Revista Brasileira de Física, Vol. 10, Nº 2, 1980

The Role of Gaussian Domination and Sum Rules in Phase Transitions — An Unpedagogical Introduction

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Recebido em 8 de abril de 1980

"A large class of classical and quantum systems exhibit phase transitions of the same nature of those observed in the Free Bose Gas and in the Spherical Model", This statement is discussed in an introductory manner.

"Uma classe ampla de sistemas clássicos e quânticos exibe transição de fase de mesma natureza daquelas observadas no Gas de Bose Livre e no Modelo Esférico". Apresentamos uma discussão introdutória dessa proposição.

1. INTRODUCTION

The purpose of this paper is to show how, from the study of two motivating examples: the Free Bose Gas and the Spherical Model, it is possible to ahstract some general features of the phase transitions associated to a large class of classical and quantum systems. These models refer to quite different physical situations (the Free Bose Gas describes quantum particles, the Spherical Model incorporates some features of classical ferromagnets or classical lattice gases). In spite of that, there are striking similarities in thair behavior, specially with respect to the associated phase transitions. The common features we want to stress are:

* Partial Financia1 support by CNPq.

a) Their *Gaussian nature*, which accounts for their explicit solvability and for a specific behavior of the two-point functions $\rho(p)$ around $p = 0$; and

b) The Sum *Rules*, which in the - Free Bose Gas corresponds to the density constraint and in the spherical model to the "sphericity" constraint.

In $v \geq 3$ dimensions and at low enough temperatures, a) and b) combine, in both models, to produce a "condensate" of zero momentum or a spontaneous symmetry breakdown.

Great progress has been achieved in the theory of phase transitions and spontaneous symmetry breakdown after the pioneering work of Fröhlich, Simon and Spencer [1,FSS] who realized that features a) and b), in the form of *inequalities* could be found in a large class of models. In particular they showed that the N-vector model (N=1 is the Ising model, N=2 is the plane rotator and N=3 is the classical Heisenberg model) in $\nu \geq 3$ dimensions have a phase transition and provided lower bounds for their critical temperatures after proving:

a') *Gaussian Domination*: the two point function $\rho(p)$ of these models are, for $p \neq 0$, dominated by the two point function of the spherical model (Infrared bound)

b') *Sum Rules*, which just express the fact that

$$\sigma^2 \equiv \sigma_1 \sigma, + \ldots + \sigma_N \cdot \sigma_N = 1, \quad N=1, \ldots$$

As in the Free Bose Gas and in the Spherical Modelat low enough temperatures, a') and b') combine again to produce a "condensate" of zero momentum or a spontaneous magnetization.

The lower bounds in the critical temperature are excellent when compared to values which are *considered* to be exact: for N=I (Ising) the errors is 14%, for N=3 (Heisenberg) 9% and as $N \to \infty$ (spherical model !) it is exact.

Extensions of these ideas, techniques and results have been obtained by Dyson, Lieb and Simon[2]. In particular they prove phase transitions and estimate critical temperatures for the x-y model and for the quantum Heisenberg *antiferromagnet* in $\nu \geq 3$ dirnensions (but not for the Heisenberg *ferromagnet*, as erroneously annouced in the original version). By another point of view, quantum systems have been also analysed by Driessler, Landau, Perez and Perez-Wrezinski[3,4,5] where, for a class of systems, the problem is reduced to a classical one after "euclideanization".

The crucial Infrared Bound is, both for quantum and classical systems, a consequence of a positivity condition known as Reflection Positivity. This property allows the introduction of a scalar product in the space of observables and Gaussian Domination follows from the associated Schwarz inequality. The general theory of Reflection Positivity and Gaussian Domination is developed in the series of papers[6] by Fröhlich, Israel, Lieb and Simon.

We are not going to prove a') here we will rather restrict ourselves to understanding its content and exemplify its applications. Therefore this paper may be viewed as an introduction and could serve as a guide and an appetizer to the reading of the original papers quoted above. The material presented originated from a series of lectures on these topics held at different places. It is a pleasure to thank Ricardo Schor for suggesting its publication. We thank also W.Wreszinski for a careful revision of the manuscript and for stimulating discussions.

2. THE FREE BOSE GAS

Let us consider a gas at a fixed density ρ and ternperatures β of non interacting particles obeying the Bose. Einstein statistics, enclosed in a volume $\Lambda \subset R^\nu$. If we take Λ to be a cubic box $\Lambda = \left[-\frac{L}{2}, +\frac{L}{2}\right]^\nu$ with periodic boundary conditions[+], the Hamiltonian H_Λ is given by

$$H_\Lambda = \sum_{k \in \Lambda^*} \left[\omega(k) - \mu\right] a^*(k)\, a(k)$$

+ This choice of the boundary conditions is just a matter of convenience.

$$= \int_\Lambda d^\nu x \ \{\tfrac{1}{2}(\nabla\psi^*(x))(\nabla\psi(x)) - \mu\psi^*(x)\psi(x)\} \qquad (2.1)$$

where:

a) $\Lambda^* = \{k = \frac{2\pi}{L} n \quad n \in Z^\nu\} =$

$$\{k = (k_1, \ldots, k_\nu),\ k_j = \frac{2\pi}{L} n_j,\ n_j \in Z\} \qquad (2.2)$$

b) $\psi(x)$ and $\psi^*(x)$ satisfy the canonical commutation rules

$$[\psi(x), \psi^*(y)] = \delta(x-y) \qquad (2.3a)$$

$$[\psi(x), \psi(y)] = 0 = [\psi^*(x), \psi^*(y)] .$$

The Fourier transforms of $\psi(x)$ and $\psi^*(x)$.

$$\begin{cases} a(k) = \frac{1}{\sqrt{\Lambda}} \int_\Lambda d^\nu x\ \psi(x)\ e^{-ik.x} \\ a^*(k) = \frac{1}{\sqrt{\Lambda}} \int d^\nu x\ \psi^*(x)\ e^{+ik.x} \end{cases} \qquad (2.3b)$$

satisfy the canonical commutation relations:

$$[a(k), a^*(k')] = \delta_{kk'} \qquad (2.3c)$$

$$[a(k), a(k')] = 0 = [a^*(k), a^*(k)]$$

c) $\omega(k) = \frac{k^2}{2}$ is the energy of a free particle of momentum ; and

d) the "chemical potential" μ is introduced as usual in the grand-canonical ensemble, in order to adjust the density ρ of the system, that is, μ is a function $\mu_\Lambda(\rho)$ defined implicitly by the equation:

$$\rho = \frac{\langle N\rangle_\Lambda}{\Lambda} = \frac{1}{\Lambda} \sum_{k \in \Lambda^*} \langle a^*(k)\ a(k)\rangle_\Lambda \qquad (2.4)$$

where the symbol $\langle A\rangle_\Lambda$ means the expectation value of the observable A in the Gibbs state at given inverse temperature B:

$$< A >_\Lambda = \frac{\mathrm{Tr}\, A\, e^{-\beta H_\Lambda}}{\mathrm{Tr}\, e^{-\beta H_\Lambda}} \tag{2.5}$$

A standard computation yields tne well-known result:

$$\rho_\Lambda(k,\mu) \equiv \frac{1}{\Lambda} <a^*(k)a(k)>_\Lambda = \frac{1}{\Lambda} \frac{1}{e^{\beta[\omega(k)-\mu]} - 1} \tag{2.6}$$

and therefore the "sum rule" (2.4) reads:

$$\rho_\Lambda(\mu) \equiv \sum_{k \in \Lambda^*} \rho_\Lambda(k,\mu) = \frac{1}{\Lambda} \sum_{k \in \Lambda^*} \frac{1}{\exp \beta[\omega(k)-\mu]-1} = \rho \tag{2.7}$$

The functions $\rho_\Lambda(k,\mu)$ have the following properties:

a) $\rho_\Lambda(k,\mu) > \rho_\Lambda(k, \mu') > 0$ (2.8)

if $0 > \mu > \mu'$

b) $\rho_\Lambda(k,\mu) \xrightarrow[\mu \to -\infty]{} 0$ (2.9)

c)
$$\begin{cases} \rho_\Lambda(k,\mu) \xrightarrow{\mu \uparrow 0} \frac{1}{\Lambda} \frac{1}{\exp \beta\omega(k) - 1}, & k \neq 0 \\ \rho_\Lambda(0,\mu) = \frac{1}{\Lambda} \frac{1}{e^{-\beta\mu} - 1} \xrightarrow[\mu \uparrow 0]{} \infty & \end{cases} \tag{2.10}$$

d) $\rho_\Lambda(\mu)$ is convex.

Therefore at fixed $\beta > 0$ and $\rho > 0$ and for all Λ finite there is unique solution

$$\mu = \mu_\Lambda(\rho) < 0 \tag{2.11}$$

of the density condition (sum rule) (2.7).*

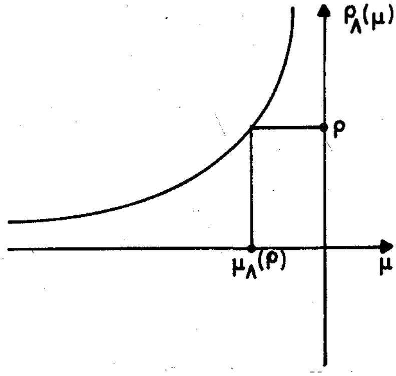

Fig.1

Defining

$$\rho_\Lambda(k) \equiv \rho_\Lambda(k, \mu_\Lambda(\rho)) \tag{2.12}$$

we obtain from (2.11) and (2.6)

$$\lim_{\Lambda\to\infty} \rho_\Lambda(k) = 0 \quad \text{if} \quad k \neq 0 \tag{2.13a}$$

and

$$\rho(0) \equiv \lim_{\Lambda\to\infty} \rho_\Lambda(0) = \lim_{\Lambda\to\infty} \frac{1}{\Lambda} \frac{1}{\exp\left[-\beta\mu_\Lambda(\rho)\right]-1} \tag{2.13b}$$

For a comprehensive discusssion of the thermodynamic limit of the Free Bose Gas see references [9] and [11].

On the other hand, from (2.7)

$$\rho_\Lambda(0) = \rho - \frac{1}{\Lambda} \sum_{\substack{k \in \Lambda \\ k \neq 0}} \frac{1}{\exp \beta\left[\omega(k) - \mu_\Lambda(\rho)\right] - 1} \tag{2.14}$$

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* From a), b) and c) listed above, the function $\rho_\Lambda(\mu) = \sum_{k\in\Lambda^*} \rho_\Lambda(k,\mu)$ has a graph looking like the plot of figure 1.

which in the thermodinamic limit $\Lambda \to \infty$ reads

$$\rho(0) = \rho - \frac{1}{(2\pi)^\nu} \int d^\nu k \; \frac{1}{\exp \beta[\omega(k) - \mu(\rho)] - 1} \tag{2.15}$$

where $\rho(0) = \lim_{\Lambda\to\infty} \rho_\Lambda(0)$ and $\mu(\rho) = \lim_{\Lambda\to\infty} \mu_\Lambda(\rho)$.

Now, for $\mu < 0$ from (2.8)

$$\frac{1}{(2\pi)^\nu} \int d^\nu k \; \frac{1}{e^{\beta[\omega(k)-\mu]} - 1} \leq \rho_{max}(\beta) \equiv \frac{1}{(2\pi)^\nu} \int d^\nu k \; \frac{1}{e^{\beta\omega(k)} - 1} \tag{2.16}$$

and,

$$\rho_{max}(\beta) < \infty \tag{2.17}$$

if and only if $\nu \geq 3$. (*)

From (2.15), (2.16) and (2.17) we then conclude: $\rho(0) > 0$ if $\rho > \rho_{max}(\beta)$, i.e. there is a macroscopic occupation of the zero-energy state which is the phenomenon of Bose-Einstein condensation.

The occurrence of condensation is connected with the spontaneous breakdown of the gauge symmetry, that is, the invariance of H_Λ under the transformation

$$\begin{aligned} a(k) &\longrightarrow e^{i\alpha}\, a(k) \\ a^*(k) &\longrightarrow e^{-i\alpha}\, a^*(k) \end{aligned} \tag{2.18}$$

To show that link we consider the Hamiltonian

(*) In fact, the singularity of the integrand in (2.16) at k=0 is integrable if and only if $\nu \geq 3$, since $e^{\beta\omega(k)} - 1 = 0(\omega(k)) = 0(k^2)$ as $k \to 0$ and

$$\int_{|k|\leq 1} \frac{d^\nu k}{k^2} < \infty \quad \text{iff} \quad \nu \geq 3 .$$

$$\tilde{H}_\Lambda = \sum_{k \in \Lambda} [\omega(k)-\mu]\ a^*(k)a(k) + \lambda\sqrt{\Lambda}\ [a^*(0) + a(0)]$$

$$= \sum_{\substack{k \in \Lambda \\ k \neq 0}} [\omega(k)-\mu]a^*(k)a(k) + [\omega(0)-\mu]\{[a^*(0) + \frac{\lambda\sqrt{\Lambda}}{[\omega(0)-\mu]}]$$

$$\times [a(0) + \frac{\lambda\sqrt{\Lambda}}{\omega(0)-\mu}]\} - \lambda^2\Lambda \qquad (2.19)$$

Introducing new variables

$$b(k) = a(k),\ b^*(k) = a^*(k) \quad \text{if} \quad k \neq 0$$

and

$$b(0) = a(0) + \frac{\lambda\sqrt{\Lambda}}{\omega(0)-\mu},\ b^*(0) = a^*(0) + \frac{\lambda\sqrt{\Lambda}}{\omega(0)-\mu}$$

which also satisfy the canonical commutation rules (2.3) we obtain

$$H_\Lambda \equiv \tilde{H}_\Lambda + \lambda^2\Lambda = \sum_{k \in \Lambda} [\omega(k)-\mu]\ b^*(k)b(k)$$

and we are back to the original problem. The two point function is therefore given by

$$\langle a^*(k)\ a(k)\rangle_\Lambda = \frac{1}{\exp\beta[\omega(k) - \mu] - 1}, \quad k \neq 0,$$

$$\langle a^*(0)\ a(0)\rangle_\Lambda = \frac{1}{e^{-\beta\mu} - 1} + \frac{\lambda^2\Lambda}{\mu} \qquad (2.20)$$

and the sum rule (2.4) reads

$$\frac{\lambda^2}{\mu^2} + \frac{1}{\Lambda}\frac{1}{e^{-\beta\mu} - 1} + \frac{1}{\Lambda}\sum_{\substack{k \in \Lambda \\ k \neq 0}} \frac{1}{\exp\beta[\omega(k)-\mu] - 1} = \rho \qquad (2.21)$$

Because of the extra term $\frac{\lambda^2}{\mu^2}$ (as compared to (2.7)), the solution $\mu_\Lambda(\lambda,\rho)$ of (2.21) for $\lambda\neq 0$ remains strictly negative even in the limit $\Lambda \to \infty$:

$$\frac{\lambda^2}{\mu^2} + \frac{1}{(2\pi)^\nu} \int d^\nu k \, \frac{1}{e^{\beta[\omega(k)-\mu]} - 1} = \rho \tag{2.22}$$

On the other hand, the one point functions are given by:

$$<\psi^*(0)>_\Lambda = \frac{1}{\Lambda} \int_\Lambda <\psi^*(x)>_\Lambda d^\nu x = \frac{1}{\sqrt{\Lambda}} <a^*(0)>_\Lambda = \frac{\lambda}{\mu_\Lambda(\lambda,\rho)}$$

$$<\psi(0)>_\Lambda = \frac{1}{\Lambda} \int_\Lambda <\psi(x)>_\Lambda d^\nu x = \frac{1}{\sqrt{\Lambda}} <a(0)>_\Lambda = \frac{\lambda}{\mu_\Lambda(\lambda,\rho)} \tag{2.23}$$

where we used in the first equality sign, the invariance under translations. In the thermodynamic limit

$$<\psi(0)> = <\psi^*(0)> = \frac{\lambda}{\mu(\lambda,\rho)} . \tag{2.24}$$

From (2.22) we get then

$$<\psi^*(0)><\psi(0)> = \rho - \frac{1}{(2\pi)^y} \int d^x p \, \frac{1}{e^{\beta|\omega(k)-\mu(\lambda,\rho)]} - 1} = \frac{\lambda^2}{\mu(\lambda,\rho)} \tag{2.25}$$

and so even when $\lambda \to 0$ (after having taken the lim $A \to \infty$!)

$$|<\psi\ (0)>|^2 = |<\psi(0)>|^2 = \rho - \rho_{max}(\beta) > 0$$

if $\rho > \rho_{max}(\beta)$. That is the syrnmetry is not restored by taking $X \to 0$ after taking $\lambda \to \infty$.

3. THE SPHERICAL MODEL

In 1952, Berlin and Kac[7] introduced a model which incorporated some features of the Ising model and had the advantage of being explicitly solvable in all dimensions. The model was coined "spherical" because of its kynematics which can be described as follows.

In a finite volume Λ in a ν-dimensional cubic lattice, i.e. $\Lambda \subset Z^{\nu}$ we consider classical "spin" variables $\phi(x) \in R$ at each site $x \in \Lambda$. For simplicity we will take Λ to be the hypercube $\Lambda = \{-L, \ldots, 0, \ldots, +L\}^{\nu}$. The variables $\phi(x)$ are however constrained by the condition

$$\frac{1}{\Lambda} \sum_{x \in \Lambda} \phi^2(x) = 1 \tag{3.1}$$

Therefore a configuration ϕ of this system is a function $\phi : \Lambda \to R$ and can be viewed as a point in the surface of a Λ dimensional sphere of radius $\sqrt{\Lambda}$ as opposed to the Ising model, where $\phi(x) = \pm 1$ and whose configurations are the vertices of hypercube of side 2 in Λ dimensions.

The energy $H_{\Lambda}(\phi)$ with periodic boundary conditions of a configuration is given by

$$H_{\Lambda}(\phi) = (\phi \left[-\frac{\Delta}{2} - \mu\right]\phi) = \sum_{x \in \Lambda} \phi(x) \left[(-\Delta-\mu)\phi\right](x) \tag{3.2}$$

where:

a) the "lattice laplacean" Δ is given by

$$(-\Delta\phi)(x) = 2\phi(x) - \sum_{i=1}^{\nu} \left[\phi(x + e_i) - \phi(x - e_i)\right] \tag{3.3}$$

the e_i, $i=1,\ldots,\nu$ being the unit vectors in the i-th direction. In (3.3) we used periodic boundary conditions in Λ.

b) the scalar product (f, g) is defined by

$$(f, g) = \sum_{x \in \Lambda} \overline{f(x)}\, g(x) \tag{3.4}$$

and

c) the "chemical potencial" $\mu = \mu_{\Lambda}(\beta)$ is introduced in order to handle the spherical constraint (3.1) in the same way we treated the density condition of the free Bose gas (grand-canonical ensemble!) i.e. $\mu_{\Lambda}(\beta)$ solves the equation

$$\frac{1}{\Lambda} <(\phi,\phi)>_{\Lambda} = \frac{1}{\Lambda} \sum_{x \in \Lambda} <\phi^2(x)>_{\Lambda} = 1 \tag{3.5}$$

where $<.>_\Lambda$ refers to the expectation value in the Gibbs' state defined by H_Λ at inverse temperature β

$$<F>_\Lambda = \frac{\int (\prod_{x \in \Lambda} d\phi(x))\, F(\phi)\, e^{-\beta H(\phi)}}{\int (\pi d\phi(x))\, e^{-\beta H(\phi)}} \tag{3.6}$$

The Fourier transformation $\hat{f}$ of a function $f = \Lambda \to C$ is defined by

$$\hat{f}(p) = \frac{1}{\sqrt{\Lambda}} \sum_{x \in \Lambda} e^{-ip.x} f(x) \tag{3.7}$$

for $p \in \Lambda^* = \{p = \frac{x\pi}{2L+1},\ x \in \Lambda\}$, that is $\hat{f} = \Lambda^* \to C$.

The Hamiltonian (3.2) is "diagonalized" by Fourier transforming of the configurations:

$$H_\Lambda(\phi) = \sum_{k \in \Lambda^*} |\omega(k) - \mu|\, \hat{\phi}^*(k)\hat{\phi}(k) \tag{3.8}$$

where

a) $\omega(k) = \sum_{i=1}^{\nu} (1 - \cos k_i)$ (3.9)

b) $\hat{\phi}^*(k)$ denotes the complex conjugate of $\hat{\phi}(k)$, and from the reality of $\phi(x)$ it follows that

$$\hat{\phi}^*(k) = \phi(-k) \tag{3.10}$$

The model is solvable, since the computation of its correlation functions involves only Gaussian integrals. The twopoint function $<\hat{\phi}^*(k)\hat{\phi}(k)>_\Lambda$ for instance is given by:

$$<\hat{\phi}^*(k)\hat{\phi}(k)>_\Lambda = \frac{1}{\beta[\omega(k) - \mu]} \tag{3.11}$$

Fhe "sum rule" (3.5) can be rewritten as

$$\frac{1}{\Lambda} \sum_{k \in \Lambda^*} \langle \hat{\phi}^*(k) \hat{\phi}(k) \rangle = 1 \tag{3.12}$$

where we used the fact that $\sum_{k \in \Lambda^*} \bar{\hat{f}}(k) \hat{g}(k) = \sum_{x \in \Lambda} \bar{f}(x) g(x)$. Therefore

$$\frac{1}{\Lambda} \sum_{k \in \Lambda^*} \frac{1}{2\beta [\omega(k) - \mu]} = 1 \tag{3.13}$$

As in section 2, for all finite Λ there is a unique solution $\mu_\Lambda(\beta) < 0$ of (3.13). (Verify that the function $f_\Lambda(\mu)$ in the left hand side satisfies: a) $f_\Lambda(\mu) > f_\Lambda(\mu')$ if $0 > \mu > \mu'$; b) $f_\Lambda(\mu) \xrightarrow[\mu \to \infty-]{} 0$ and c) $f_\Lambda(\mu) \xrightarrow[\mu \to 0]{} \infty$, d) $f_\Lambda(.)$ is convex.

In the thermodynamic limit, we have

$$\begin{aligned} \rho(0) &= \lim_{\Lambda \to \infty} \frac{1}{\Lambda} \langle \hat{\phi}^*(0) \hat{\phi}(0) \rangle_\Lambda = \\ &= 1 - \lim_{\Lambda \to \infty} \frac{1}{\Lambda} \sum_{\substack{k \in \Lambda^* \\ k \neq 0}} \frac{1}{2\beta [\omega(k) - \mu_\Lambda(\beta)]} = \\ &= 1 - \frac{1}{(2\pi)^\nu} \int_{B_\nu} d^\nu k \, \frac{1}{2\beta [\omega(k) - \mu(\beta)]} \end{aligned} \tag{3.14}$$

where $B_\nu = [-\pi, +\pi]^\nu$ and $\mu(\beta) = \lim_{\Lambda \to \infty} \mu_\Lambda(\beta)$. Now, for $\mu(\beta) \leq 0$

$$\frac{1}{(2\pi)^\nu} \int_{B_\nu} \frac{d^\nu(k)}{2[\omega(k) - \mu(\beta)]} \leq \frac{1}{(2\pi)^\nu} \int_{B_\nu} \frac{d^\nu k}{2\omega(k)} \equiv I(\nu) \tag{3.15}$$

and so

$$\rho(0) \geq 1 - \frac{I(\nu)}{\beta} \tag{3.16}$$

Notice that $I(\nu) < \infty$ iff $\nu \geq 3$. This implies that for $\beta > I(\nu)$, $\nu \geq 3$ we have

$$\rho(0) > 0 \tag{3.17}$$

i.e. there is "condensation" in the zero energy mode.

The occurence of "condensation" in the spherical model is, as in the free Bose gas, connected with the existence of spontaneous "magnetization". To see that, let us introduce an uniform external field h, by considering the new Hamiltonian

$$\tilde{H}_\Lambda(\phi) = H_\Lambda(\phi) - h \sum_{x \in \Lambda} \phi(x) =$$

$$= H_\Lambda(\phi) - \sqrt{\Lambda}\, h\, \hat{\phi}(0) =$$

$$= \sum_{\substack{k \neq 0 \\ k \in \Lambda^*}} [\omega(k)-\mu]\hat{\phi}^*(k)\phi(k) + [\omega(0)-\mu]\left[\hat{\phi}(0) - \frac{h\sqrt{\Lambda}}{2[\omega(0)-\mu]}\right]^2 - \frac{h^2\Lambda}{4[\omega(0)-\mu]} \tag{3.18}$$

If we introduce new variables $\psi(x)$ given by

$$\begin{aligned} \hat{\psi}(k) &= \hat{\phi}(k) \;,\; k \neq 0 \\ \hat{\psi}(0) &= \hat{\phi}(0) - \frac{h\sqrt{\Lambda}}{2[\omega(0)-\mu]} \end{aligned} \tag{3.19}$$

we are back to the original problem:

$$\tilde{H}_\Lambda(\phi) + \frac{h^2\Lambda}{4[\omega(0)-\mu]} = H_\Lambda(\psi) \tag{3.20}$$

The two point function of the a variables are then given by:

$$\begin{aligned} <\phi^*(k)\phi(k)>_\Lambda &= \frac{1}{\beta}\, \frac{1}{2[\omega(k)-\mu]} \;,\; k \neq 0 \\ <\phi^*(0)\,\phi(0)>_\Lambda &= \frac{1}{\beta}\, \frac{1}{2[\omega(0)-\mu]} + \frac{h^2\Lambda}{4[\omega(0)-\mu]^2} \end{aligned} \tag{3.21}$$

The sum rule reads then

$$\frac{h^2}{4[\omega(0)-\mu]^2} + \frac{1}{\Lambda} \sum_{k \in \Lambda^*} \frac{1}{2[\omega(k)-\mu]} = 1 \tag{3.22}$$

Due to the extra term on the left hand side of (3.22), the unique, solution $\mu_\Lambda(\beta,h) < 0$ remains strictly negative even in the limit $\Lambda\to\infty$ if $k \neq 0$:

$$\lim_{\Lambda\to\infty} \mu_\Lambda(\beta,h) \equiv \mu(\beta,h) < 0 \tag{3.23}$$

In this limit (3.22) reads

$$\frac{h^2}{4[\omega(0)-\mu(\beta,h)]^2} + \frac{1}{\beta} \int \frac{d^\nu k}{2[\omega(k) - \mu(\beta,h)]} = 1 \tag{3.24}$$

since from (3.21) we have

$$\lim_{\Lambda\to\infty} \frac{1}{\Lambda} <\phi^*(0)\phi(0)> = \frac{h^2}{4[\omega(0)-\omega]^2}$$

On the other hand, the one point function is given by

$$<\phi(0)>_\Lambda = \frac{1}{\Lambda} \sum_{x\in\Lambda} <\phi(0)> = \frac{1}{\sqrt{\Lambda}} <\hat{\phi}(0)>_\Lambda = + \frac{h}{2[\omega(0) - \mu_\Lambda(\beta,h)]} \tag{3.25}$$

where again, we used translation invariance.

In the thermodynamic limit

$$m(h) \equiv \lim_{\Lambda\to\infty} <\phi(0)>_\Lambda = \frac{h}{2[\omega(0) - \mu(\beta,h)]} \tag{3.26}$$

and so from (3.24)

$$m(h)^2 \equiv 1 - \frac{1}{\beta} \frac{1}{(2\pi)^2} \int_{B_\nu} \frac{d^\nu k}{2[\omega(k) - \mu(\beta,h)]} \tag{3.27}$$

Therefore, since $\mu(\beta,h) \to 0$ as $h \to 0$ (so as to keep (3.24) valid) if $\beta > \beta_c = I(\nu)$ we have

$$m(0)^2 = 1 - \frac{I(\nu)}{\beta} \tag{3.28}$$

that is the spontaneous magnetization squared is equal to the "density of condensate" $\rho(0)$ given by (3.14). It is important to keep in mind that for $\beta > \beta_c = I(\nu)$, $\rho(0) \neq 0$ if we first set $h = 0$ and then take the limit $\Lambda \to \infty$. If we first fix $h \neq 0$, take the limit $\Lambda \to \infty$ and *then* the limit $h \to$ we get $\rho(0) = 0$ by the remark following (3.24). For $m(h)$ it is just the opposite: if we first take Λ finite and set $h = 0$ then $m(0) = 0$.

4. THE N-VECTOR MODEL

This model for $N \geq 2$ is a generalization of the Ising model ($N=1$). At each lattice site $x \in Z^\nu$ we have a continuous "spin" variable $\phi(x) = (\phi, \quad _N(x)) \in R$ subject to the constraint

$$\phi(x)^2 \equiv \sum_{i=1}^{N} \phi_i(x)^2 = 1 \qquad (4.1)$$

That is a configuration of the system in a finite volume $\Lambda = \{-L,\ldots, L\}^\nu$ is a function $\phi = \Lambda \to S_{N-1}$ where S_{N-1} is the sphere of radius 1 in R^N.

The Hamiltonian is the usual Ising Hamiltonian:

$$H_\Lambda(\phi) = (\phi, -\frac{\Delta}{2} \phi) \qquad (4.2)$$

where the lattice laplacean, and the scalar product (,) are as defined in §3.

A "sum rule" is trivially obtained from (4.1):

$$\frac{1}{\Lambda}(\phi,\phi) = \frac{1}{\Lambda} \sum_{x \in \Lambda} \phi(k)^2 = \frac{1}{\Lambda} \sum_{k \in \Lambda^*} \hat{\phi}^*(k)\hat{\phi}(k) = 1 \qquad (4.3)$$

Notice that (4.3), in contradistinction to (3.5), holds without taking expectation value.

As is well known this model is not explicitly solvable in $\nu \geq 2$ dimensions the only exception being the case $N=1$, $\nu=2$ (two dimensional Ising model). Numerical calculations however, suggested the existence of phase transitions with spontaneous breakdown of the $O(N)$ symmetry for $\nu \geq 3$

and $N \geq 2$. The existence of phase transitions in the Ising model, $N = 1$, $\nu \geq 3$ follows from Griffiths' inequalities. For $N \geq 2$, $\nu \leq 2$ no spontaneous breakdown of a *continuous* symmetry for short-range interactions is possible, by a theorem of Dobrushin and Shlosman[10]. (Another result of Mermin and Wagner[8] forbids spontaneous magnetization).

The central results of FSS [1] is the estimate

$$Z_\Lambda(h) \leq Z_\Lambda(0) \tag{4.4}$$

where:

1) h is a function

$$h : \Lambda \to R^N$$
$$x \mapsto h(x) \in R^N$$

and

2) $Z_\Lambda(h)$ is the partition function of a modified N-vector model obtained after the replacement in the Hamiltonian (4.1) of $\phi(x)$ by $\phi(x) + h(x)$, $x \in \Lambda$; i.e.

$$Z_\Lambda(h) = \int \{ \prod_{x \in \Lambda} \delta(\phi^2(x) - 1) d\phi(x) \} \exp\{-\beta H_\Lambda(\phi + h)\} \tag{4.5}$$

Since

$$H_\Lambda(\phi + h) = H_\Lambda(\phi) + (h, -\Delta\phi) + (h, -\frac{\Delta}{2} h) \tag{4.6}$$

we can rewrite (4.4) in the form

$$\frac{Z_\Lambda(h)}{Z_\Lambda(0)} = \langle e^{-(h,\Delta\phi)} \rangle \; e^{(h, \frac{\Delta}{2} h)} \leq 1 \tag{4.7}$$

i.e.

$$\langle e^{-(h,\Delta\phi)} \rangle \leq e^{+(h, -\frac{\Delta}{2} h)} \tag{4.8}$$

If in (4.8) we replace h by λh and consider the Taylor expansion in λ of both members we have

$$1 - \lambda \langle (h,\Delta\phi) \rangle + \lambda^2 \langle (h,-\Delta\phi)(h,-\Delta\phi) \rangle \leq 1 + \lambda^2 (h, -\frac{\Delta}{2} h) \tag{4.9}$$

since, by translation invariance

$$\langle h,\Delta\phi\rangle = \sum_{x\in\Lambda} (-\Delta h)(x)\ \langle\phi(x)\rangle = \langle\phi(0)\rangle \sum_{x\in\Lambda} (-\Delta h)(x) = 0 \ .$$

Therefore:

$$\langle(h,-\Delta\phi)(h,-\Delta\phi)\rangle \leq (h,\ -\frac{\Delta}{2}h) \tag{4.10}$$

If we take

$$h_j(x) = \frac{1}{\sqrt{2}}(e^{ip.x} + e^{-ip.x})\,\delta_{ji} \quad , \quad j = 1,\dots,\nu \tag{4.11}$$

we get

$$\langle\hat{\phi}_i(p)^*\ \hat{\phi}_i(p)\rangle \leq \frac{1}{2\beta E(p)} \qquad p \neq 0 \tag{4.12}$$

and summing over $i = 1,\dots,\nu$

$$\langle\hat{\phi}^*(p)\cdot\hat{\phi}(p)\rangle \leq \frac{N}{2\beta E(p)} \tag{4.13}$$

This is the so called Infrared Bound which expresses the phenomenon of gaussian domination. In order to understand this nomenclature compare (4.13) with the exact two-point function (3.10) of the (gaussian) spherical model. (The factor N in (4.13) accounts for the number of components of$).

If we now combine, as we did in § 2 and in § 3 , the *sum rule* (4.3) and the *Infrared bound* (4.13) we have:

$$\frac{1}{\Lambda}\langle\hat{\phi}^*(0)\,\hat{\phi}(0)\rangle \geq 1 - \frac{1}{\Lambda}\sum_{\substack{p\in\Lambda^*\\ p\neq 0}} \frac{1}{2\beta E(p)} \tag{4.14}$$

and in the thermodynamic limit

$$\rho(0) = \lim_{\Lambda\to\infty} \frac{1}{\Lambda}\langle\phi^*(0)\phi(0)\rangle \geq 1 - \frac{N}{\beta} I(\nu) \tag{4.15}$$

with $I(\nu)$ given by (3.15).

From (4.15) we see that if $\nu \geq 3$ and

$$\beta > \bar{\beta}_c = NI(\nu) < \infty \qquad (4.16)$$

then

$$\rho(0) > 0$$

i.e. then is condensation.

The relation between condensation and spontaneous magnetization is in this case more subtle than in the previous examples. However in[2] then is a proof (a generalization of an argument by Griffths) of the relation

$$m(0)^2 \geq \rho(0) \qquad (4.17)$$

The method requires $\nu \geq 3$, since $\nu \leq 2$ we have $I(\nu) = \infty$. However if suitable long range interactions are allowed then, it is possible to prove with the same technique to prove phase transitions also for $\nu = 1,2$.

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