

# On the Equivalence of Renormalizable Massive QED in 2-Dimensional Space-Time and Massive QED in the Unitary Gauge (Non-Renormalizable)

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In the framework of BPHZ renormalization procedure, we discuss the equivalence between 4-dimensional renormalizable massive quantum electrodynamics (Stueckelberg lagrangian), and massive QED in the unitary gauge.

No contexto do método BPHZ de renormalização, discute-se a equivalência entre a eletrodinâmica quântica bidimensional com massa (lagrangeano de Stueckelberg), e a eletrodinâmica quântica com massa no 'gauge' unitário.

## 1. INTRODUCTION

Massive QED can be described by the Proca-Wentzel<sup>1</sup> Lagrangian:

$$L_{Pr} = \frac{1}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi - M \bar{\psi} \psi - \frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2} m^2 A_\mu A^\mu + \frac{1}{2} (\partial_\mu A^\mu)^2 + e \bar{\psi} \gamma_\mu A^\mu \quad (1)$$

However, this is a non-renormalizable theory, due to the ultra-violet behavior of the vector meson propagator:

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$$\tilde{\Delta}_{\mu\nu}^{-1}(k) = i \frac{g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}}{k^2 - m^2 + i\epsilon} \quad (2)$$

As in massless QED, we introduce a term proportional to  $(\partial_\mu A^\mu)^2$ , which improves the ultra-violet behavior. In this way we are led to the Stueckelberg<sup>1,2</sup> Lagrangian:

$$L_{St} = L_{Pr} - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 \quad (3)$$

which describes the interaction of a vector meson with a fermion, in an indefinite metric Hilbert space.

We define the physical subspace by the same procedure as used Gupta-Bleuler:

$$\partial_\mu A^{\mu\dagger} | \Psi \rangle = 0 \quad (4)$$

If we now make the following formal gauge transformation:

$$U^\mu = A^\mu + \partial^\mu \Lambda \quad (5a)$$

$$\phi(x) = \exp[-ie\Lambda(x)] \psi(x) \quad (5b)$$

$$\Lambda(x) = \frac{1}{m^2} \partial_\mu A^\mu(x) \quad (5c)$$

we obtain the separation of the dynamics:

$$L_{St}(A_\mu, \psi) = L_{Pr}(U_\mu, \phi) + \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 + \text{surface terms} \quad (6)$$

so that the dynamics of the physical fields is separated from that of the ghost  $\partial_\mu A^\mu$  (which has negative metric).

However, all this separation is only formal, first because the renormalization of the Proca lagrangian requires an infinite number of counterterms<sup>3</sup>; and also  $\exp[-ie\Lambda(x)]$  is not well defined<sup>2</sup>, in such a way that if  $\psi(x)$  defines a operator valued distribution,  $\phi(x)$  does not<sup>2</sup>.

In the present paper we prove the equivalence between the theory in the Stueckelberg's lagrangian and that one with Proca's lagran-

gian with counterterms (which we shall call unitary gauge), taking into account the problem of renormalization. In 2-dimensions this has been done in Ref. (3).

Our paper is divided as follows:

Having stated the problem in section 2, we adjust the parameters (which appear in unitary gauge's lagrangian) in section 3 so that Green's functions in the renormalizable case be independent of  $m_0^2$  ( $= am^2$ ) on the mass shell.

In section 4 we prove that parameters can be fixed in such a way that Green's functions be equivalent in both theories, in such a way that they differ only by a renormalization.

In section 5 we give an explicit example which shows that the parameters become infinite in the limit  $m_0^2 \rightarrow \infty$ .

## 2. STATEMENT OF THE PROBLEM

With the Proca lagrangian the photon propagator turns out to be

$$\Delta_F^{\mu\nu} = - \frac{i}{k^2 - m^2} \left[ g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2} \right] \quad (7)$$

The interaction is given by

$$e \bar{\Psi} \gamma_\mu \Psi A^\mu \quad (8)$$

yielding a superficial degree of divergence

$$\delta_\mu(\gamma) = 4 - \frac{3F}{2} - 2B + N_\gamma(v) \quad (9)$$

$$N_\gamma(v) = n? \text{ of vertices in } \gamma$$

Since  $\delta_\mu(\gamma)$  depends on  $N_\gamma(v)$ , any Green's function will eventually turn out to be divergent. Consequently we have a non-renormalizable theory, and an infinite number of counterterms is generated. In

order to define a unique theory an infinite number of renormalization conditions is required. This arbitrariness will be fixed by the following criterion: the Green's functions of the unitary gauge (non-renormalizable theory) must be equivalent to those of the renormalizable gauges.

Having in mind this aim, we define the following  $\lambda$ -dependent Lagrangian ( $\lambda$ -Lagrangian):

$$\begin{aligned}
 L_{ef}(i) = & \frac{1}{2} (1+a) N_4 [\bar{\Psi} \not{\partial} \Psi] - (M-e) N_4 [\bar{\Psi} \Psi] - \frac{1}{4} N_4 [F_{\mu\nu} F^{\mu\nu}] + \frac{1}{2} (m^2+a) A_\mu A^\mu - \\
 & - \frac{1}{2} \frac{m^2+a}{m_0^2} N_4 [(\partial_\mu A^\mu)^2] + \frac{1}{4} b [(1-\lambda) N_6 [F_{\mu\nu} F^{\mu\nu}] + \lambda N_4 [F_{\mu\nu} F^{\mu\nu}]] + \\
 & + (e+f) \times [(1-\lambda) N_5 [\bar{\Psi} A \Psi] + \lambda N_4 [\bar{\Psi} A \Psi]] + L_g = L_0 + L_{int} \quad (10)
 \end{aligned}$$

with

$$L_g = \sum f_{mnpq}^P N_\delta [\Omega_{mnpq}^P]$$

where  $\Omega_{mnpq}^P$  is a complete, linearly independent set of formally gauge-invariant counterterms. The prescription to find the finite part is the BPHZ renormalizable procedure, with degree:

$$\delta_\lambda(\gamma) = 4 - \frac{3}{2} F_\gamma - B_\gamma - 2\bar{B}_\gamma + \sum_{\alpha \in \gamma} (\delta_\alpha - 4)$$

$B = n?$  of external boson lines attached to vertices  $N_5 [\bar{\Psi} A \Psi]$ , or  $N_\delta [\Omega_\alpha]$  but not to  $A_\mu$  from  $(\partial_\mu - i e A_\mu)$ ,  $B = n?$  of the other internal boson lines. In such a way that

$$\begin{aligned}
 R(g) = & \text{Finite part of } \langle 0 | T \Psi(x_1) \dots \Psi(x_N) \bar{\Psi}(y_1) \dots \bar{\Psi}(y_N) \times \\
 & \times A_{\mu_1}(z_1) \dots A_{\mu_L}(z_L) e^{i \int d^4x L_{int}} | 0 \rangle
 \end{aligned}$$

$\Omega_{mnpq}^P$  can be parametrized as follows:

$$\Omega_{mnp p_1 q}^P = M_{(\mu)(\nu)(\lambda)(\sigma)} \partial^{(i_1)} F_{\mu_2 \nu_2} \dots \partial^{(i_n)} F_{\mu_n \nu_n} (\bar{\psi}_{\alpha_1} \bar{o}_{\lambda_1} o_{\lambda_1} \psi_{\beta_1}) \dots$$

$$\dots (\bar{\psi}_{\alpha_m} o_{\lambda_m} o_{\lambda_m} \psi_{\beta_m})$$

where

$$o_{\lambda_j} = \sum_{\ell=1}^{k_j} \frac{k_j}{i\ell} (\partial \eta_{\ell} - i e A \eta_{\ell})$$

$$\bar{o}_{\lambda_j} = \sum_{\ell=1}^{k_j} \frac{k_j}{i\ell} (\bar{\partial} \eta_{\ell} + i e A \eta_{\ell})$$

$$\sum_1^m k_j = p, \quad \sum_1^m k_{j'} = p_1, \quad \sum_1^n i_{\ell} = q$$

$\partial^{(n)}$  is the same as  $\partial_{\sigma_1} \dots \partial_{\sigma_m}$

P refers to any of the possible permutations of  $k_j, i_{\ell}, j_{\ell}, k_{j'}, o_{\lambda_j}, \bar{o}_{\lambda_j}$ , such that  $m, n, p, p_1, q$  stay invariant, besides integration by parts.

$M_{(\mu)(\nu)(\lambda)(\sigma)}$  is matrix to contract the Lorentz indices, and finally

$$\delta = q + 2p + 2p_1 + 2n + 3m$$

We call attention to the fact that if  $X=0$  and  $m_0^2 = \infty$  the A-lagrangian becomes the Proca-Lagrangian (plus, of course, an infinite number of counterterms). If  $f_{\lambda}(\lambda=1) = 0$ , we have for  $A=1$  the Stueckelberg lagrangian.

The problem now is to show that Green's functions are independent of X.

### 3. DEPENDENCE ON $m_0^2$

Now we shall fix the counterterms in such a way that the usual relation is ensured<sup>1,4</sup>:

$$\frac{\partial G}{\partial m_0^2} = \frac{1}{m^2 + a} \Delta_0 G \quad (11)$$

where  $\Delta_0 G$  vanishes in the mass-shell.

For simplicity, we define

$$\Omega_{mnp_1q}^P = \Omega_y \quad , \quad y > 6 \quad (12a)$$

$$f_1 = a \quad , \quad f_2 = b \quad , \quad f_3 = c \quad , \quad f_4 = d \quad , \quad f_5 = f$$

$$f_6 = -b - \frac{m^2+a}{m_0^2} \quad , \quad \text{and} \quad f_y = f_{mnp_1q}^P \quad \text{for} \quad y > 6 \quad (12b)$$

We define furthermore:

$$\begin{aligned} \Delta_1 &= \frac{i}{2} \int d^4x N_4 [A_\mu A^\mu] (x) \\ \Delta_2 &= \frac{i}{4} \int d^4x N_4 [F_{\mu\nu} F^{\mu\nu}] (x) \\ \bar{\Delta}_2 &= \frac{i}{4} \int d^4x N_6 [F_{\mu\nu} F^{\mu\nu}] (x) \\ \Delta_3 &= i \int d^4x N_4 [\bar{\psi}\psi] (x) \\ \Delta_4 &= -\frac{1}{2} \int d^4x N_4 [\bar{\psi}\gamma^\mu \overleftrightarrow{\partial}_\mu \psi] (x) \\ \Delta_5 &= i \int d^4x N_4 [\bar{\psi}\gamma^\mu \psi A_\mu] (x) \\ \bar{\Delta}_5 &= i \int d^4x N_6 [\bar{\psi}\gamma^\mu \psi A_\mu] (x) \\ \Delta_6 &= \frac{i}{2} \int d^4x N_4 [\partial_\mu A^\mu]^2 (x) \end{aligned} \quad (13)$$

$$\begin{aligned} \Delta_0 G &= i \int d^4x \left\{ \sum_{i \neq j} \partial_{\nu_i} \Delta_F(x-z_i, m_0^2) \partial_{\nu_j} \Delta_F(x-z_j, m_0^2) \langle 0 | T X_{\hat{i}\hat{j}} | 0 \rangle + \right. \\ &+ i \frac{e+f}{T+d} \sum_{i=1}^L \sum_{j=1}^N \partial_{\nu_i} \Delta_F(x-z_i, m_0^2) [\Delta_F(x-x_j, m_0^2) - \Delta_F(x-y_j, m_0^2)] \langle 0 | T X_{\hat{i}} | 0 \rangle + \end{aligned}$$

$$\begin{aligned}
& + i \left( \frac{e+f}{1+d} \right)^2 \sum_{i \neq j}^N [\Delta_F(x-x_i) \Delta_F(x-x_j) + \Delta_F(x-y_i) \Delta_F(x-y_j)] G + \\
& + 2 \left( \frac{e+f}{1+d} \right)^2 \sum_{i,j} \Delta_F(x-x_i) \Delta_F(x-x_j) G \} \tag{14}
\end{aligned}$$

From the Gell-Mann Low formula we have:

$$\begin{aligned}
\frac{\partial G}{\partial m_0^2} = \sum_{y \neq 2,5} \frac{\partial f}{\partial m_0^2} \Delta_y G + \frac{\partial b}{\partial m_0^2} [(1-\lambda) \bar{\Delta}_2 + \lambda \Delta_2] G + \frac{\partial f}{\partial m_0^2} [(1-\lambda) \bar{\Delta}_5 + \\
+ \lambda \Delta_5] G \tag{15}
\end{aligned}$$

In order to use this formula let us establish a relation between  $\Delta_0$  and  $A_6$  (see figure (1)).

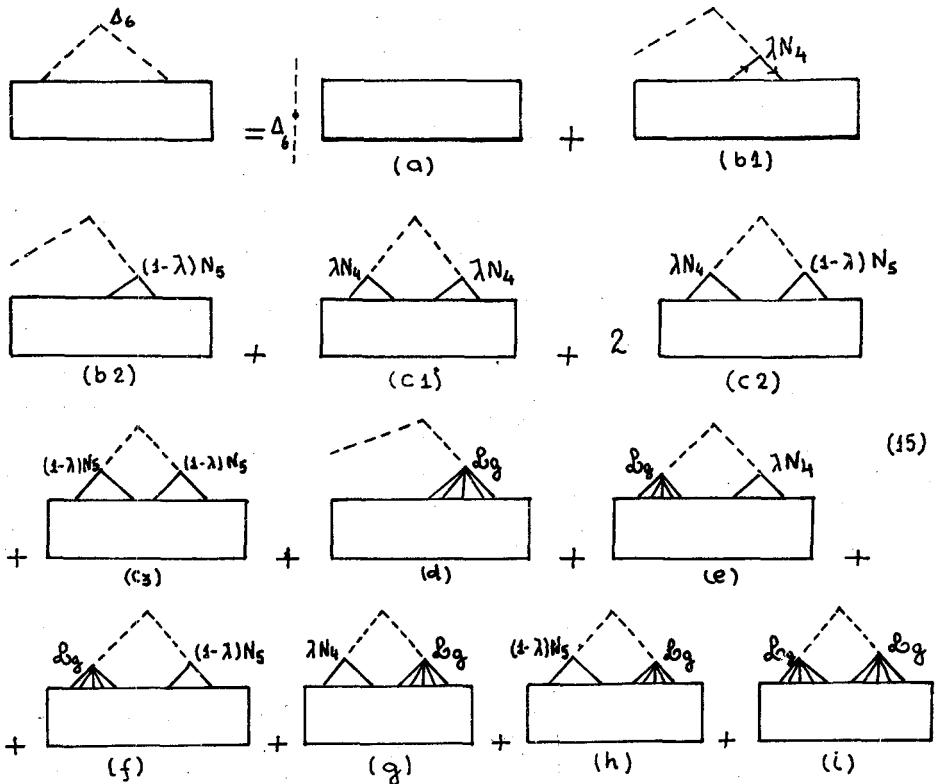
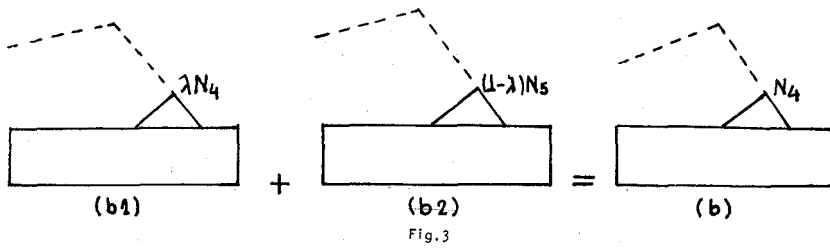
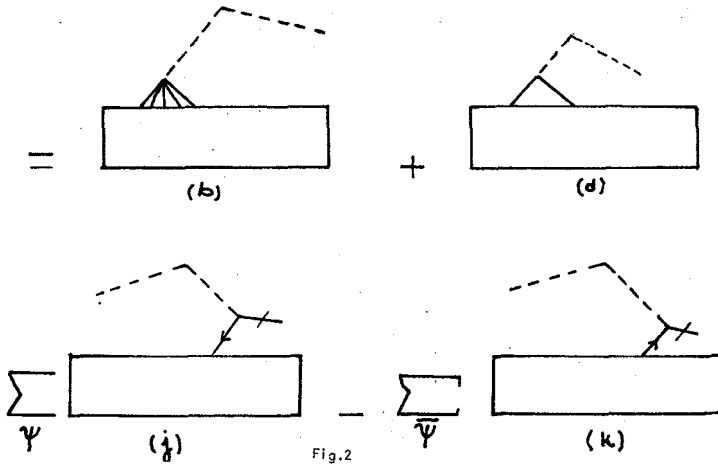
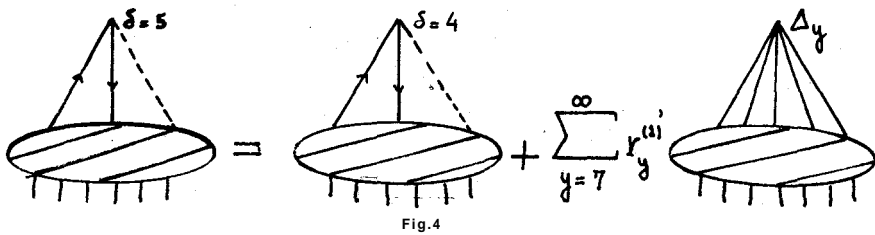


Fig. 1

Using now figures (2), (3) and (4)



and



we find the relation shown in figure (5)



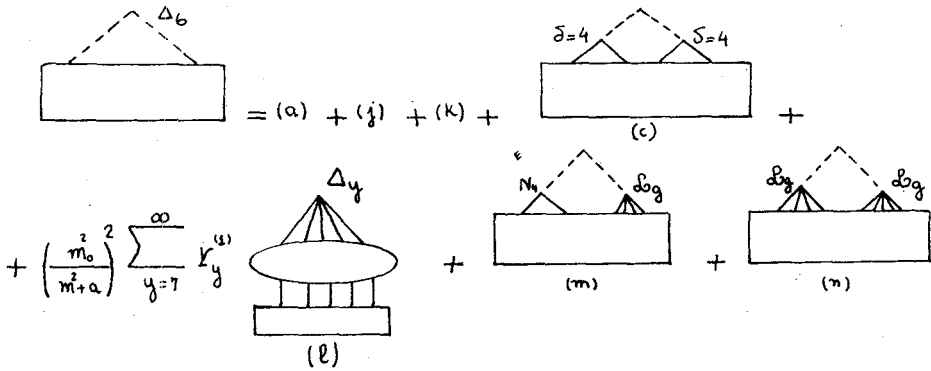


Fig.5

where (a), (j), (k), were given in figures (1) and (3).

Due to gauge invariance  $L_g$ ; we have the relation shown in figure (6)

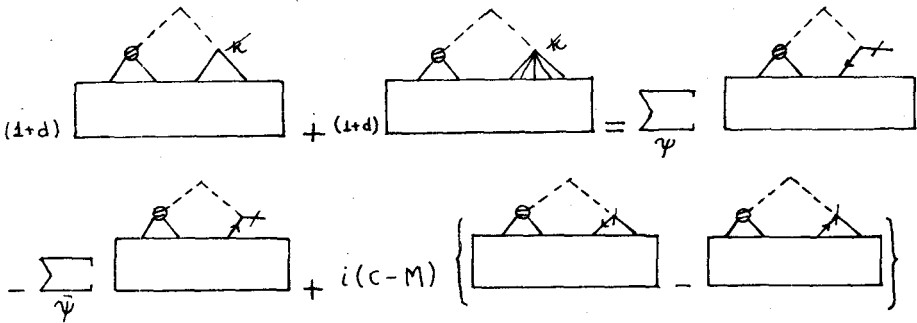


Fig.6

where  is equal to  or  and  in

the last term means: if a subgraph is all contained in one side of the bar, we have minimal subtraction if the bar divides the subgraph in two pieces, one must make one extra subtraction (see ref.1).

Iterating the above process for the other vertex we find:

$$\Delta_0 G = \left[ \frac{m^2 + a}{m_0^2} \right]^2 \Delta_6 G + 2i(c-M)R_1 - (c-M)^2 R_2 - \sum r_y^{(1)} \Delta_y G \quad (16)$$

with  $R_1$  coming from the extra subtraction of figure (7)

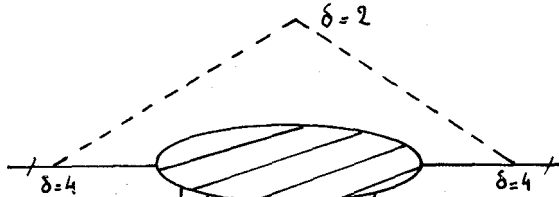


Fig.7

and  $R_2$  from the 2 extra subtractions from figure (8)

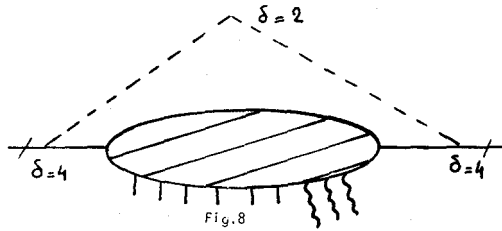


Fig.8

They can be written as  $\sum r_Y^{(2)} A_Y G$  so that

$$\Delta_0 G = \sum_{Y \geq 3} r_Y \Delta_Y G \quad (17)$$

Now inserting

$$\begin{aligned} \bar{\Delta}_5 - \Delta_5 &= \sum \xi_{5y} \Delta_y \\ \bar{\Delta}_2 - \Delta_2 &= \sum \xi_{2y} \Delta_y \end{aligned} \quad (18)$$

in (15) and comparing with (17) we obtain

$$a = a_0(\lambda) \quad (19a)$$

$$b = b_0(\lambda) \quad (19b)$$

$$f = f_0(\lambda) + \frac{1}{m^2 + a} \int_0^{m_0^2} r_5 (m_0'^2) dm_0'^2 \quad (20)$$

$$f_y = f_y^0(\lambda) + \frac{1}{m^2 + a} \int_0^{m_0^2} [(1-\lambda)r_5 \xi_{5y} + r_y] dm_0'^2 \quad (21)$$

So that (11) holds.

#### 4. THE EQUIVALENCE

In this section we shall prove that the Green's functions calculated with the X-lagrangian (10) are X-independent, i.e. we shall prove that<sup>3</sup>

$$\frac{\partial G}{\partial \lambda} = 0 \quad (22)$$

From the Gell-Man-Low formula:

$$\begin{aligned} \frac{\partial G}{\partial \lambda} = & \left\{ \sum_{y \neq 2,5} \frac{\partial f_y}{\partial \lambda} \Delta_y + (e+f) (\Delta_5 - \bar{\Delta}_5) + b(\Delta_2 - \bar{\Delta}_2) + \right. \\ & \left. + \frac{\partial f}{\partial \lambda} [(1-\lambda)\bar{\Delta}_5 + \lambda\Delta_5] + \frac{\partial b}{\partial \lambda} [(1-\lambda)\bar{\Delta}_2 + \lambda\Delta_2] \right\} G \end{aligned} \quad (23)$$

Inserting (18) in (23)

$$\begin{aligned} \frac{\partial G}{\partial \lambda} = & \sum_y \left[ \frac{\partial f_y}{\partial \lambda} + \frac{\partial f}{\partial \lambda} (1-\lambda) \xi_{5y} + \frac{\partial b}{\partial \lambda} (1-\lambda) \xi_{2y} - \right. \\ & \left. - (e+f) \xi_{5y} - b \xi_{2y} \right] \Delta_y G \end{aligned} \quad (24)$$

Now it is our aim to find  $\bar{\Delta}_y G$  gauge independent, that is:

$$\left[ \frac{\partial}{\partial m_0^2} - \frac{1}{m^2 + a} \Delta_0 \right] \bar{\Delta}_y G = 0 \quad (25)$$

We can construct  $\tilde{\Delta}_Y$  by the following procedure: gauge invariant  $\tilde{\Delta}_Y$  are constructed, taking linear combinations of the  $\Delta_Y$ 's.

$$\tilde{\Delta}_Y^{(n)} = \Delta_Y + \sum_{j=1}^n \epsilon_{YY'} \beta_{YY'}^{(j)} \Delta_Y \quad (26)$$

$$\beta_{yy'}^{(n)} = \int_0^{m_0^2} \alpha_{yy'}^{(n)} dm_0'^2 \quad (27)$$

$$\alpha_{yy'}^{(n)} = \sum_{y''} \beta_{yy''}^{(n-1)} \alpha_{y''y'}^{(n-1)} \quad (28)$$

$$\left( \frac{\partial}{\partial m_0^2} - \frac{1}{m^2 + a} \Delta_0 \right) \Delta_Y^G = - \sum \alpha_{yy'}^{(1)} \Delta_Y^G \quad (29)$$

Then we can write:

$$\Delta_Y = \sum \omega_{yy'} \tilde{\Delta}_Y$$

such that

$$\frac{\partial G}{\partial \lambda} = \sum_{yy'} \left\{ \frac{\partial f}{\partial \lambda} + \frac{\partial f}{\partial \lambda} (1-\lambda) \epsilon_{5y} + \frac{\partial b}{\partial \lambda} (1-\lambda) \epsilon_{2y} - (e+f) \epsilon_{5y} - b \epsilon_{2y} \right\} \omega_{yy'} \tilde{\Delta}_Y^G \quad (30)$$

where the coefficient of  $\tilde{\Delta}_Y$  is independent of  $m^2$ . We put  $m_0^2 = 0$ , and impose  $\frac{\partial G}{\partial \lambda} = 0$ . ( $\frac{\partial G}{\partial \lambda} = 0$  holds independently of  $m^2$ , since the coefficient of  $\tilde{\Delta}_Y$  is  $m^2$  independent). By the fact that  $\det[\omega] \neq 0$ :

$$\frac{\partial f_y^{(0)}}{\partial \lambda} + (1-\lambda) \frac{\partial f^{(0)}}{\partial \lambda} \epsilon_{5y} + (1-\lambda) \frac{\partial b^{(0)}}{\partial \lambda} \epsilon_{2y} - (e+f^{(0)}) \epsilon_{5y} - b^{(0)} \epsilon_{2y} = 0 \quad (31)$$

$f_y^{(0)}$  can be calculated in perturbation theory, using:

$$f_y^{(0)}(\lambda=1) = 0, \quad y > 6$$

and the normalization conditions of  $\mathcal{G}$  for  $y < 6$ . Equation (31) implies independence of the Green's functions with respect to  $A$ , eq. (22)<sup>5</sup>.

## 5. AN EXAMPLE

In this chapter we take the explicit Green's function  $G(2,0)$  and prove that

$$1. G^{(2,0)}(\lambda=1) = G^{(2,0)}(\lambda)$$

2. The counterterms diverge for  $\lambda=1$  as it should be, because in this case, the graphs are explicitly finite, but Green's function must be infinite, because in the case  $X=0$  the Green's functions are infinite.

Let us take  $G$  as in figure (9)

$$G = \frac{1-\lambda}{\delta=5} \text{ (wavy line)} \frac{1-\lambda}{\delta=5} - 2 \frac{\lambda}{\delta=4} \text{ (wavy line)} \frac{1-\lambda}{\delta=5} + \frac{\lambda}{\delta=4} \text{ (wavy line)} \frac{\lambda}{\delta=4} + \frac{f_3 + f_4 + f_4^2 + f_8^2}{\times}$$

Fig.9

An straightforward calculation shows that:

$$G(\lambda) = G(1) \tag{32}$$

where

$$f_Y = f_Y^{(0)}(A) + \frac{1}{m^2 + a} \int_0^{m_0^2} r_Y(m_0'^2) dm_0'^2 \tag{33}$$

$$r_Y = r_Y^{(1)} + r_Y^{(2)}$$

$r_Y^{(2)}$  comes from anisotropies, and  $r_Y^{(1)}$  from the graphs shown in figure (10)

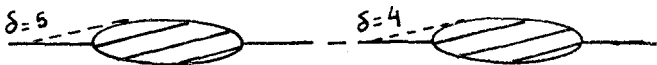


Fig.10

we note that

$$r_3^{(1)} = r_4^{(1)} = 0 \quad (35)$$

$$r_7^{(2)} = r_8^{(2)} = 0 \quad (36)$$

$$r_3 = M^3 \int_0^1 \frac{y d_y}{ym_0^2 + (1-y)M^2} \quad (37)$$

$$r_4 = -3M^2 \int \frac{d^4k}{(k^2 - m_0^2)(k^2 - M^2)} - 2M^4 \int \frac{d^4k}{(k^2 - m_0^2)^2 (k^2 - M^2)^3} \quad (38)$$

$$r_7 = (1 - \lambda^2)M \int \frac{d^4k(k^2 + M^2)}{(k^2 - m_0^2)(k^2 - M^2)^2} \left[ 1 + \frac{M^2}{k^2 - M^2} \right] \quad (39)$$

For  $m_0^2 \gg M^2$  we find:

$$r_3 \approx \frac{M^3}{m_0^2} \quad (40)$$

$$r_4 \approx -\frac{3M^2}{m_0^2} \quad (41)$$

$$r_7 \approx \frac{\pi}{m_0^2} \quad (42)$$

which diverge logarithmically when integrated.

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5. The gauge independence of  $\tilde{A}_Y$  will be used to put  $m_0^2 = 0$  in equation (31). This is important in order not to have contradiction between

$$\frac{\partial G}{\partial m_0^2} = \frac{1}{m^2 + a} \Delta_0 G \text{ and } \frac{\partial G}{\partial \lambda} = 0$$