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The Principle of General Covariance and the Principle of Equivalence: Two Distinct Concepts

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It is shown how to construct a theory with general covariance but without the equivalence principle. Such a theory is in disagreement with experiment, but it serves to illustrate the independence of the former principle from the latter one.

Mostramos como construir uma teoria com covariância geral mas sem o principio de equivalência. Tal teoria está em desacordo com a experiência, mas serve para ilustrar a independência do primeiro princípio com relação ao segundo.

1. INTRODUCTION

Some texbooks on the theory of general relativity (GRT) introduce the equivalence principle and the general covariance principle almost in the same breath (see, for example, Einstein[1], p.56; Pauli[2], pp.143-144; Tonnelat[3], p.262; Møller[4], pp.250-251).

What happened[2,3] is that soon after Einstein's discovery of the theory of special relativity, two problems presented themselves: (1) how to incorporate gravity in the new theory, and (2) how to generalize covariance in inertial frames of reference into covariance in accelerated frames. As is well known, Einstein coupled and solved these two pro-

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blems simultaneously, precisely through a *close association* of the two principles under discussion.

But then we feel that the student may fail to appreciate the import of general covariance *per se*, seeing it perhaps as a mere consequence of the equivalence principle.

Therefore, we make some conjectures based on the assumption of general covariance *without* equivalence. This serves to illustrate the independence of the former concept from the latter one[5].

In this spirit we discuss in Sec.2, the law of motion of a particle under the influence of a force, and in Sec. 3 we present the equations of electromagnetism as well as a hypothetical set of equations for the gravitational field. We conclude remembering the pedagogical nature of this paper.

2. THE MOTION OF A MASSIVE PARTICLE UNDER THE INFLUENCE OF A FORCE

In a Lorentz system of reference (x^μ) with metric

$$ds = \bar{g}_{\mu\nu}\, d\bar{x}^\mu\, d\bar{x}^\nu \tag{1}$$

$$(\bar{g}_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2}$$

the equation of motion of a particle of inertial mass $m \neq 0$, under the influence of a 4-force $\bar{F}^\mu$ is

$$m\,\frac{d\bar{u}^\mu}{ds} = \bar{F}^\mu \tag{3}$$

where s is the particle's proper time (c–1) and

$$\bar{u}^\mu \equiv \frac{d\bar{x}^\mu}{ds} \tag{4}$$

is its 4-velocity. $\bar{F}^\mu$ is the net externa1 force applied on the particle, for example the electromagnetic force as usual, but here also the gravitational force.

If we now go to an arbitrary system (x^μ), with

$$x^\mu = x^\mu(\bar{x}) \tag{5}$$

the metric invariant becomes

$$ds^2 = g_{\mu\nu}(x)\ dx^\mu\ dx^\nu \tag{6}$$

with

$$g_{\mu\nu}(x) = \bar{g}_{\alpha\beta}\ \frac{\partial \bar{x}^\alpha}{\partial x^\mu}\ \frac{\partial \bar{x}^\beta}{\partial x^\nu} \tag{7}$$

according to the general rule for covariant tensors[6]. The 4-velocity is defined as in Eq.(4),

$$u^\mu \equiv \frac{dx^\mu}{ds} \tag{8}$$

and u^μ and F^μ are given by

$$u^\mu = \frac{\partial x^\mu}{\partial \bar{x}^\alpha}\ \bar{u}^\alpha \tag{9}$$

and

$$F^\mu = \frac{\partial x^\mu}{\partial \bar{x}^\alpha}\ \bar{F}^\alpha \tag{10}$$

Equation (3), however, takes a different look, because we have to replace du^μ by the covariant differential

$$Du^\mu = du^\mu + \Gamma^\mu_{\alpha\beta}\ u^\alpha\ dx^\beta \tag{11}$$

The equation of motion is therefore

$$m\ \frac{Du^\mu}{ds} = F^\mu \tag{12}$$

or

$$\frac{du^{\mu}}{ds} + \Gamma^{\mu}_{\alpha\beta}\, u^{\alpha}\, u^{\beta} = \frac{1}{m}\, F^{\mu} \tag{13}$$

In the absence of forces ($F^{\mu} = 0$) Eq.(13) becomes the geodesic equation

$$\frac{d^2x^{\mu}}{ds^2} + \Gamma^{\mu}_{\alpha\beta} \frac{dx^{\alpha}}{ds} \frac{dx^{\beta}}{ds} = 0 \tag{14}$$

In the absence of matter, Eq. (14) is the same as in GRT. Thus, for instance, the questions discussed by Adler *et al.*[7], Sec. 4.2, about space and time in a uniformly rotating system remain valid here. An interesting fact in this connection is that, on the one hand, space-time is flat in the rotating system (t,r,ϕ,z), since curvature is an invariant. On the other hand, the 3-spatial geometry is curved. The reason is that the definition of distances demands the concept of simultaneity, and this is obtained by $dt^* = 0$, where t^* is a non-integrable time-coordinate[7].

For completeness, let us calculate the Gaussian curvature of a plane z = const. in the rotating frame. The line element $d\ell$ is given by

$$d\ell^2 = dr^2 + \frac{r^2}{1-\omega^2r^2}\, d\phi^2$$

Hence[8]

$$K = - \left(\frac{1-\omega^2r^2}{r^2}\right)^{\frac{1}{2}} \frac{d^2}{dr^2} \left(\frac{r^2}{1-\omega^2r^2}\right)^{\frac{1}{2}}$$

$$= - \frac{3\omega^2}{(1-\omega^2r^2)^2} < 0 \quad .$$

We observe here that our theory predicts a rotational red shift (see Ref.7, p.126), but no gravitational red shift, since gravity does not influence the metric. This is why we say that the theory iscontradicted by experiment.

3. THE ELECTROMAGNETIC AND GRAVITATIONAL FORCES AND FIELDS

The equations of electromagnetism are expressed in the same way as in GRT[6,7]. Thus the force on a particle of charge q in a field $F^{\mu\nu}$ is

$$F^{\mu} = q\, F^{\mu\nu}\, u_{\nu} \tag{15}$$

Maxwell's equations and the continuity equation are

$$F^{\mu\nu}{}_{||\nu} = 4\pi\, j^{\mu} \tag{16}$$

$$*F^{\mu\nu}{}_{||\nu} = 0 \tag{17}$$

$$j^{\mu}{}_{||\mu} = 0 \tag{18}$$

where $||\nu$ indicates the absolute derivative with respect to x^{ν}, j^{μ} is the 4-current density, and $*F^{\mu\nu}$ is the tensor dual to $F^{\mu\nu}$

But of course there is a difference with GRT: our $g_{\mu\nu}$'s are not influenced by the presence of the field $F^{\mu\nu}$,

Now we must face the problem of dealing with gravitation. We shall consider only the simplest situation, i.e., we assume the gravity field to be derived from a scalar potential $\psi(x)$. We generalize Poisson's equation

$$\nabla^2\phi = 4\pi\, G_p \tag{19}$$

into

$$g_{\mu\nu}\, \psi_{||\mu||\nu} = -\, 4\pi\, G_p \tag{20}$$

where the 4-scalar $\rho(x)$ is the density of gravitational mass.

Eq. (20) is formaliy similar to the equation for Brans and Dicke's scalar field[7], but there are two important differences: our coupling constant G is the usual gravitational constant, and our source cannot be the energy-momentum tensor, because this tensor is related to inertial mass.

Finally, the law of force on a particle of gravitational mass μ is a generalization of the Newtonian law

$$\vec{F} = -\mu\vec{\nabla}\phi \tag{21}$$

which goes into

$$F^{\alpha} = -\mu g^{\alpha\beta}\,\psi_{||\beta} = -\mu g^{\alpha\beta}\frac{\partial\psi}{\partial x^{\beta}} \tag{22}$$

Inserting Eq. (22) into Eq.(13), we obtain the expression

$$\frac{du^{\alpha}}{ds} + \Gamma^{\alpha}_{\beta\gamma}\,u^{\beta}\,u^{\gamma} = -\frac{\mu}{m}\,\psi^{||\alpha} \tag{23}$$

for the motion of a particle under the action of gravity.

4. CONCLUSION

We emphaslze that we have no intention of invalidating the equivalence principle. Our purpose should rather be viewed as a pedagogical effort to clarify the independence of the general covariance principle.

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3. M. A. Tonnelat, *Les Principes* de *la Théorie Électromagnétique* et de *la Relativité*, Masson, Paris 1959.
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5. S. Weinberg argues that general covariance follows from equivalence: see his *Gravitation and Cosmology*, John Wiley, New York, 1972. We do not deny this, but rather affirm that general covariance would still be possible without equivalence.

6. For this and other general matters see, for example, L. Landau and E. Lifshitz, *Classical Theory of Fields*, English translation by M. Hamermesh, Addison-Wesley, Cambridge, 1951.

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