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Topological Solitons and Magnetic Monopoles*

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The mathematics of the topological solitons is studied in the framework of a non abelian gauge field theory for three spatial dimensions. The topological charges are identified with the magnetic charge of monopoles satisfying the Schwinger's quantization rule. A specific solution with symmetry under the diagonal subgroup of O(3) % 0(3), known as it Hoof-Polyakov ansatz, is explicitly exhibited.

Estuda-se a matemática dos sólitons topológicos no contexto de uma teoria de gauge não abeliana para três dimensões espaciais. As cargas topológicas são identificadas à carga magnética de monopolos que satisfazem a regra de quantização de Schwinger. Uma solução especifica com simetria correspondente ao subgrupo diagonal de O(3) % 0(3) conhecida como o "Ansatz" de it Hooft-Polyakov, é exibida explicitamente.

1. INTRODUCTION

After the discovery by Maxwell of the dynamical equations of the electromagnetic fields, physicists started to speculate that the theories of electricity and magnetismo would be treated in parallel by the proposal of the existence of the magnetic charges. The symmetries between electricity and magnetism become manifest by the following dual transformations

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$$\vec{E} \to \vec{B} \; , \tag{1a}$$

$$\rho_e \to \rho_m \; , \tag{1b}$$

$$\vec{j}_e \to \vec{j}_m \; , \tag{1c}$$

and

$$\vec{B} \to -\vec{E} \; , \tag{2a}$$

$$\rho_m \to -\rho_e , \tag{2b}$$

$$\vec{j}_m \to -\vec{j}_e \; . \tag{2c}$$

Yet this is incornpatible with the Lorentz covariant formulations of the electrodynamics **if** one restricts oneself to the non singular solutions of the covariant four-vector potentials $A_\mu(\vec{r},t)$ in the presence of the magnetic charges. The problem was investigated by admitting the singular solutions of A_μ, known as Dirac string, in which the magnetic source term appeared as a mnopole situated at the end of a magnetic dipole line or a tightly wound solenoid.

As to the non singular solutions of the Maxwell equations, one needs at least two kinds of vector potentials $A^\mu_e(\vec{r},t)$ and $A^\mu_m(\vec{r},t)$, in response respectively to the electric and mgnetic charges and currents, defined as follows

$$A^\mu_e(\vec{r},t) = \frac{1}{4\pi} \int \frac{j^\mu_e(\vec{r}\,',t)}{|\vec{r} - \vec{r}\,'|} \, d^3\vec{r}\,' \; , \tag{3a}$$

with

$$j^\mu_e(\vec{r},t) = (\rho_e(\vec{r},t) \; , \; \vec{j}_e(\vec{r},t)) \; , \tag{3b}$$

and

$$A^\mu_m(\vec{r},t) = \frac{1}{4\pi} \int \frac{j^\mu_m(\vec{r}\,',t)}{|\vec{r} - \vec{r}\,'|} \, d^3\vec{r}\,' \; , \tag{4a}$$

with

$$j^\mu_m(\vec{r},t) = (\rho_m(\vec{r},t) , \; \vec{j}_m(\vec{r},t)) \; . \tag{4b}$$

The fundamental problem of. the above assurnptions is whether the electric fields produced by the static charge ρ_e are physically indistinguishable to the fields produced by the mtion of the magnetic charges, namely $\vec{j}_m$. It seems that one lacks the ingenuity in differenciating one field from the other. For this reason, one might just simply accept the primitive version of the Maxwell equations, i.e. the Maxwell equations describing the nature without the magnetic sources, because a proper choice of the rotation in the charge space, the theory of two kinds of vector potentials is mathematically equivalent to the original Maxwell theory of electrodynamics. The statement can be proved quite easily. Let

$$\vec{E} = -\nabla A_e^0 - \nabla\times\vec{A}_m \quad , \tag{5a}$$

and

$$\vec{B} = \nabla\times\vec{A}_e - \nabla A_m^0 \quad , \tag{5b}$$

be the electric field and magnetic field due to the presence of the electric and magnetic charges, then it allows one to construct a covariant field tensor

$$G^{\mu\nu} = F_e^{\mu\nu} + \tilde{F}_m^{\mu\nu} \,, \tag{6}$$

with

$$F_{e,m}^{\mu\nu} = \partial^\mu A_{e,m}^\nu - \partial^\nu A_{e,m}^\mu \quad . \tag{7}$$

Let us consider a dual rotation of the complex field tensor defined in the following form

$$g'^{\mu\nu} = e^{-i\theta} g^{\mu\nu} \tag{8}$$

with

$$g^{\mu\nu} = G^{\mu\nu} + i\tilde{G}^{\mu\nu} \tag{9}$$

then by the proper choice of $\tan\theta = j_e^\mu / j_m^\mu$ for all μ, one concludes that the rotated field tensor $g'^{\mu\nu} = G'^{\mu\nu} + i\tilde{G}'^{\mu\nu}$ satisfies the following equations,

$$\partial_\nu G'^{\mu\nu} = j'_e \tag{10}$$

$$\partial_\nu \tilde{G}'^{\mu\nu} = 0 \tag{11}$$

The above equations are the Maxwell equations without magnetic charges if one regards $G'^{\mu\nu}$ as the physical observables of the field tensor.

The connections between the magnetic charges (or monopoles) and the topological charges by means of the non-Abelian gauge fields [2] has been investigated since 1974. The results obtained by t'Hooft and Polyakov have met with a great success in three spatial dimensions. The non-Abelian Yang-Mills fields[3] and the spontaneous symmetry breaking of the Higgs mechanism[4] will be reviewed in the following section. Section 3 will deal with the technical aspects of the topological soliton and its relation to the magnetic monopole. A particular solution based upon 't Hooft-Polyakov ansatz was explicitly demonstrated in Section 4. The possible developements on topics of relevance are briefly discussed in the last section.

2. YANG-MILLS FIELUS AND GEORGI-GLASHOW MODEL[5]

Since the introduction of two kinds of the vector fields in EM theory was proven to be equivalent to the usual Maxwell theory, it is therefore natural for one to speculate that fields like Yang-Mills type may help in solving the problem. At first thought, one may fear that such a bold approach will make the theory more complicated, or even absurd because charged photons have not been observed so far. Fortunately the worry was shown to be unnecessary. The reason is due to the celebrated Higgs mechanism of the spontaneous symmetry breaking, which enables the charged components of the triplet gauge particles to gain mass, meanwhile leaving the photon field to remain massless if the neutra1 component of the Higgs particle develops a non vanishing vacuum expectation value. The grand architecture in building up the photon with the massive vector boson is a very attractive and beautiful piece of work in the recent attempt for the unification of the EM interaction and the weak interaction. In lieu of adopting the Georgi-Glashow model literally, we replace the charge heavy lepton doublet by a triplet of Higgs fields, and search for the soliton solution from the following Lagrangian,

$$L = -\frac{1}{4}\vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu} - \frac{1}{2}D_\mu\vec{\phi}\cdot D^\mu\vec{\phi} - \frac{\lambda}{4}(\vec{\phi}\cdot\vec{\phi}-a^2)^2 , \tag{12}$$

where

$$\vec{F}_{\mu\nu} = \partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu - e\vec{A}_\mu\times\vec{A}_\nu, \tag{13}$$

$$D_\mu \vec{\phi} = \partial_\mu \vec{\phi} - e\vec{A}_\mu \times \vec{\phi} \tag{14}$$

$\vec{A}_\mu$ and $\vec{\phi}$ are isotriplets of gauge field and Higgs field respectively. One also takes $\lambda \geqslant 0$, because of the positive definite condition for potential $V(\vec{\phi}) = \frac{\lambda}{4}(\vec{\phi}.\vec{\phi}-a^2)^2$. The minimum of $V(\vec{\phi})$ defines the vacuum at $|\vec{\phi}| = a$.

The Lagrangian of eq. (12) is locally gauge invariant under the following infinitesimal transformation

$$\vec{A}_\mu{}' = \vec{A}_\mu + \vec{\varepsilon} \times \vec{A}_\mu + \frac{1}{e}\partial_\mu \vec{\varepsilon} \ , \tag{15}$$

$$\vec{\phi}' = \vec{\phi} + \vec{\varepsilon} \times \vec{\phi} \tag{16}$$

With eq. (14) and Eq. (15), one verifies easily that

$$\begin{aligned} D_\mu \vec{\phi}' &= \partial_\mu \vec{\phi}' - e\vec{A}_\mu' \times \vec{\phi}' \\ &= D_\mu \vec{\phi} + \vec{\varepsilon} \times D_\mu \vec{\phi} \ , \end{aligned} \tag{17}$$

and

$$\begin{aligned} \vec{F}_{\mu\nu}' &= \partial_\mu \vec{A}_\nu' - \partial_\nu \vec{A}_\mu' - e\vec{A}_\mu' \times \vec{A}_\nu' \\ &= \vec{F}_{\mu\nu} + \vec{\varepsilon} \times \vec{F}_{\mu\nu} \ , \end{aligned} \tag{18}$$

and hence the invariance of L.

Let us assume that ϕ_3 develops a non vanishing vacuurn expectation value a, which also simultaneously rninimizes the potential $V(\vec{\phi})$. If one redefines $\vec{\psi}$ by

$$\vec{\psi} = (\phi_1, \phi_2, \phi_3 - \langle\phi_3\rangle_0) \ , \tag{19}$$

$$= (\phi_1, \phi_2, \phi_3 - a) \ , \tag{20}$$

then the Lagrangian becomes

$$\begin{aligned} L = &-\frac{1}{4}\vec{F}_{\mu\nu} \cdot \vec{F}^{\mu\nu} - \frac{1}{2}D_\mu\vec{\psi} \cdot D^\mu\vec{\psi} - \frac{1}{2}(2\lambda a^2)\psi_3^2 \\ &- \frac{1}{2}(ea)^2 \{A_{+\mu}A_-^\mu + A_{-\mu}A_+^\mu + L \text{ int } (\vec{A},\vec{\phi}) \end{aligned} \tag{21}$$

where

$$A_{\pm\mu} = \frac{1}{\sqrt{2}} (A_{1\mu} \pm iA_{2\mu})$$

correspond to the two charged vector bosons.

The spontaneous breaking of the $O(3)$ gauge symmetry, but leaving the $U(1)$ symrnetry about z-axis to remain intact, allows us to recognize that the charged gauge particles become massive with

$$M_{A\pm} = ea \quad , \tag{22}$$

which is estimated to be larger than 53 GeV on account of its role of intermediate vector boson in weak interactions.

3. THE TOPOLOGICAL CHARGES

In the previous section, we have dernonstraded that the third component of gauge particles, namely the physical photon, acquires no rnass through the spontaneous breaking of $O(3)$ gauge symrnetry. Let us define a space-time region, for instance the space at infinite, $|\vec{r}| \to \infty$ in a pre-specified direction, such that the Higgs fields tend to a constant in z-direction with the length equal to a, so that the potential is minimized. Such a region is called the supervacuurn[6] by 't Hooft, in which one can easily convince oneself that the electro-magnetic field tensor is related to the photon field A_3^μ by,

$$F^{\mu\nu} = \partial^\mu A_3^\nu - \partial^\nu A_3^\mu \tag{23}$$

Since the Lagrangian is symmetric with respect to $O(3)$ gauge group, one may define equivalently a supervacuurn in the space time region in which the scalar Higgs field will point in a particular direction $\hat{\phi}_0$ with $|\vec{\phi}|=a$ by a gauge rotation. In this case, the physical observable of the electro-magnetic field tensor $F^{\mu\nu}$ can be taken in the following expression,

$$F^{\mu\nu} = \frac{1}{a} \vec{\phi} \cdot \vec{F}^{\mu\nu} + \frac{1}{ea^3} \vec{\phi} \cdot (D^\mu\vec{\phi} \times D^\nu\vec{\phi}) \quad , \tag{24}$$

which reduces to eq. (23) in the region of supervaccum if $\vec{\phi}(\vec{r}) = a\hat{k}$ at $|\vec{r}| \to \infty$.

The meanings of the last equation can be understood by the application of the equations of motion,

$$D_\mu \vec{F}^{\mu\nu} = + e\vec{\phi} \times D^\nu \vec{\phi} \tag{25}$$

$$D_\mu D^\mu \vec{\phi} = - \lambda(\vec{\phi}\cdot\vec{\phi}-a^2)\vec{\phi} \tag{26}$$

and rewrite the $F^{\mu\nu}$ in eq. (24) as

$$F^{\mu\nu} = (\partial^\mu B^\nu - \partial^\nu B^\mu) + \frac{1}{ea^3}\vec{\phi}\cdot(\partial^\mu\vec{\phi} \times \partial^\nu\vec{\phi}) \tag{27}$$

$$= H^{\mu\nu} + M^{\mu\nu} \tag{28}$$

where

$$B_\mu = \frac{1}{a}\vec{\phi}\cdot\vec{A}_\mu \quad , \tag{29}$$

$$H^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu \ , \tag{30}$$

and

$$M^{\mu\nu} = \frac{1}{ea^3}\vec{\phi}\cdot \ \partial^\mu\vec{\phi} \times \partial^\nu\vec{\phi} \ . \tag{31}$$

If we consider the case, in which the B_μ is singular nowhere, then the divergence of the dual field tensor of eq.(30) automatically vanishes, i.e.,

$$\partial_\mu \tilde{H}^{\mu\nu} = 0 \quad . \tag{32}$$

Defining the topological current κ^μ by taking the divergence of the dual of eq. (28), on reaches

$$\kappa^\mu = \partial_\nu \tilde{F}^{\mu\nu} = \frac{1}{2ea^3}\varepsilon^{\mu\nu\alpha\beta}\partial_\nu \ \{\vec{\phi}\cdot(\partial_\alpha\vec{\phi} \times \partial_\beta\vec{\phi}) \ , \tag{33}$$

which by no means equals zero. Since $F^{\mu\nu}$ is the physical EM field tensor, κ^μ in eq.(33) can be interpreted as the magnetic source term, which allows one to calculate the total magnetic charge

$$g = \int k^0 d^3\vec{r} = \frac{1}{2ea^3}\,\varepsilon^{0\nu\alpha\beta}\int\partial_\nu\{\vec{\phi}\cdot(\partial_\alpha\vec{\phi}\times\partial_\beta\vec{\phi})\}d^3\vec{r} \quad . \tag{34}$$

Making use of the **Gauss** theorem and $\varepsilon^{0\nu\alpha\beta} = \varepsilon^{\nu\alpha\beta}$ for ν,α,β running from 1 to 3, one can simplify eq.(34) to

$$g = \frac{1}{2ea^3}\,\varepsilon^{\nu\alpha\beta}\,\varepsilon_{ijk}\,.\int_{S^2_\infty}\phi^i\,\partial_\alpha\phi^j\,\partial_\beta\phi^k\,d\sigma_\nu \tag{35}$$

where the surface integration is taken on the two dimensional sphere of radius tending to Infinite.

To evaluate the surface integral explicitly[7], let us parametrize the surface element of integration on S^2 by two new variables ξ_1 and ξ_2, and use the identity

$$d\sigma_\nu = \frac{1}{2}\,\varepsilon_{\nu\alpha\beta}\,\varepsilon_{mn}\,\frac{\partial\xi_\alpha}{\partial\xi_m}\,\frac{\partial\xi_\beta}{\partial\xi_n}\,d^2\xi \tag{36}$$

to re-express the charge g,

$$g = \frac{1}{4ea}\int_{S^2_\infty}\varepsilon_{mn}\varepsilon^{\nu\alpha\beta}\varepsilon_{\nu\alpha'\beta'}\varepsilon^{ijk}\,\phi_i\partial_\alpha\phi_j\partial_\beta\phi_k\times\frac{\partial\xi_{\alpha'}}{\partial\xi_m}\,\frac{\partial\xi_{\beta'}}{\partial\xi_n}\,d^2\xi$$

$$\frac{1}{4ea^3}\int\varepsilon_{mn}\,\varepsilon^{ijk}\,\phi_i\partial^m\phi_j\partial^n\phi_k\,d^2\xi = \frac{1}{2ea^3}\int D\,d^2\xi \tag{37}$$

where

$$D = \begin{vmatrix} \phi_1 & \phi_2 & \phi_3 \\ \dfrac{\partial\phi_1}{\partial\xi_1} & \dfrac{\partial\phi_2}{\partial\xi_1} & \dfrac{\partial\phi_3}{\partial\xi_1} \\ \dfrac{\partial\phi_1}{\partial\xi_2} & \dfrac{\partial\phi_2}{\partial\xi_2} & \dfrac{\partial\phi_3}{\partial\xi_2} \end{vmatrix} \tag{38}$$

Since the condition of the vacuum $|\vec{\phi}|^2 - a^2 = 0$ at $|\vec{r}|\to\infty$, one has

$$\phi_i\,\frac{\partial\phi_i}{\partial\xi_m} = 0 \;, \tag{39}$$

therefore

$$D^2 = D\,D^T = \begin{vmatrix} a^2 & 0 & 0 \\ 0 & E & F \\ O & F & G \end{vmatrix} \tag{40}$$

with

$$E = \frac{\partial}{\partial \xi_1}\vec{\phi} \cdot \frac{\partial}{\partial \xi_1}\vec{\phi} \quad , \tag{41a}$$

$$F = \frac{\partial}{\partial \xi_1}\vec{\phi} \cdot \frac{\partial}{\partial \xi_1}\vec{\phi} \quad , \tag{41b}$$

$$G = \frac{\partial}{\partial \xi_2}\vec{\phi} \cdot \frac{\partial}{\partial \xi_2}\vec{\phi} \quad . \tag{41c}$$

One recognizes that E, F and G are the three first fundamental forms of the surface in ϕ, henceforth, one can rewrite g imnediately in the expression

$$g = \frac{1}{ea^2}\int \sqrt{EG-F^2}\, d^2\xi = \frac{1}{ea^2}\int_{S^2_{|\phi|=a}} d^2\sigma_\phi = \frac{4\pi n}{e} \; , \tag{42}$$

where the integral d^2O_ϕ is taken over $S^2_{|\phi|=a}$, the two dimensional sphere with **radius** $|\phi|=a$. The integer

$$n = 0,\ \pm 1,\ \pm 2,\ \ldots.. \tag{43}$$

is the wrapping number, or Kronecker index, which characterizes the homotopic classes[8] in mapping $S^2_\infty \to S^2_{|\phi|=a}$. Since in natural unit a = = $e^2/4\pi$ = 1/137, one obtains the Schwinger condition for g

$$g = 137\, en \tag{44}$$

4. 't HOOFT-POLYAKOV ANSATZ FOR THE D = 3 SOLITON

Let us consider the energy of the system derived from the Lagrangian given by Eq. (12),

$$\theta^{00} = \frac{1}{2}\{ \sum_{i<j} \vec{F}_{ij} \cdot \vec{F}_{ij} + \vec{F}_{0i} \cdot \vec{F}_{0i} + \vec{\pi}\cdot\vec{\pi} + \partial_i\vec{\phi}\cdot\partial_i\vec{\phi} \}$$
$$+ \frac{\lambda}{4}(\vec{\phi}\cdot\vec{\phi} - a^2)^2 \quad , \tag{45}$$

where

$$\vec{\pi} = D^0\vec{\phi} \tag{46}$$

Since all the terrns on the right hand side of eq.(45) are positive definite, the vacuurn solution, in which $\theta^{00} = 0$, can be obtained in and only if

$$\vec{F}_{\mu\nu} = 0 \ , \tag{47a}$$
$$D_\mu\vec{\phi} = 0 \ , \tag{47b}$$
$$\vec{\phi}\cdot\vec{\phi}-a^2 = 0 \ . \tag{47c}$$

What we are looking for is the solution less trivial than that stated above. In order to simplify our discussion, we also restrict ourselves to the stationary solution for the electrically neutra1 system. The density for this particular consideration becomes

$$\theta^{00} = \frac{1}{2}\{ \sum_{i<j} \vec{F}_{ij} \cdot \vec{F}_{ij} + \partial_i\vec{\phi}\cdot\partial_i\vec{\phi} + \frac{\lambda}{4}(\vec{\phi}\cdot\vec{\phi}-a^2)^2\} \quad . \tag{48}$$

The expression given above has the rotational symmetry both in the coordinate space and in the isospin space. It has also the discrete symmetry in space and isospin space inversions. Let us denote the two discrete syrnrnetry operators by P and a respectively, and assume $\vec{\phi}$ is a scalar and pseudoscalar with respect to P and a, while A_i is a vector and axial vector under the P and a inversions. Narnely

$$P\vec{\phi}(\vec{r})P^{-1} = \vec{\phi}(-\vec{r}) \tag{49a}$$

$$\pi\vec{\phi}(\vec{r})\pi^{-1} = -\vec{\phi}(\vec{r}) \tag{49b}$$

$$P\vec{A}_i(\vec{r})P^{-1} = -\vec{A}_i(-\vec{r}) \tag{50a}$$

$$\pi\vec{A}_i(\vec{r})\pi^{-1} = \vec{A}_i(\vec{r}) \tag{50b}$$

By the definitions of the discrete transformation given above, one can easily verify that

$$P\vec{F}_{ij}(\vec{r})P^{-1} = \vec{F}_{ij}(-\vec{r}) \tag{51a}$$

and

$$\pi\vec{F}_{ij}(\vec{r})\pi^{-1} = -\vec{F}_{ij}(\vec{r}) \tag{51b}$$

Therefore the θ^{00} of the stationary system with electrostatic neutrality is invariant under the group $O(3) \times O(3)$ as well as the discrete group $P\times\pi$. To ensure the the Lagrangian describes the system of magnetic monopole, one shall not expect the discrete symmetry to hold separately for P and a. The reason is obvious: if one takes the divergence of the magnetic field $\vec{B}$, which is identified with the third component of the gauge field tensor F^3_{ij}, one has

$$\partial_i \, \varepsilon^{ijk} F^3_{jk} = k^0 \tag{52}$$

The equation can be invariant under P and π transformation separately only when $k^0 = 0$, which certainly is not the case of our prime interest.

't Hoof and Polyakov approached the problem with much atricter condition. Instead of considering the symmetry of $O(3) \times O(3)$ and $P\times\pi$, they investigated the soliton with the symmetry under the diagonal subgroup of $O(3) \times O(3)$ in space and isospin space transformation, as well as the diagonal subgroup in the discrete transformation[9]. In other words, the solution which transforms with equal rotation of angle in space and isospin space. To meet all these conditions, the solution can be cast into the general expressions of the following form,

$$\phi(\vec{r})^1 = a \frac{r^1}{|r|} H(|r|) \ , \tag{53a}$$

$$A_i = -\frac{1}{e}\varepsilon_{1ij}\frac{r_j}{r^2} K(|r|) + B(|r|)\,\delta_{1i} + C(|r|) r_i r_1 \tag{53b}$$

The invariaiice under the product of the discrete symmetry $P\pi$ forces one to put $B(r) = C(r) = 0$. The unknown function $H(r)$ and $K(r)$ are all of spherical symmetry. With this ansatz

$$\phi_1(\vec{r}) = a\,\frac{r_1}{r}\,H(r)\ , \tag{54a}$$

$$A_i^1 = -\frac{1}{e}\,\varepsilon_{1ij}\,\frac{r_j}{r^2}\,K(r)\ , \tag{54b}$$

one finds

$$\theta^{00} = \frac{1}{2e^2}\left\{\frac{1}{r^4}\left(K^2(K-2)^2 + K'^2\right)\right\} + \frac{a^2}{r^2}\left(H^2(k-1)^2 + \frac{H'^2}{2}\right)$$

$$+ \frac{\lambda}{4}\,a^4\,(H^2-1)^2 \tag{55}$$

The energy can be calculated by the spatial integration,

$$E = \int\theta^{00}d^3\vec{r} = -\int L\ d^3\vec{r}$$

$$= \frac{4\pi a}{e}\int\frac{dr}{r^2}\left\{\frac{1}{ae}\left(K^2(K-2)^2 + K'^2r^2\right) + ae\left(H^2(K-1)^2\right.\right.$$

$$\left.\left. + \frac{1}{2}H'^2\right)r^2 + \frac{ae}{4}\,\frac{\lambda}{e^2}\,(H^2-1)^2r^2\right\}\ . \tag{56}$$

Defining the dimensionless variables[10]

$$k(x) = K\left(\frac{x}{ea}\right)\ , \tag{57}$$

$$h(x) = H\left(\frac{x}{ea}\right)\ , \tag{58}$$

with

$$x = ear \tag{59}$$

one rewrites

$$E = \frac{4\pi a}{e}\int\frac{dx}{x^2}\left\{k^2(k-2)^2 + k'^2x^2 + h^2(k-1)^2x^2\right.$$

$$\left. + \frac{1}{2}h'^2x^2 + \frac{1}{4}\,\frac{\lambda}{e^2}\,x^2(h-1)^2\right\}. \tag{60}$$

The equations of motion are

$$x^2k'' = 2(k-1)\ (k-2) + h^2x^2(k-1)\ , \tag{61}$$

$$x^{-2}(x^2h')' = 2h(k-1)^2 + \frac{1}{2}\ \frac{\lambda}{e^2}\,h(h^2-1)\ . \tag{62}$$

If the solution is referred to the localized lump of energy, the finiteness of E requires the following boundary conditions

$$\begin{matrix} h \sim x \\ k \sim x^2 \end{matrix} \quad \text{as} \quad x \to 0 \tag{63}$$

and

$$\begin{matrix} h \sim 1 \\ k \sim 1 \end{matrix} \quad \text{as} \quad x \to \infty . \tag{64}$$

The numerical calculations for different values of λ/e^2 are shown in figure 1-3.

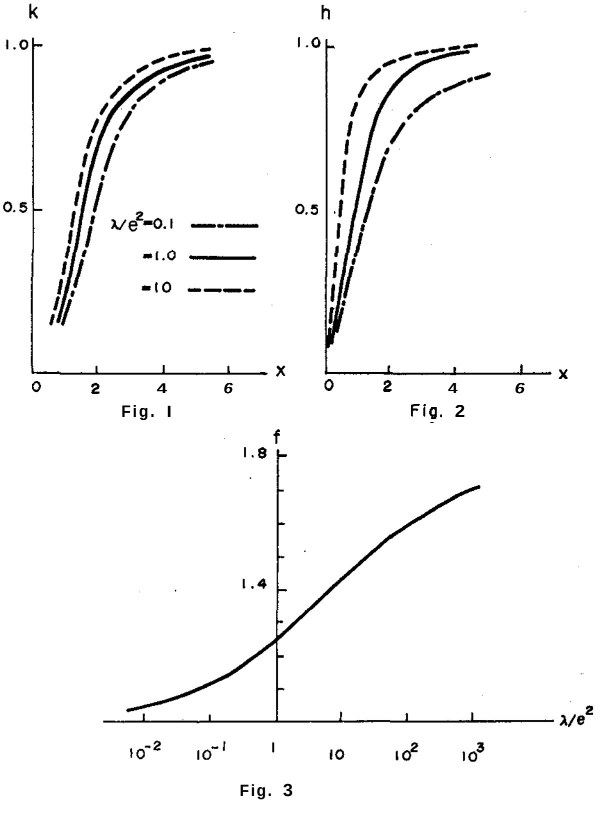

Fig. 1 Fig. 2 Fig. 3

The total energy, taken as the mass of soliton, is found to be a smooth and slowly varying function $f(\lambda/e^2)$, i.e.

$$M_s = \frac{\pi a}{e} f\left(\frac{\lambda}{e^2}\right) , \tag{65}$$

$$\simeq 137 M_A \quad . \tag{66}$$

The extraordinarily large mass of soliton, between 700 GeV to 1300 GeV lies beyond the limit of present experiments.

5. DISCUSSION

'The soliton given in the last section is a static, spherically symmetric one. It is also the solution for the single charged magnetic monopole. Since the gauge field tensor

$$F_{ij} = \frac{1}{e} \{\varepsilon_{ljk} \partial_i (\frac{r_k}{r} K) - \varepsilon_{lip} \partial_i (\frac{r_p}{r} K) + $$
$$+ \varepsilon_{ijk} \frac{r_l r_k}{r^4} K^2 \} \tag{67}$$

is related to the magnetic field B_i by

$$B_i = \varepsilon_{ijk} \frac{1}{a} \vec{F}_{ijk} \cdot \vec{\phi} \tag{68}$$

which reduces to

$$B_i \simeq \frac{1}{e} \frac{r_i}{r^3} \tag{69}$$

at the spatial infinite.

Therefore one will conclude that the soliton behaves like a magnetic monopole with pole strength $Q_m = 4\pi/e$, a particular case of Kronecker index $n = 1$ for the topological charge g. This result is not surprising because the 8t Hooft-Polyakov ansatz corresponds to the class 1 of the homotopy that maps the S^2_∞ of the sphere at spatial infinite to $S^2_{\phi=a}$ of the Higgs field in vacuum configuration with wrapping number 1. A direct consequence of the theory invariant under the diagonal subgroup of $O(3) \times O(3)$.

The solitons belonging to other homotopic classes have not yet been solved, except for the case for Yang-Mills field in space dimension equal to 4. The topological charges in the latter case will characterize the distinct classes of vacua state. The discussions of the structure of vacuum and the tunnelling between the distinct but equivalent classes of vacua are beyond the scope of this note.

As mentioned in Section 4, the 't Hooft-Polyakov ansatz is based upon the particular gauge $\vec{A}_0(\vec{r}) = 0$. The solution for the stationary, spherical symmetry of non vanishing $\vec{A}_0(r)$ does exist, and is referred to as the dyon solution. The study of the dyon solution and its relation to the duality rotation will be pursued further.

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