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The Sturm-Liouville Expansion for the Kummer Green Function

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The Green function for the Kummer differential equation is calculated by means of the Sturm-Liouville method. As by-product we obtain Laguerre, Hermite and Whittaker Green functions.

Calcula-se a função de Green para a equação diferencial de Kummer pelo método de Sturm-Liouville. Como casos particulares obtêm-se também as funções de Green para as equações diferenciais de Laguerre, Hermite e Whittaker.

1. INTRODUCTION

In a recent paper[1] we have presented a sistematic Sturm-Liouville expansion for the Green function for a number of special functions, the common feature for all of them being that the corresponding differential equation could be derived from the hypergeometric differential equation. In this paper we extend this calculation to those special function which are special cases of Kummer or confluent hypergeornetric equation. Particular cases of Kummer functions are Laguerre, Hermite and Whitakker functions. As a by-product of Whittaker functions we also derive the Green functions for Bessel functions.

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The present paper is organized in the following way: in the second section we calculate the Green function for Kummer differential eqcation; in the third we obtain Green functions for Laguerre, Hermite and Whittaker differential equations. Discussions are presented in thefourth section.

2. KUMMER GREEN'S FUNCTION

Kummer's differential operator[2] reads

$$L_z = z \frac{d^2}{dz^2} + (c-z) \frac{d}{dz} - a \quad . \tag{1}$$

The Green functions for this differential operator satisfiesthe following inhomogeneous differential equation

$$L_z \, G(z,z') = \delta(z-z') \tag{2}$$

which is bounded on the dornain $0 \leq z < \infty$.

Restrictions for the parameters are: i) $c \neq -m$, m non-negative integer and all'values of a; ii) $a \neq -n$, $c = -m$ or $a = -n$, $c = -m$ and $m < n$, where m, n are non-negative integers. For the special case $c \neq -m$, $a = -n$, m and n non-negative integers, we have polynomials solutions.

The Sturm-Liouville method[3] consists in writing the Green function as the product of two linearly independent solutions of the corresponding homogeneous differential equation $L_z \, \Psi(z) = 0$. These linearly independent solutions are Kummer functions ${}_1F_1(a,c;z)$ and $U(a,c;z)$, the first being regular at the origin and the second at infinity.

We write the Green function, then, as

$$G(z,z') = \frac{\Gamma(a)}{\Gamma(c)} \, {}_1F_1(a,c;z_<) U(a,c;z_>) \tag{3}$$

where z and z are min (z,z') and max (z,z') respectively.

3. PARTICULAR CASES

a) Laguerre Green Function

Laguerre functions[2] are particular polynomial cases of Kummer functions with a = -ν, ν non-negative integer and $c = 1+\alpha$, $a > -1$. The relation between Kummer and Laguerre functions is

$$L_\nu^{(\alpha)}(z) = \frac{\Gamma(\nu+\alpha+1)}{\Gamma(\alpha+1)\Gamma(\nu+1)} {}_1F_1(-\nu,\ 1+\alpha;\ z) \ . \qquad (4)$$

The second solution for the Laguerre differential equation can be defined in terms of the $U(a,c;z)$ Kummer function in the following way

$$N_\nu^{(\alpha)}(z) = \frac{\Gamma(\nu+1)\Gamma(-\nu)}{\Gamma(\nu+\alpha+1)} U(-\nu,\ 1+\alpha;z) \ . \qquad (5)$$

Using Eq. (3), we obtain the Green function for the associated Laguerre differential equation

$$G(z,z') = L_\nu^{(\alpha)}(z_<)\ N_\nu^{(\alpha)}(z_>) \ . \qquad (6)$$

b) Herrnite Green Function

Hermite functions* are particular cases of Laguerre functions with a = ±1/2, depending on the parity.

Relation between Laguerre and Hermite functions are

$$L_n^{(1/2)}(z) = \frac{(-1)^n}{\Gamma(n+1)} z^{-1/2}\, 2^{-2n-1}\, H_{2n+i}(\sqrt{z})$$

$$L_n^{(-1/2)}(z) = (-1)^n \frac{2^{-2n}}{\Gamma(n+1)} H_{2n}(\sqrt{z}) \qquad (7)$$

and for the second solution we define $H_n(\sqrt{z})$, related with the second Laguerre solution by

$$
\begin{aligned}
N_{\nu}^{(1/2)}(z) &= z^{-1/2}\, 2^{2n+1}\, \Gamma(n+1)\, H_{2n+1}(\sqrt{z}) \\
N_{\nu}^{(-1/2)}(z) &= \Gamma(n+1)\; 2 \quad H_{2n}(\sqrt{z}) \; .
\end{aligned}
\tag{8}
$$

Then using Eq. (6) we write the Green functions as

$$
\begin{aligned}
G^{-}(z,z') &= (-1)^{n}(zz')^{-1/2}\, H_{2n+1}(\sqrt{z_<})\, H_{2n+1}(\sqrt{z_>}) \\
G^{+}(z,z') &= (-1)^{n}\, H_{2n}(\sqrt{z_<})\, H_{2n}(\sqrt{z_>})
\end{aligned}
\tag{9}
$$

where $G^{-}(z,z')$ and $G^{+}(z,z')$ are odd and even parity Green function respectively for thc Hermite differential equation.

c) Whittaker Green Function

The Kummer differential equation can be put into the auto-adjunct form where the differential equation does not involve the first derivative term.

Making the exchange $\omega(z) = e^{z/2} z^{-c/2} u(z)$ in the Kummer differential equation, with $a = \mu-\nu+1/2$ and $c = 2\mu+1$ we obtain the Whittaker differential equation[2].

The Whittaker differential operator L_z is

$$
L_z = \frac{d^2}{dz^2} + \left(-\frac{1}{4} + \frac{\nu}{z} + \frac{1/4-\mu^2}{z^2}\right) . \tag{10}
$$

The two linearly independent solutions of the homogeneous differential equation are $M_{\nu,\mu}(z)$ and $W_{\nu,\mu}(z)$, the first is regular at the origin and the second is regular at the infinit respectively. The Green function for the Whittaker differential equation is

$$
G(z,z') = \frac{\Gamma(1/2+\mu-\nu)}{\Gamma(1+2\mu)}\, M_{\nu,\mu}(z_<)\, W_{\nu,\mu}(z_>) . \tag{11}
$$

As special case of this Green function we obtain the Green function for the modified Bessel equation. We put $\nu=0$ in Eq.(11) and using relations which involve the Whittaker functions and the modified Bessel function[2] given by

$$M_{0,\mu}(z) = \Gamma(1+\mu)\ 2^{2\mu}\ z^{1/2}\ I_{\mu}\ (z/2)$$
$$W_{0,\mu}(z) = \pi^{-1/2}\ z^{1/2}\ K_{\mu}(z/2) \tag{12}$$

we obtain

$$G(z,z') = (zz')^{1/2}\ \ I_{\mu}(z_{<}/2)\ K_{\mu}(z_{>}/2)\ . \tag{13}$$

The Green function for others Bessel functions can be easily obtained from this expression, using the well known relations between the modified Bessel function and others Bessel functions.

4. DISCUSSIONS

In this paperwe have calculated the Green function for the Kummer differential operator and for particular cases of this operator. Among thsse, the most important is the Whittaker case.

There are many problems in quantum mechanics having differential equati'ons which reduce to a Whittaker equation. Problems where the differential equation shows spherical or cylindrical symmetry, have a radial equation, that can be reduced to a Whittaker equation. The Coulomb problem[4], the isotropic harmonic oscillator[3] and the charged particle in a uniform magnetic field[5] are examples of these problems.

The same is true for relativistic problems[6]. The parameter v in the Whittaker function is related to the square of the energy, and, as Eq. (11) is analytic in all complex plane, the representation is valid for all possible values of the energy.

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