S, the cylinder's mass must be, from (1):

$$m = m_0/(1-u_z^2/c^2)^{1/2} \qquad (20)$$

When the box acquires the speed v, the speed of C relative to S becomes

$$w = (u_z'^2 + v^2)^{1/2} \qquad (21)$$

so that its mass should be

$$m' = m_0/(1-w^2/c^2)^{1/2}$$

$$m' = m_0/[1-(u_z'^2+v^2)/c^2]^{1/2} \qquad (22)$$

Relation (22) is deduced from (1). Is it compatible with (14)? Yes. A simple manipulation allows us to deduce (14) from (9), (20) and (22). The proof will not be reproduced here.

5.3 - The great conceptual difficulty of the derivations shown in this paper (and which are also present, under disguised form in the usual didactic derivations) is the physical support of (P3) - or (P3'). First, let us notice that (P2) - or (P2') - is just a definition which allows (P3) - or (P3') - to make any sense. The physical propostion is not contained in the definition, which presents no conceptual problem. The question is: what supports the supposition that p - or the angular momentum - remains constant in the studied examples?

One could be tempted to use the classical constancy of these magnitudes as evidence, but this is not valid. In fact, we may easily show that there are other mechanical magnitudes which should remain constant according to classical mechanics and which our "intuitive feelings" would not allow to change - but which do not remain constant, according to special relativity. Let us take as an instance the magnitude K defined as

$$K_z = m.u_z^2/2 \; . \qquad (23)$$

This magnitude is the contribution of the vertical motion of a moving body to its classical kinetic energy. According to classical mechanics, this magnitude remains constant, in the case of the bouncing cylinder. There is no reason to suppose that K_z changes when the box is accelerated, because there is no vertical component of any force acting on the cylinder, and so there is no work done on that direction. But if we suppose that this magnitude is also constant in relativistic mechanics, we obtain the wrong result analogous to (2):

$$m' = m/(1-v^2/c^2) \tag{24}$$

The same relation would be deduced if we tried to separate the gyroscope's kinetic energy in two parts (translaticnal and rotational) and then supposed that its rotational kinetic energy K_r

$$K_r = I.(2\pi/T)^2/2 \tag{25}$$

remained constant.

Although this kind of premises lead to the wrong result, it seems very natural and "intuitive" to allow the kinetic energy to be decomposed into several independent contributions. This was (wrongly) used by Epstein[21], who, in a paper about special relativity, considers the motion of a simple pendulum and says:

"In this case, the kinetic energy of oscillations can be readily separated from the kinetic energy of the translatory motion and the usua! theory of the pendulum can he applied."

From this, he obtains the wrong transformations formula for forces in special relativity.

In a simple derivation of the mass-speed relation, Bondi[22] tried to justify the constancy of p by the following argument:

"To visualize this sort of toy experiment we may consider a case where B fires bullets at pieces of armor plating and we shall make the assuption that the penetrating power of the bullets depends solely on the component of momentum at right angles to the pieces of armor plating. (see our Fig.3)

"We assume that A and B agree about distances at right angles to their motion. If they do, then since the thickness of the armor plating is at right angles to B's motion, they agree on this. They agree

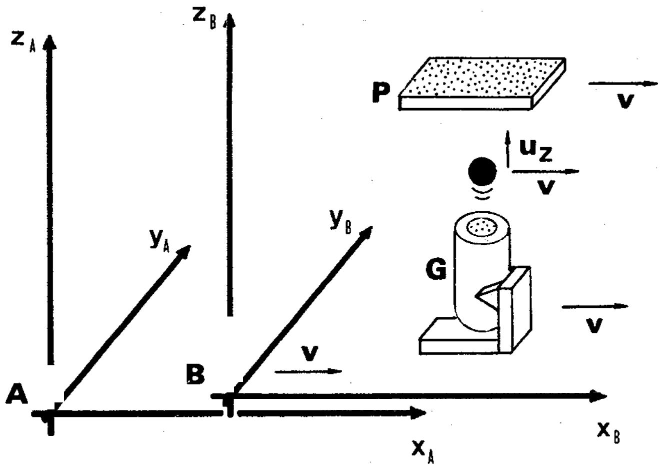

Fig.3 - The method proposed by H. Bondi for comparison of transverse linear nwmnta: a gun G fires a bullet towards an armor plating P; Bondi supposes that the penetrating power of the bullet depends only on its transverse momentum.

also on the distance from the muzzle of B's gun to the armor plating. Furthermore, whether or not the bullet penetrates the armor plating is something that A can see and if the penetrating power depends wholly on the normal momentum, then by using different bits of armor plating and seeing whether the bullets penetrate them or not, A and B can reach entire agreement on the normal rnomentum of B's bullets."

Now, there are two important faults in this argument. First: in classical physics, the penetrating power of the bullets does not depend on the component of rnomentum at right angles to the pieces of armor plating; it depends (and is directly proporcional, in this case) to the "component" of kinetic energy K_z at right angies to the armor plate. Second: he assumes that the motion of the armor plating does not change its intrinsic resistance to the bullet, because its thickness did not change. That is not correct. A relativistic analysis which will not be shown in this paper proves that the intrinsic resistance of the armor plating changes with its speed (it is easier to penetrate a shield at rest relative to the gun than one moving relative to the gun, perpendicular to the bullet's direction). Although Bondi's argument is wrong, he obtains the correct relation between mass and speed, because the two errors mutually cancel.

So, why should we prefer (1) and (14) to *(2)* and (24) ? Our discussion has showed that there remains a kind of arbitrariness in relativistic dynamics if we try to build it from sirnple analogies of classical dynamics. This arbitrariness arised from the tentative of deriving a relation between mass and velocity without an explicit elucidation of the dynarnical concepts, such as force, momentum, energy. There is no arbitrariness when operational meanings are given to at least one of the dynamical magnitudes. This will be shown in a following paper, which will present a detailed examination of this methodological aspect of relativity, which has not deserved up to now the close scrutinity that it deserves.

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