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On the Local Properties of the "Metric Tensor" $G_{\mu\nu}$ of the Unified Matrix theory of Gravitation, Electromagnetism and the Yang-Mills Field.*

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The vierbeins and the generalized γ-matrices associated to a modification and extension of Einstein's nonsymmetric field theory which includes the gravitational, electromagnetic and Yang-Mills fields are determined. The vierbein associated to the exact spherically symmetric solution of the unified field equations without Yang-Mills field is determined. Some algebraic properties of the unified matrix tensor $G_{\mu\nu}$ are discussed. A generalization of this formalism involving octonions is proposed.

Os vierbeins e as matrizes γ generalizadas associadas a uma modificação e extensão da teoria de campos não-simétrica de Einstein que inclui os campos gravitacional, eletromagnético e de Yang-Mills, são determinados. O vierbein associado a uma solução exata esfericamente simétrica das equações de campo unificado sem os campos de Yang-Mills é determinado. Algumas propriedades algébricas da matriz tensorial unificada $g_{\mu\nu}$ são discutidas. Propõe-se uma generalização deste formalismo envolvendo octonions.

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1. INTRODUCTION

An attempt to include the electromagnetic field as a part of the geometry in a extension of the theory of general relativity was proposed by Einstein in his nonsymmetric (complex) theory . In this theory the *metric tensor* is decomposed into its symmetric and antisymmetric parts as

$$h_{\mu\nu} = h_{(\mu\nu)} + h_{[\mu\nu]}$$

$$h_{(\mu\nu)} = g_{\mu\nu} \, , \; h_{[\mu\nu]} = i \, f_{\mu\nu} \qquad (1.1)$$

where the $g_{\mu\nu}$ and $f_{\mu\nu}$ are real quantities. The generalized *metric* $h_{\mu\nu}$ is a 4×4 Hermitian matrix, $h^*_{\mu\nu} = h_{\nu\mu}$. This condition generalizes the symmetry property of the metric in general relativity.

Recently this theory has been reviewed and modified by Moffat and Boal. They have proposed an unified theory of gravitation and electromagnetism which tends to the Einstein-Maxwell theory in the limit where *a* fundamental constant $k=iK$ tends formally to zero[2]. The constant K is given in terms of a length L as $K=L^2/e$, where e is the charge of the electron and L may be shosen as the Planck length $L_P = (\hbar G/c^3)^{1/2} = 1.62$ 10^{-33}cm [3]. The introduction of this constant is obtained by observing thãt the $h_{\mu\nu}$ may be taken as dimensionless quantities: $ds^2 = h_{\mu\nu}dx^\mu dx^\nu =$ $= g_{\mu\nu}dx^\mu dx^\nu$. Thus, $f_{\mu\nu}$ is dimensionless and may be written as $f_{\mu\nu}= KF_{\mu\nu}$, where $F_{\mu\nu}$ has the dimension of an electrornagnetic íield. In c.g.s. units

$$[F_{\mu\nu}] = g^{+1/2} \text{ cm}^{-1/2} \text{ sec}^{-1} \; , \; [K] = 1/[F_{\mu\nu}] = L^2/e$$

The length L may be defined in terms of the fundamental constants $\hbar$, G and c as the Planck length L_P. A pure "classical" length may also be obtained on dimensional grounds as $L_c = e\, G^{1/2}/c^2 = 1.38\times10^{-34}$cm. The values assumed by K for these two choices are $K_P = 5.44\times10^{-57}$ cm, $K_c =$ $= 3.95\times10^{-59}$ cm, giving the ratio $K_P/K_c = 137$. Thus, the constant K for these two cases differ only by a factor $\alpha^{-1} = \hbar c/e^2$.

The field equations of the generalized theory are derived from a variational principle. Defining a nonsymmetrical affine connection by

$$\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{(\mu\nu)} + \Gamma^{\lambda}_{[\mu\nu]}$$

where $\Gamma^{\lambda}_{[\mu\nu]}$ is a imaginary third rank antisymmetric tensor, it is possible to introduce another connection $W^{\lambda}_{\mu\nu}$ by the equation

$$\Gamma^{\lambda}_{\mu\nu} = W^{\lambda}_{\mu\nu} + 2/3\ \delta^{\lambda}_{\mu} W_{\nu}$$

with

$$W_{\nu} = 1/2(W^{\sigma}_{\nu\sigma} - W^{\sigma}_{\sigma\nu}) .$$

The variational principle is written as

$$\delta \int L\, d^4x = 0 \tag{1.2}$$

$$L = \sqrt{-h}\, h^{\mu\nu} R_{\mu\nu}(W) - \frac{4\pi G}{k^2 c^4} \sqrt{-h}\, h^{[\mu\nu]} h_{[\mu\nu]} \tag{1.3}$$

where $R_{\mu\nu}(W)$ is the Ricci tensor of the connection $W^{\lambda}_{\mu\nu}$ and $h^{\mu\nu}$ is defined by

$$h^{\mu\nu} h_{\sigma\nu} = h^{\nu\mu} h_{\nu\sigma} = \delta^{\mu}_{\sigma}$$

with $h^{\mu\nu} = h^{(\mu\nu)} + h^{[\mu\nu]}$. In (1.2) $\mathfrak{h}^{\mu\nu} = \sqrt{-h}\, h^{\mu\nu}$ and $W^{\lambda}_{\mu\nu}$ are to be varied independently of one another (Palatini's method). The first term in (1.3) was used by Einstein in his formulation of the nonsymmetrical theory. The present formulation contains the additional quadratic term $\sqrt{-h}\, h^{[\mu\nu]} h_{[\mu\nu]}$. The field equations obtained from the variational principle (1.2) tend to the Einstein-Maxwell equations in the limit where $K \to 0$, and this property gives a direct correspondence principle between the generalized unitary theory and the Einstein theory of gravitation (general relativity). This theory was generalized in order to include the Yang-Mills field theory[4]. In this generalization of the Moffat- Boal theory the functions $h_{\mu\nu}(x)$ and $\Gamma^{\lambda}_{\mu\nu}(x)$ become 2×2 matrices. Under the action of local SU(2) transformations these matrices transform as

$$h_{\mu\nu} \to G_{\mu\nu},\ \bar{G}_{\mu\nu}(x) = M(x) G_{\mu\nu}(x) M^{-1}(x),\ M = M^{-1},$$

$$\bar{\Gamma}^{\lambda}_{\mu\nu}(x) = M(x)\Gamma^{\lambda}_{\mu\nu}(x)\ M^{-1}(x) + \delta^{\lambda}_{\mu} M(x)\ M^{-1}_{,\nu}(x)$$

In general $G_{\mu\nu} = q_{\mu\nu i}\omega_i$ $(i=1,2,3,4)$, where ω_4 is the 2x2 identity matrix and the ω_i $(i=1,2,3)$ are the Pauli matrices. The field equations, derived from a variational principle, tend to the EMYM (Einstein- Maxwell Yang-Mills) field equations in the limit $K \to 0$ [4]. The coefficients $q_{\mu\nu i}$ have the form

$$
\begin{aligned}
q_{\mu\nu 4} &= 1/2\ h_{\mu\nu} \\
q_{\mu\nu \underline{i}} &= iK\mu\ f_{\mu\nu \underline{i}}
\end{aligned}
\tag{1.4}
$$

where $f_{\mu\nu i} = \overset{*}{f}_{\mu\nu i}$, $f_{\mu\nu i} = -f_{\nu\mu i}$ is the Yang-Mills field tensor of the unified theory. The constant μ is given by $\mu = 1/2\ Ka$, where $a = \varepsilon/\hbar$, ε being the elementary isotopic charge. We mention that our first equation (1.4) differs from the Borchsenius choice by a factor 1/2 in front of the $h_{\mu\nu}$. The relations (1.4) imply that Tr $G_{\mu\nu} = h_{\mu\nu}$. For $K \to 0$ this relation gives the metric of general relativity.

The Hermitian symmetry condition now holds for all coefficients $q_{\mu\nu i}$, namely $q_{\mu\nu i} = q^{*}_{\nu\mu i}$. With these conditions we have

$$
G^{\dagger}_{\mu\nu} = G_{\nu\mu} \tag{1.5}
$$

The symbol $\dagger$ indicates Hermitian conjugation in the space of complex 2x2 matrices ($G^{\dagger}_{\mu\nu} = q^{*}_{\mu\nu i}\ \omega_i$). The condition (1.5) generalizes the Hermitian symmetry conditions satisfied by the $h_{\mu\nu}$.

2. VIERBEIN REPRESENTATION OF THE MATRIX FORMULATION OF THE NONSYMMETRIC THEORY

It is well known that a complex vierbeine formalism may describe a nonsymmetric *metric* of the form (1.1) [5]. Presently we extend this result to the case where the $h_{\mu\nu}$ of (1.1) become a set of sixteen 2 × 2 matrices. Thus, associated to the matrices $G_{\mu\nu}$ a field of complex matrix tetrad is defined by

$$
G_{\mu\nu}(x) = H^{\dagger(\alpha)}_{\mu}(x)\ \Sigma_{(\alpha)(\beta)} H^{(\beta)}_{\nu}(x) \tag{2.1}
$$

where

$$H_{\mu}^{(\alpha)} = \rho_{\mu i}^{(\alpha)} \omega_{i} \qquad (2.2)$$

$$\Sigma_{(\alpha)(\beta)} = \sigma_{(\alpha)(\beta)i} \omega_{i} \qquad (2.3)$$

From (1.5) and (2.1) it follows that the matrices $\Sigma_{(\alpha)(\beta)}$ satisfy

$$\Sigma^{\dagger}_{(\alpha)(\beta)} = \Sigma_{(\alpha)(\beta)} \qquad (2.4)$$

$$\Sigma_{(\alpha)(\beta)} = \Sigma_{(\beta)(\alpha)} \qquad (2.5)$$

For arbitrary 2x2 matrices $A = a_i \omega_i$, $B = b_i w_i$ the commutator and anti-commutator are given by

$$[A,B] = a_i b_j 1_{ijk} \omega_i \qquad (2.6)$$

$$\{A,B\} = a_i b_j f_{ijk} \omega_i \qquad (2.7)$$

with

$$1_{\underline{i}44} = 1_{\underline{i}4\underline{r}} = 1_{\underline{ir}4} = 0 \;, \; 1_{\underline{irk}} = 2i\varepsilon_{\underline{irk}} \qquad (2.8)$$

$$f_{\underline{irk}} = f_{\underline{i}44} = f_{44\underline{i}} = 0 \;, \; f_{\underline{ik}4} = f_{\underline{i}4\underline{k}} = 2\delta_{\underline{ik}} \;, \; f_{444} = 2 \qquad (2.9)$$

Using these relations it may be shown that

$$\omega_i \omega_j \omega_k = 1/4 \left(1_{ijs} 1_{skm} + 1_{ijs} f_{skm} + f_{ijs} 1_{skm} + f_{ijs} f_{skm}\right) \omega_m \qquad (2.10)$$

From (2.1), (2.2), (2.3) and (2.10) we find

$$q_{\mu\nu m} = 1/4 \, \rho_{\mu i}^{*(\alpha)} \, \sigma_{(\alpha)(\beta)} \, \rho_{\nu k}^{(\beta)} \left(1_{ijs} 1_{skm} + 1_{ijs} f_{skm} + f_{ijs} 1_{skm} + f_{ijs} f_{skm}\right) \qquad (2.11)$$

for $m=4$ these relations give

$$q_{\mu\nu 4} = 1/2\, h_{\mu\nu} = i\vec{\rho}_{\nu}^{(\beta)} \cdot (\vec{\rho}_{\mu}^{*(\alpha)} \times \vec{\sigma}_{(\alpha)(\beta)}) + (\rho_{\nu 4}^{(\beta)} \vec{\rho}_{\mu}^{*(\alpha)}$$

$$+ \rho_{\mu 4}^{*(\alpha)} \vec{\rho}_{\nu}^{(\beta)}) \cdot \vec{\sigma}_{(\alpha)(\beta)} + (\rho_{\mu 4}^{*(\alpha)} \rho_{\nu 4}^{(\beta)} + \vec{\rho}_{\mu}^{*(\alpha)} \cdot \vec{\rho}_{\nu}^{(\beta)}) \sigma_{(\alpha)(\beta)4} \tag{2.12}$$

and form m = $\underline{m}$ we get

$$\vec{q}_{\mu\nu} = (\rho_{\mu 4}^{*(\alpha)} \rho_{\nu 4}^{(\beta)} - \vec{\rho}_{\mu}^{*(\alpha)} \cdot \vec{\rho}_{\nu}^{(\beta)}) \vec{\sigma}_{(\alpha)(\beta)}$$

$$+ (\vec{\rho}_{\nu}^{(\beta)} \cdot \vec{\sigma}_{(\alpha)(\beta)} + \sigma_{(\alpha)(\beta)4} \rho_{\nu 4}^{(\beta)}) \vec{\rho}_{\mu}^{*(\alpha)}$$

$$\rho_{\mu 4}^{*(\alpha)} \sigma_{(a)(\beta)4} \vec{\rho}_{\nu}^{(\beta)} + i(\vec{\rho}_{\mu}^{*(\alpha)} \times \vec{\rho}_{\nu}^{(\beta)}) \sigma_{(\alpha)(\beta)4}$$

$$+ i(\vec{\rho}_{\mu}^{*(\alpha)} \times \vec{\sigma}_{(\alpha)(\beta)}) \rho_{\nu 4}^{(\beta)} + i(\vec{\sigma}_{(\alpha)(\beta)} \times \vec{\rho}_{\nu}^{(\beta)}) \rho_{\mu 4}^{*(\alpha)} \tag{2.13}$$

These equations can be **simplified** by noting that matrices $\Sigma_{(\alpha)(\beta)}$ satisfying the conditions (2.4) and (2.5) may be written as $\Sigma_{(\alpha)(\beta)} = \eta_{\alpha\beta}\, \omega_4$, where $\eta_{\alpha\beta}$ is the Minkowski tensor. Using this expression for the matrices $\Sigma_{(\alpha)(\beta)}$ the equations (2.12) and (2.13) are rewritten as

$$1/2\, h_{\mu\nu} = (\rho_{\mu 4}^{*(\alpha)} \rho_{\nu 4}^{(\beta)} + \vec{P}_{\mu}^{*(\alpha)}, \vec{\rho}_{\nu}^{(\beta)}) \eta_{\alpha\beta} \tag{2.14}$$

$$\vec{q}_{\mu\nu} = (\rho_{\nu 4}^{(\beta)} \vec{\rho}_{\mu}^{*(\alpha)} + \rho_{\mu 4}^{*(\alpha)} \vec{\rho}_{\nu}^{(\beta)}) \eta_{\alpha\beta} + i(\vec{\rho}_{\mu}^{*(\beta)} \times \vec{\rho}_{\nu}^{(\beta)}) \eta_{\alpha\beta} \tag{2.15}$$

Writing

$$\rho_{\mu 4}^{(\alpha)} = \frac{1}{\sqrt{2}} k_{\mu}^{(\alpha)} \quad ; \quad \vec{\rho}_{\mu}^{(\alpha)} = \frac{1}{\sqrt{2}} \vec{\tau}_{\mu}^{(\alpha)} \quad ,$$

$$k_{\mu}^{(\alpha)} = r_{\mu}^{(\alpha)} + i\, s_{\mu}^{(\alpha)} \quad , \quad \vec{\tau}_{\mu}^{(\alpha)} = \vec{\theta}_{\mu}^{(\alpha)} + i\vec{\pi}_{\mu}^{(\alpha)}$$

we have for (2.14), (2.15) and (1.1)

$$g_{\mu\nu} = (r_\mu^{(\alpha)} r_\nu^{(\beta)} + s_\mu^{(\alpha)} s_\nu^{(\beta)} + \vec{\theta}_\mu^{(\alpha)} \cdot \vec{\theta}_\nu^{(\beta)} + \vec{\pi}_\mu^{(\alpha)} \cdot \vec{\pi}_\nu^{(\beta)}) \eta_{\alpha\beta} \tag{2.16}$$

$$K\, F_{\mu\nu} = (r_\mu^{(\beta)} s_\nu^{(\alpha)} - s_\mu^{(\alpha)} r_\nu^{(\beta)} + \vec{\pi}_\nu^{(\beta)} \cdot \vec{\theta}_\mu^{(\alpha)} - \vec{\pi}_\mu^{(\alpha)} \cdot \vec{\theta}_\nu^{(\beta)}) \eta_{\alpha\beta} \tag{2.17}$$

$$\mathrm{Re}\, \vec{q}_{\mu\nu} = 1/2 (r_\nu^{(\beta)} \vec{\theta}_\mu^{(\alpha)} + r_\mu^{(\beta)} \vec{\theta}_\nu^{(\alpha)} + s_\nu^{(\beta)} \vec{\pi}_\mu^{(\alpha)} + s_\mu^{(\beta)} \vec{\pi}_\nu^{(\alpha)}$$

$$- \vec{\pi}_\mu^{(\alpha)} \times \vec{\theta}_\nu^{(\beta)} + \vec{\theta}_\mu^{(\beta)} \times \vec{\pi}_\nu^{(\alpha)}) \eta_{\alpha\beta} \tag{2.18}$$

$$\mathrm{Im}\, \vec{q}_{\mu\nu} = K\mu \vec{f}_{\mu\nu} = 1/2 (r_\mu^{(\beta)} \vec{\pi}_\nu^{(\alpha)} - r_\nu^{(\beta)} \vec{\pi}_\mu^{(\alpha)} + s_\nu^{(\beta)} \vec{\theta}_\mu^{(\alpha)}$$

$$- s_\mu^{(\beta)} \vec{\theta}_\nu^{(\alpha)} + \vec{\theta}_\mu^{(\alpha)} \times \vec{\theta}_\nu^{(\beta)} + \vec{\pi}_\mu^{(\alpha)} \times \vec{\pi}_\nu^{(\beta)}) \eta_{\alpha\beta} \tag{2.19}$$

The quantity Re $\vec{q}_{\mu\nu}$ vanishes due to the choice (1.4) . It should be noted that the Hermitian symmetry condition still allows for the presence of a field with spin 2 and isotopic spin 1 of the form Re $\vec{q}_{\mu\nu} = \vec{s}_{(\mu\nu)}$ (*). In this paper this contribution will not be considered. We use the notation

$$r_\mu^{(\alpha)} r_\nu^{(\beta)} \eta_{\alpha\beta} = \overset{\circ}{g}_{\mu\nu} \tag{2.20}$$

the tensor $\overset{\circ}{g}_{\mu\nu}$ describes the pure gravitational field, that means, it corresponds to the gravitational potentials of Einstein's general relativity theory. Writing

$$\Delta g_{\mu\nu} = g_{\mu\nu} - \overset{\circ}{g}_{\mu\nu}$$

it follows from (2.16) that in the limit where $K \to 0$ the tensor $\Delta g_{\mu\nu}$ also tends to zero. From (2.17) and (2.19) we see that the quantities $s_\mu^{(\alpha)}$, $\vec{\theta}_\mu^{(\alpha)}$ and $\vec{\pi}_\mu^{(\alpha)}$ tend to zero in this limit (the same conclusion follows from (2.16)). Thus the tetrads $s_\mu^{(\alpha)}$, $\vec{\theta}_\mu^{(\alpha)}$ and $\vec{\pi}_\mu^{(\alpha)}$ are proportional to K.

The unified field theory of gravitation and electromagnetism descrlbed in terms of a Hermitian field tensor $h_{\mu\nu}$ may be formulated in terms of a complex tetrad by

(*) in this general case $G_{\mu\nu} = 1/2(g_{\mu\nu} + iKF_{\mu\nu})\omega_4 + iK\mu\vec{f}_{\mu\nu} \cdot \vec{\omega} + \vec{s}_{(\mu\nu)} \cdot \vec{\omega}$ -4

$$h_{\mu\nu} = k^{*(\alpha)}_{\mu} k^{(\beta)}_{\nu} \eta_{\alpha\beta} .$$

For given $h_{\mu\nu}$ the tetrads are determined up to a pseudo-unitary transformation:

$$k'^{(\alpha)}_{\mu} = C^{(\alpha)}{}_{(\beta)} k^{(\beta)}_{\mu} , \quad C^{\Pi} \cdot \eta \cdot C = \eta \; (*) \qquad (2.21)$$

The matrix elements of this transformation are arbitrary functions of the coordinates. For infinitesimal transformations $C=1+U$, and from (2.21) we have

$$\eta \cdot U = R + iS , \quad R^{\pi} = -R , \quad S^{\pi} = S .$$

Thus, there are sixteen degrees of freedom in the transformations (2.21). In this theory we have sixtaen field functions (the components of the Hermitian tensor $h_{\mu\nu}$) and 32 real independent components for the tetrads. Then, sixteen conditions can be imposed on the $k^{(\alpha)}_{\mu}$ in such form that $C\to 1$ (fixation of the orientation of the complex tetrad).

In the present formulation of the unitary field theory of gravitation, electromagnetism and the Yang-Mills field the generalized metric $G_{\mu\nu}$ which satisfies the symmetry condition (1.5) is written in terms of amatrix tetrad by the equations (2.1)(**). Forgiven $G_{\mu\nu}$ the matrix tetrads are determined up to a local generalized pseudo-unitary transformation

$$H'^{(\alpha)}_{\mu} = L^{(\alpha)}{}_{(\beta)} H^{(\beta)}_{\mu} , \quad H'^{\dagger(\alpha)}_{\mu} = H^{\dagger(\beta)}_{\mu} L^{\dagger(\alpha)}{}_{(\beta)} ,$$

$$L^{\dagger(\alpha)}{}_{(\rho)} \eta_{\alpha\beta} L^{(\beta)}{}_{(\tau)} = \eta_{\rho\tau} \omega_4 \qquad (2.22)$$

(the $L^{(\alpha)}{}_{(\beta)}$ are a set of sixteen 2×2 matrices). Writing

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(*) the symbol Π indicates Hermitian conjugation in the space of 4×4 matrices of the form $C=(C^{(\alpha)}{}_{(\beta)})$. Then, $C^{\Pi} = C^{\pi *}$.

(**) here we use the condition $C_{(\alpha)(\beta)} = \eta_{\alpha\beta}\, \omega_4$.

$$H'^{(\alpha)}_{\mu} = \rho'^{(\alpha)}_{\mu i}\ \omega_i = L^{(\alpha)}_{(\beta)}\ \rho^{(\beta)}_{\mu i}\ \omega_i \quad ,$$

$$L^{(\alpha)}_{(\beta)} = B^{(\alpha)}_{(\beta)i}\ \omega_i$$

we find

$$\rho'^{(\alpha)}_{\mu i} = 1/2(B^{(\alpha)}_{(\beta)j}\ \rho^{(\beta)}_{\mu k}\ f_{jki} + B^{(\alpha)}_{(\beta)j}\ \rho^{(\alpha)}_{\mu k}\ 1_{jki}) . \qquad (2.23)$$

Taking

$$B^{(\alpha)}_{(\beta)4} = D^{(\alpha)}_{(\beta)} + i\ E^{(\alpha)}_{(\beta)} \ , \quad \vec{B}^{(\alpha)}_{(\beta)} = \vec{P}^{(\alpha)}_{(\beta)} + i\ \vec{N}^{(\alpha)}_{(\beta)}$$

and using **the** previous decomposi**tion of the** $\rho^{(\alpha)}_{\mu i}$ **into real and** imaginary parts, we **obtain** from **(2.23)**

$$r'^{(\alpha)}_{\mu} = D^{(\alpha)}_{(\beta)}\ r^{(\beta)}_{\mu} - E^{(\alpha)}_{(\beta)}\ s^{(\beta)}_{\mu} + \vec{P}^{(\alpha)}_{(\beta)}\cdot\vec{\theta}^{(\beta)}_{\mu} - \vec{N}^{(\alpha)}_{(\beta)}\cdot\vec{\pi}^{(\beta)}_{\mu} \qquad (2.24)$$

$$s'^{(\alpha)}_{\mu} = E^{(\alpha)}_{(\beta)} r^{(\beta)}_{\mu} + D^{(\alpha)}_{(\beta)}\ s^{(\beta)}_{\mu} + \vec{P}^{(\alpha)}_{(\beta)}\cdot\vec{\pi}^{(\beta)}_{\mu} + \vec{N}^{(\alpha)}_{(\beta)}\cdot\vec{\theta}^{(\beta)}_{\mu} \qquad (2.25)$$

$$\vec{\theta}'^{(\alpha)}_{\mu} = D^{(\alpha)}_{(\beta)}\vec{\theta}^{(\beta)}_{\mu} - E^{(\alpha)}_{(\beta)}\vec{\pi}^{(\beta)}_{\mu} + \vec{P}^{(\alpha)}_{(\beta)} r^{(\beta)}_{\mu} - \vec{N}^{(\alpha)}_{(\beta)}\ s^{(\beta)}_{\mu}$$

$$- \vec{P}^{(\alpha)}_{(\beta)} \times \vec{\pi}^{(\beta)}_{\mu} - \vec{N}^{(\alpha)}_{(\beta)} \times \vec{\theta}^{(\beta)}_{\mu} \qquad (2.26)$$

$$\vec{\pi}'^{(\alpha)}_{\mu} = D^{(\alpha)}_{(\beta)}\vec{\pi}^{(\beta)}_{\mu} + E^{(\alpha)}_{(\beta)}\vec{\theta}^{(\alpha)}_{\mu} + \vec{P}^{(\alpha)}_{(\beta)}\ s^{(\beta)}_{\mu} + \vec{N}^{(\alpha)}_{(\beta)} r^{(\beta)}_{\mu}$$

$$+ \vec{P}^{(\alpha)}_{(\beta)} \times \vec{\theta}^{(\beta)}_{\mu} - \vec{N}^{(\alpha)}_{(\beta)} \times \vec{\pi}^{(\beta)}_{\mu} \qquad (2.27)$$

The relations (2.22) may also be written as

$$C^{\Pi}.\eta.C + B^{\Pi}_{\underline{i}}.\eta.B_{\underline{i}} = \eta \ , \quad C^{\Pi}.\eta.B_{\underline{k}} + B^{\Pi}_{\underline{k}}.\eta.C + iB^{\Pi}_{\underline{s}}.\eta.B_{\underline{j}}\ \varepsilon_{\underline{sjk}} = 0$$

In the limit $B_{i} \to 0$ they assume the form of pseudo-unitary transformations given by (2.21). Presently we have 34 real field components in the $G_{\mu\nu}$, namely the quantities $g_{\mu\nu}$, $F_{\mu\nu}$ and $\vec{f}_{\mu\nu}$. The matrices $H_{\mu}^{(\alpha)}$ contain a total of 128 real independent components. These components are subjected to 30 conditions given by Re$\vec{q}_{\mu\nu} = 0$ (eq.2.18). Thus, only 94 components in the $H_{\mu}^{(\alpha)}$ remain independent: It follows that 64 conditions may still be imposed on the components of $H^{(\alpha)}$ (the difference betweeo the total number of field components in $H_{\mu}^{(\alpha)}$ and the total number of field components in $G_{\mu\nu}$). The conditions represent here a given fixation of the *orientation* for the complex matrix tetrad $H_{\mu}^{(\alpha)}$. Considering infinitesimal tetrad transformations

$$C = 1 + U \ , \ B_{i} = \varepsilon_{i} \ ,$$

where U and ε_{i} are first order matrices, and $C = B_4 = (B^{(\alpha)}{}_{(\beta)4})$, we have

$$\eta \cdot U = R + i\, S \ , \quad \eta \cdot \varepsilon_{k} = T_{k} + i\, Q_{k}$$

with

$$R^{\pi} = -R \ , \ S^{\pi} = S \ , \ T_{k}^{\pi} = -T_{k} \ , \ Q_{k}^{\pi} = Q_{k} \ .$$

Therefore the generalized local tetrad transformations (2.22) contain 64 real independent parameters. The values for these parameters may be fixed by considering a particular orietantion for the tetrads $H_{\mu}^{(\alpha)}$.

3. GENERALIZED γ-MATRICES ASSOCIATED TO THE VIERBEIN $\mathbf{H}_{\mu}^{(\alpha)}$

With the set of internal matrices $H_{\mu}^{(\alpha)} = (H_{\mu}^{(\alpha)a}{}_{b}) = \rho_{\mu i}^{(\alpha)} \omega_{i}$, and with the choice of the matrices $\Sigma_{(\alpha)(\beta)}$ considered in the previous section the matrices $G_{\mu\nu}$ take the form

$$G_{\mu\nu} = \eta_{\alpha\beta}\, H_{\mu}^{\dagger(\alpha)} H_{\nu}^{(\beta)} \tag{3.1}$$

In this section we extend the conventional formulas connecting the vierbein to the y-matrices to the present definition of the vierbein as a set of 2x2 matrices. Thus, we introduce the generalized "y-matrices" as

$$\Gamma_\mu(x) = H_\mu^{(\alpha)}(x)\, \overset{o}{\gamma}_\alpha = (\Gamma^{a\ A}_{\mu\ bB}) = (H_\mu^{(\alpha)a}\, {}_b\overset{o}{\gamma}_\alpha{}^A{}_B) \tag{3.2}$$

capital Roman letters indicate fourdimensional spinor degrees of freedom. The $\overset{o}{\gamma}_a$ are the Dirac matrices. Then

$$\mathrm{I} \qquad = 2\eta_{\alpha\beta} 1_4 \ , \quad 1_4 = (\delta^A{}_B) \tag{3.3}$$

The object $\Gamma_\mu^\dagger$ is defined as the direct contracted product of the matrices $H_\mu^{\dagger(\alpha)}$ by the Dirac matrices $\overset{o}{\gamma}$:

$$\Gamma_\mu^\dagger = H_\mu^{\dagger(\alpha)}\, \overset{o}{\gamma}_\alpha$$

For the anticommutador of $\Gamma_\mu^\dagger$ and Γ_ν we find

$$\{\Gamma_\mu^\dagger, \Gamma_\nu\} = H_\mu^{\dagger(\alpha)}\, \overset{o}{\gamma}_\alpha\, H_\nu^{(\beta)}\, \overset{o}{\gamma}_\beta + H_\nu^{(\beta)}\, \overset{o}{\gamma}_\beta\, H_\mu^{\dagger(\alpha)}\, \overset{o}{\gamma}_\alpha$$

$$= H_\mu^{\dagger(\alpha)}\, H_\nu^{(\beta)}\, \overset{o}{\gamma}_\alpha\, \overset{o}{\gamma}_\beta + H_\nu^{(\beta)}\, H_\mu^{\dagger(\alpha)}\, \overset{o}{\gamma}_\beta\, \overset{o}{\gamma}_\alpha \tag{3.4}$$

where use have been made of the property that $H_\mu^{(\alpha)}$ and $\overset{o}{\gamma}_\lambda$ commute since they belong to different spaces. From (2.2) and *(2.6)* we have

$$[H_\mu^{\dagger(\alpha)}\, ,\, H_\nu^{(\beta)}] = \rho_{\mu i}^{*(\alpha)}\, \rho_{\nu j}^{(\beta)}\, 1_{ijk}\, \omega_k \tag{3.5}$$

Eqs.(3.4) and *(3.5)* give

$$\{\Gamma_\mu^\dagger, \Gamma_\nu\} = H_\mu^{\dagger(\alpha)}\, H_\nu^{(\beta)}\, (\overset{o}{\gamma}_\alpha\, \overset{o}{\gamma}_\beta + \overset{o}{\gamma}_\beta\, \overset{o}{\gamma}_\alpha) + \rho_{\mu i}^{*(\alpha)}\, \rho_{\nu j}^{(\beta)}\, 1_{jik}\omega_k \cdot \overset{o}{\gamma}_\beta \overset{o}{\gamma}_\alpha$$

Thus, from *(3.3)* and *(3.1)* we may write this equation as

$$\{\Gamma_\mu^\dagger\, \Gamma_\nu\} = 2G_{\mu\nu} \cdot 1_4 + \rho_{\mu i}^{a}\, \rho_{\nu j}^{(\beta)}\, 1_{jik}\, \omega_k\, \overset{o}{\gamma}_\beta\, \overset{o}{\gamma}_a \tag{3.6}$$

The extra term in the r.h.s. of this relation has the explicit form

$$\rho_{\mu i}^{*(\alpha)}\, \rho_{\nu j}^{(\beta)}\, 1_{jik}\, \omega_k\, \overset{o}{\gamma}_\beta\, \overset{o}{\gamma}_\alpha = 1/2\ \vec{\omega}.(\vec{\tau}_\nu^{(\beta)} \times \vec{\tau}_\mu^{*(\alpha)})\overset{o}{\gamma}_\beta\, \overset{o}{\gamma}_\alpha$$

The expression *(3.6)* generalizes the anticommutation relation of the matrices $\gamma_\mu(x) = r_\mu^{(\alpha)} \overset{o}{\gamma}_a$ used in general ralitivity. In the limit of $K\to 0$ this relation is recovered by taking trace on the internal indices:

$$\mathrm{Tr} \lim_{K\to 0} \{\Gamma_\mu^\dagger, \Gamma_\nu\} = 2g_{\mu\nu} \cdot 1_4$$

4. THE VIERBEIN FOR THE EXACT SPHERICALLY SYMMETRIC SOLUTION

In this section we consider the problem of determination of the vierbeins $r_\mu^{(\alpha)}$ and $s_\mu^{(\alpha)}$ for a spherically symmetric exact solution of the field equations of the Moffat-Boal theory (without Yang-Mills fields). Exact static solutions of the theory in this case have been derived[6], and a static field with spherical symmetry in polar coordinated $x^1 = r, x^2 = \theta$, $x^3 = \phi$, $x^4 = ct$ assumes in general the form

$$h_{\mu\nu} = \begin{bmatrix} -\alpha & 0 & 0 & w \\ 0 & -\beta & f\sin\theta & 0 \\ 0 & -f\sin\theta & -\beta\sin^2\theta & 0 \\ -w & 0 & 0 & \gamma \end{bmatrix},$$

where α, β, γ, f and w are functions of r only. The functions w and f are imaginary quantities[7]. For a charged massive particle at the origin of co-ordinates these functions are given by[2]

$$\begin{aligned}
\alpha &= \left(1 - \frac{2Gm}{c^2r} + \frac{4\pi G q^2}{c^4r^2}\right)^{-1} \\
\beta &= r^2 \\
\gamma &= \left(1 - \frac{2Gm}{c^2r} + \frac{4\pi G q^2}{c^4 r}\right)\left(1 - \frac{K^2q^2}{r^4}\right) \qquad (4.1) \\
w &= \frac{iKq}{r^2} \\
f &= 0
\end{aligned}$$

The field $-i\, w/K = E_r = q/r^2$ is the static electric field due to the charge q at the origin. In this case the tensor $\overset{\circ}{g}_{\mu\nu}$ is the Reissner-Nordstrom metric with value

$$\overset{\circ}{g}_{\mu\nu} = \begin{pmatrix} -\overset{\circ}{\alpha} & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \\ 0 & 0 & -\beta \sin^2\theta & 0 \\ 0 & 0 & 0 & \overset{\circ}{\alpha}^{-1} \end{pmatrix} \tag{4.2}$$

where

$$\overset{\circ}{\alpha} = \left(1 - \frac{2Gm}{c^2 r} + \frac{4\ G\ q^2}{c^4 r^2} \right)^{-1} = \alpha \tag{4.3}$$

(for signature -2). Here the relation (2.16) and (2.17) take the form

$$\Delta g_{\mu\nu} = s_\mu^{(\alpha)}\ s_\nu^{(\beta)}\ \eta_{\alpha\beta} \tag{4.4}$$

$$K\ F_{\mu\nu} = (r_\mu^{(\alpha)}\ s_\nu^{(\beta)} - s_\mu^{(\beta)}\ r_\nu^{(\alpha)})\eta_{\alpha\beta} \tag{4.5}$$

As it is easily seen the 4x4 matrix $\Delta g_{\mu\nu}$ is singular. A possible solution for the equations (4.4) and (4.5) for given $\Delta g_{\mu\nu}$ and $F_{\mu\nu}$ (with the values written in (4.1) and (4.2)) is of the form

$$s_4^{(\alpha)} = - \frac{Kq}{\alpha r^2}\, r_1^{(\alpha)} \quad , \quad s_{\underline{i}}^{(\alpha)} = 0 \tag{4.5}$$

and the value for $r_1^{(\alpha)}$ is known in the literature (the tetrad for diagonal metric tensors, such as the Schwarzschild and Reissner - Nordström metrics in polar coordinates has a form well known in the literature). The simple form assumed by the tetrad $s_\mu^{(\alpha)}$ for this particular solution (Eqs. (4.6)) holds only for a partial fixation of the orientation of the tetrad directions which corresponds to take in the transformation equations (2.24) and (2.25) the condition $E^{(\alpha)}{}_{(\beta)} = 0$. In this case it is possible to impose that the $s_\mu^{(\alpha)}$ are proportional to the $r_\mu^{(\alpha)}$, namely $s_\mu^{(\alpha)} = \lambda_\mu{}^\nu\ r_\nu^{(\alpha)}$ which has the form of the solutions (4.6) for

$$\lambda_4{}^1 = - \frac{Kq}{\alpha r^2}$$

with all remaining $X_\mu^{\ \nu}$ equal to zero. Geometrically this condition means that the internal tetrad space has dimension 4 for this chosen orientation of the complex tetrad $k_\mu^{(\alpha)}$, wherear the dimensionallity of this internal space is 8 for arbitrary orientations of the complex tetrad.

5. ALGEBRA OF THE MATRIX TENSOR FIELD $G_{\mu\nu}$

Given the matrices $G_{\mu\nu}$ defined by the coefficients (1.4) we define $G^{\mu\nu}$ by

$$G_{\sigma\nu} \cdot G^{\mu\nu} = G^{\nu\mu} \cdot G_{\nu\sigma} = \delta^\mu_\sigma \, \omega_4 \tag{5.1}$$

The matrices $G_{\mu\nu}$ and $G^{\mu\nu}$ are used to lower and raise indices in the unified matrix theory. Defining

$$U_\lambda = U^\alpha \cdot G_{\alpha\lambda} = U^\alpha \cdot G^\dagger_{\lambda\alpha} \tag{5.2}$$

we get from (5.1)

$$U^\lambda = U_\alpha \cdot G^{\lambda\alpha} = U_\alpha \cdot G^{\dagger\alpha\lambda} \tag{5.3}$$

in general the U_α and U^α are four 2x2 matrices. Similarly

$$U^\dagger_\lambda = G_{\lambda\alpha} \cdot U^{\dagger\lambda} \quad , \quad U^{\dagger\lambda} = G^{\alpha\lambda} \cdot U^\dagger_\alpha$$

By analogy with the choice (1.4) the matrices $G^{\mu\nu}$ may be written as

$$G^{\mu\nu} = a \, g^{\mu\nu}\omega_4 + i \, K \, F^{\mu\nu}\omega_4 + i \, K\mu \vec{f}^{\mu\nu} \cdot \vec{\omega} \tag{5.4}$$

We note that $\vec{f}^{\mu\nu}$ and $F^{\mu\nu}$ are defined by (5.4) and not by raising the indices of $\vec{f}_{\mu\nu}$ and $F_{\mu\nu}$ with the help of the metric tensor $g^{\mu\nu}$. These two definitions become identical in the limit $K \to 0$. In this limit (1.4), (5.1) and (5.4) imply in the usual definition used in general relativity if the constant <u>a</u> is taken equal to 2.

Associated to the matrices $G^{\mu\nu}$ we define vierbeins $H^\mu_{(\alpha)}$ such

$$H^{(\alpha)}_\nu \cdot H^{\dagger\nu}_{(\beta)} = \delta^{(\alpha)}_{\ \ (\beta)}\omega_4 \; , \quad H^{\dagger(\alpha)}_\mu \cdot H^\nu_{(\alpha)} = \delta^\nu_\mu \omega_4 \tag{5.5}$$

Writing $H_{(a)} = \rho^{\mu}_{(\alpha)i}\omega_i$ with $\rho^{\mu}_{(\alpha)4} = \sqrt{2}\, k^{\mu}_{(\alpha)}$, we obtain in the limit $K \to 0$:

$$H^{\mu}_{(\alpha)} \to \sqrt{2}\ \mathrm{Re}\ k^{\mu}_{(\alpha)}\omega_4 = \sqrt{2}\, r^{\mu}_{(\alpha)}\omega_4$$

$$H^{(\alpha)}_{\mu} \to \frac{1}{\sqrt{2}}\, r^{(\alpha)}_{\mu}\,\omega_4 \ .$$

Thus, the equations (5.5) in this limit become the usual orthonormallity conditions for the tetrad in general relativity. With the help of (5.5) we may write the expression of $G^{\mu\nu}$ in terms of vierbeins as

$$G^{\mu\nu} = H^{\dagger\nu}_{(\alpha)} \cdot H^{\mu}_{(\beta)}\ \eta^{\alpha\beta}$$

It is also possible to introduce another process of raising and loweririg indices with the use of the matrix tensors $G^{(\mu\nu)}$ and $G_{(\mu\nu)}$, where

$$G_{(\mu\nu)} = 1/2\ g_{\mu\nu}\ \omega_4 \qquad , \qquad G^{(\mu\nu)} = 2\ g^{\mu\nu}\ \omega_4$$

For general matrices of the form $U_{\alpha\ldots\lambda\ldots\gamma}$ and $V^{\alpha\ldots\lambda\ldots\gamma}$ the raising and lowering of the index λ is indicated by placing a dot over the new index

$$U_{\alpha\ldots\lambda\ldots\gamma}\, G^{(\lambda\beta)} = U_{\alpha\ldots}{}^{\dot\beta}{}_{\ldots\gamma}$$

$$V^{\alpha\ldots\lambda\ldots\gamma}\, G_{(\lambda\beta)} = V^{\alpha\ldots}{}_{\dot\beta}{}^{\ldots\gamma}$$

In particular, the double application of this operation generates an index without dot, as for instance $U_{\beta} = U^{\dot\alpha}\, G_{(\alpha\beta)}$. This process of lowering and raising indices is similar to the process used by Einstein in his non-symmetric unitary theory, the only difference is that here we deal with matrices instead of functions. Thus, we may write

$$G_{[\mu\nu]} \cdot G^{(\nu\alpha)} = G_{\mu}{}^{\dot\alpha}$$

which gives

$$i\ K\ g^{\nu\alpha}\ F_{\mu\nu}\ \omega_4 + i\ K\mu\ \vec{f}_{\mu\nu}\cdot\omega_4\ g^{\nu\alpha} = G_{\mu}{}^{\dot\alpha} \qquad (5.6)$$

It is possible to extend to matrices the concept of eigenvectors associated to arbitrary tensors, namely the extension of equations of the type $P^{\mu\nu} v_\nu = h\, v_\mu$. Here we are directly interested in the case where $P^{\mu\nu}$ corresponds to the skew symmetric part of the matrix tensor $G_{\mu\nu}$ with the index ν raised by the previous convention. Consider the equation

$$\mathrm{Tr}(G_\mu{}^{\dot{\alpha}} \lambda^\mu - \rho \lambda^\alpha) = 0 \tag{5.7}$$

where

$$G_\mu{}^{\dot{\alpha}} = q_{\mu i}^{\dot{\alpha}} \, \omega_i \quad , \quad \rho = \rho_i \, \omega_i \quad , \quad \lambda^\mu = \tau_i^\mu \, \omega_i = \phi^\mu \, \omega_4 + \vec{\psi}^\mu \cdot \vec{\omega}$$

It can be shown from (5.6) and (5.7) and from the assumption that the eigenvectors ϕ^μ and $\vec{\psi}^\mu$ are linearly independent that

$$i\, K\, F_\mu{}^{\dot{\alpha}} \phi^\mu = \rho_4 \, \phi^\alpha \quad , \quad F_\mu{}^{\dot{\alpha}} = F_{\mu\nu}\, g^{\nu\alpha}$$

$$2i\, K\, \mu \vec{f}_\mu{}^{\dot{\alpha}} \cdot \vec{\psi}^\mu = \vec{\rho} \cdot \vec{\psi}^\alpha \quad , \quad \vec{f}_\mu^{\dot{\alpha}} = \vec{f}_{\mu\nu}\, g^{\nu\alpha}$$

These equations are the eigenvalue equations for the Maxwell and Yang-Mills field tensors (redefining $\rho_4 \to iK\alpha_4$, $\vec{\rho} \to 2iK\mu\vec{\alpha}$). The last equation is the sum of three equations which correspond to the individual eigenvalue equations for each isotopic component of the Yang-Mills field tensor. As it is clear, all properties concerning the structure of the eigenvalues and eigenvectors of antisymmetric tensors hold here. We just write some of these properties[8]:

1) If $\phi^\mu(\vec{\psi}^\mu)$ is an eigenvector of $F_\mu{}^{\dot{\alpha}}(\vec{f}_\mu{}^{\dot{\alpha}})$ corresponding to the eigenvalue $\rho_4 \neq 0 (\vec{\rho} \neq 0)$ then $\phi^\mu(\vec{\psi}^\mu)$ is a null vector for the metric $g_{\mu\nu}$.

2) Let ϕ_1^μ and ϕ_2^μ (ψ_1^μ and ψ_2^μ) be two eigenvectors of $F_\mu{}^{\dot{\alpha}}(\vec{f}_\mu{}^{\dot{\alpha}})$ and ${}_{(1)}\rho_4$, ${}_{(2)}\rho_4$ ($\vec{\rho}_{(1)}$, $\vec{\rho}_{(2)}$) the corresponding eigenvalues such that ${}_{(2)}\rho_4 \neq -{}_{(1)}\rho_4$ ($\vec{\rho}_{(2)} \neq -\vec{\rho}_{(1)}$). Then ϕ_1^μ and ϕ_2^μ (ψ_1^μ and ψ_2^μ) are orthogonal for the metric $g_{\mu\nu}$.

3) For each one of the above eigenvalue equations if two roots are non null then it will exist two linearly independent eigenvectors.

4) If all roots vanish (radiation field) there will be a one-parameter set of eigenvectors associated to the null eigenvalue.

It is also possible to introduce the analog of a scalar product in terms of matrices. Defining the real quantity

$$v^2 = \mathrm{Tr}\ (V^\mu \cdot G_{\mu\nu} \cdot V^{\dagger\nu}) = \mathrm{Tr}(V^\mu \cdot V^\dagger_\mu)$$

we find by a direct calculation, taking $V = v^\mu{}_i \omega_i$, $V^{\dagger\mu} = v^*{}_i\ \omega_i$

$$v^2 = -2K\mu\vec{v}^{*\alpha} \cdot (\vec{v}^\beta \times \vec{f}_{\beta\alpha}) + \vec{v}^\beta \cdot \vec{v}^{*\alpha} h_{\beta\alpha} + 2iK\mu v_4^{*\alpha} \vec{v}^\beta \cdot \vec{f}_{\beta\alpha}$$

$$+ 2iK\mu v_4^\beta \vec{v}^{*\alpha} \cdot \vec{f}_{\beta\alpha} + v_4^\alpha v_4^{*\beta} h_{\alpha\beta}$$

If $\vec{v}^\alpha = 0$ this expression degenerates in the law of scalar product of the Moffat-Boal theory. For $K \to 0$ it goes over the usual definitions of scalar product in a Riemannian spacetime. The expression for v^2 can also be written under the form

$$v^2 = v_i^\beta\ \ v^{*\alpha}_k\ p_{\beta\alpha ik}\ \ ,\ \ p_{\beta\alpha ik} = 1/2(1_{ijs} + f_{ijs}) q_{\beta\alpha j}\ f_{sk4}$$

With the matrices $G_{\mu\nu}$ it is possible to construct c-number functions by two different ways: by taking trace of the matrices or by indroducing the functions

$$\bar{G}_{\mu\nu} = \frac{\psi^\dagger\ G_{\mu\nu}\ \psi}{\psi^\dagger \psi}$$

where ψ is some given field with isotopic spin 1/2. Then

$$\bar{G}_{\mu\nu} = \bar{G}_{[\mu\nu]} + \bar{G}_{(\mu\nu)}$$

with

$$\bar{G}_{(\mu\nu)} = 1/2\ g_{\mu\nu}\ \ ,\ \ -i\bar{G}_{[\mu\nu]} = K/2\ F_{\mu\nu} + K\mu\vec{f}_{\mu\nu} \cdot \vec{\omega}$$

With the use of these functions we may define the square of the line element either by

$$ds^2 = \bar{G}_{\mu\nu} dx^\mu dx^\nu = \bar{G}_{(\mu\nu)} dx^\mu dx^\nu$$

or by

$$ds^2 = \mathrm{Tr}(G_{\mu\nu})\ dx^\mu\ dx^\nu$$

(the coordinates are c-numbers). These two definitions of the square of the line element differ only by a factor 1/2.

6. A GENERALIZATION OF THE MATRIX TENSOR FIELD $G_{\mu\nu}$

The matrix tensor field considered in the previous sections corresponds algebraically to a quaternion field. From the previous definition of $G_{\mu\nu}$ we have

$$G_{\mu\nu} = \vec{q}_{\mu\nu} \cdot \vec{\omega} + q_{\mu\nu 4}\, \omega_4 = i\, \vec{q}_{\mu\nu} \cdot \vec{\omega} + q_{\mu\nu 4}\, \beta_4$$

with $\vec{\beta} = 1/i\, \vec{\omega}$, $\beta_4 = \omega_4$. The set of 2x2 matrices $\vec{\beta}$ and β_4 satisfy the quaternion multiplication law

$$\beta_{\underline{i}}\, \beta_{\underline{j}} = -\delta_{\underline{ij}}\, \beta_4 + \varepsilon_{\underline{ijk}}\, \beta_{\underline{k}} \qquad (6.1)$$

$$\beta_{\underline{i}}\, \beta_4 = \beta_4\, \beta_{\underline{i}} = \beta_{\underline{i}}$$

In what follows it will be of interest to rewrite this multiplication law in the form

$$\beta_{\underline{i}}\, \beta_{\underline{j}} = -\beta_{\underline{i}} * \beta_{\underline{j}} + \beta_{\underline{i}} \wedge \beta_{\underline{j}} \qquad (6.2)$$

where the operations indicated by the symbols $*$ and $\wedge$ (scalar and wedge product of the quaternion basis) are given by

$$\beta_{\underline{i}} * \beta_{\underline{j}} = -1/2\, \{\beta_{\underline{i}}, \beta_{\underline{j}}\} = \delta_{\underline{ij}}\, \beta_4 \qquad (6.3)$$

$$\beta_{\underline{i}} \wedge \beta_{\underline{j}} = 1/2\, [\beta_{\underline{i}}, \beta_{\underline{j}}] = \varepsilon_{\underline{ijk}}\, \beta_{\underline{k}}$$

In this notation the equation (5.1) which defines the matrix $G^{\mu\nu}$ takes the form

$$q_{\sigma\nu\underline{i}}\, s^{\mu\nu}_{\underline{j}}\, \beta_{\underline{i}} * \beta_{\underline{j}} + q_{\sigma\nu 4}\, s^{\mu\nu}_4\, \beta_4 = \delta^{\mu}_{\sigma}\, \beta_4$$

$$-q_{\sigma\nu\underline{i}}\, s^{\mu\nu}_{\underline{j}}\, \beta_{\underline{i}} \wedge \beta_{\underline{j}} + i(q_{\sigma\nu\underline{i}}\, s^{\mu\nu}_4 + q_{\sigma\nu 4}\, s^{\mu\nu}_{\underline{i}})\, \beta_{\underline{i}} = 0$$

where $G^{\mu\nu} = s^{\mu\nu}_{\underline{i}}\, \omega_{\underline{i}}$.

A generalization of the quaternion algebra is given by the algebra of the complex Cayley numbers (Octonions) which are defined by[9]

$$A = a\, u_4^* + b\, u_4 + x_{\underline{i}}\, u_{\underline{i}}^* + y_{\underline{i}}\, u_{\underline{i}} \qquad (6.5)$$

where $(u_4, u_{\underline{i}})$ are the basis **elements** of the algebra satisfying the multiplication **table**

$$u_{\underline{i}}^* u_{\underline{j}}^* = \epsilon_{\underline{ijk}}\, u_{\underline{k}} \;,\; u_{\underline{i}} u_{\underline{j}}^* = -\delta_{\underline{ij}}\, u_4 \;,\; u_{\underline{i}} u_4 = 0$$

$$u_{\underline{i}} u_4^* = u_{\underline{i}} \;,\; u_4 u_{\underline{i}} = u_{\underline{i}} \;,\; u_4^* u_{\underline{i}} = 0 \,,\; u_4 u_4 = u_4 \;,\; u_4 u_4^* = 0 \;. \qquad (6.6)$$

In general the complex Cayley algebra contains seven quaternion subalgebras. This property follows from (6.6) and from the definition $u_4 = 1/2(\beta_4 + i\beta_7)$, $u_{\underline{i}} = 1/2\,(\beta_{\underline{i}} + i\beta_{\underline{i}+3})$. These seven quaternion subalgebras correspond to the variation of the index $A=1\ldots7$ over the triads (1,2,3), (5,1,6), (6,2,4), (4,3,5), (6,7,3), (4,7,1) and (5,7,2).

The algebra of the complex Cayley numbers may be represented in terms of Zorn matrices[10] defined by[11]

$$Z(A) = aZ(u_4^*) + bZ(u_4) + x_{\underline{i}} Z(u_{\underline{i}}^*) + y_{\underline{i}} Z(u_{\underline{i}}) \qquad (6.7)$$

with

$$Z(u_4^*) = \begin{pmatrix} \beta_4 & 0 \\ 0 & 0 \end{pmatrix} \;,\quad Z(u_4) = \begin{pmatrix} 0 & 0 \\ 0 & \beta_4 \end{pmatrix}$$

$$Z(u_{\underline{i}}) = \begin{pmatrix} 0 & 0 \\ \beta_{\underline{i}} & 0 \end{pmatrix} \;,\quad Z(u_{\underline{i}}^*) = \begin{pmatrix} 0 & -\beta_{\underline{i}} \\ 0 & 0 \end{pmatrix} \qquad (6.8)$$

giving

$$Z(A) = \begin{pmatrix} a\,\beta_4 & -x_{\underline{i}}\,\beta_{\underline{i}} \\ y_{\underline{i}}\,\beta_{\underline{i}} & b\,\beta_4 \end{pmatrix} \qquad (6.9)$$

The product of *matrices* of this form is defined in such waythat it reproduces the multiplication table (6.6) of the complex Cayley basis. Taking another Zorn matrix $Z(B)$ similar to (6.9) with the replacements, $a\to c$, $b\to d$, $x\to z$, $y\to w$, we indicate this product with the symbol Θ :

$$Z(A) \odot Z(B) = \begin{pmatrix} ac - x * w & -az - dx - y \wedge w \\ cy + bw + x \wedge z & bd - y * z \end{pmatrix} \quad (6.10)$$

where for short we have written $x = x_{\underline{i}}\, \beta_{\underline{i}}$, $a = a\, \beta_4$ with the same abbreviation for y, z and w and for b, c and d.

The conjugation operation in the Zorn representation of octonions os defined by the replacements: $Z(u_4) \leftrightarrow Z(u_4^*)$, $Z(u_{\underline{i}}) \to -Z(u_{\underline{i}})$, $Z(u_{\underline{i}}^*) \to -Z(u_{\underline{i}}^*)$. Thus, in (6.9) we have

$$Z(A) \to Z(\bar{A}) = \begin{pmatrix} b\, \beta_4 & x_{\underline{i}}\, \beta_{\underline{i}} \\ -y_{\underline{i}} \beta_{\underline{i}} & a\, \beta_4 \end{pmatrix} \quad (6.11)$$

The norm of a octonion is defined as

$$N(A) = Z(A) \odot Z(\bar{A})$$

From (6.10) it follows that

$$N(A) = (ab + x_{\underline{i}} y_{\underline{i}})\, \mathbb{1}$$

where $\mathbb{1} = Z(u_4) + Z(u_4^*)$ is the identity element of Zorn's algebra.

The octonion formalism in the Zorn representation is now applied as a generalization of the quaternion formalism used in the previous sections. The quaternion tetrad *(2.2)* is generalized to a octonion tetrad of the form

$$H_\mu^{(\alpha)} = \begin{pmatrix} ih_{\mu 4}^{(\alpha)}\, \beta_4 & -h_{\mu \underline{i}}^{(\alpha)}\, \beta_{\underline{i}} \\ k_{\mu \underline{i}}^{(\alpha)}\, \beta_{\underline{i}} & ik_{\mu 4}^{(\alpha)}\, \beta_4 \end{pmatrix} = \begin{pmatrix} a_\mu^{(\alpha)} & -x_\mu^{(\alpha)} \\ y_\mu^{(\alpha)} & b_\mu^{(\alpha)} \end{pmatrix} \quad (6.12)$$

By convenience we have used the diagonal matrix elements in (6.12) with a multiplicative factor $\underline{i}$. Presently this modification is not relevant since all matrix elements are complex guantities. We also define the octonion Hermitian conjugation operation as

$$H_\mu^{\dagger(\alpha)} = \begin{pmatrix} b_\mu^{\dagger(\alpha)} & y_\mu^{\dagger(\alpha)} \\ -x_\mu^{\dagger(\alpha)} & a_\mu^{\dagger(\alpha)} \end{pmatrix} \tag{6.13}$$

The 2x2 matrices β_4 and β_i satisfy the properties $\beta_4^\dagger = \beta_4$, $\beta_i^\dagger = -\beta_i$, with the $\dagger$ operation having the usual meaning for matrices, thus

$$b_\mu^{\dagger(\alpha)} = -ik_{\mu 4}^{(\alpha)*}\ \beta_4$$

with similar result for the remainder *matrix elements* in (6.13).

A straightforward calculation using (6.10), (6.12) and (6.13) gives

$$G_{\mu\nu} = H_\mu^{\dagger\ \alpha} \odot \bar{H}_\nu^{\ \beta}\ \eta_{\alpha\beta} = \begin{pmatrix} S_{\mu\nu} & -P_{\mu\nu} \\ Q_{\mu\nu} & R_{\mu\nu} \end{pmatrix} \tag{6.14}$$

where

$$\begin{aligned}
S_{\mu\nu} &= (k_{\mu 4}^{(\alpha)*}\ k_{\nu 4}^{(\beta)} + \vec{k}_\mu^{(\alpha)*} \cdot \vec{k}_\nu^{(\beta)})\eta_{\alpha\beta}\ \omega_4 = s_{\mu\nu 4}\ \beta_4 \\
P_{\mu\nu} &= (h_{\nu 4}^{(\beta)}\ \vec{k}_\mu^{(\alpha)*} + k_{\mu 4}^{(\alpha)*}\ \vec{h}_\nu^{(\beta)} + i\vec{h}_\mu^{(\alpha)*} \times \vec{k}_\nu^{(\beta)})\eta_{\alpha\beta}\cdot\vec{\omega} = \vec{s}_{\mu\nu}\cdot\vec{\beta} \\
R_{\mu\nu} &= (h_{\mu 4}^{(\alpha)*}\ h_{\nu 4}^{(\beta)} + \vec{h}_\mu^{(\alpha)*} \cdot \vec{h}_\nu^{(\beta)})\eta_{\alpha\beta}\ \omega_4 = r_{\mu\nu 4}\ \beta_4 \\
Q_{\mu\nu} &= (k_{\nu 4}^{(\beta)}\ \vec{h}_\mu^{(\alpha)*} + h_{\mu 4}^{(\alpha)*}\ \vec{k}_\nu^{(\beta)} + i\vec{k}_\mu^{(\alpha)*} \times \vec{h}_\nu^{(\beta)})\eta_{\alpha\beta}\cdot\vec{\omega} = \vec{r}_{\mu\nu}\cdot\vec{\beta}
\end{aligned} \tag{6.15}$$

The equation (6.14) generalizes the quartenion matrix representation given by (2.1). Thus, we may call the $G_{\mu\nu}$ of (6.14) as aoctonion representation of the *metric*. In the limit where $h \to k$ we reobtain theprevious expression (2.14), (2.15). The $G_{\mu\nu}$ in this limit takes the form

$$G_{\mu\nu} = \begin{bmatrix} q_{\mu\nu 4}\ \omega_4 & -\vec{q}_{\mu\nu}\cdot\vec{\omega} \\ \vec{q}_{\mu\nu}\cdot\vec{\omega} & q_{\mu\nu 4}\ \omega_4 \end{bmatrix}$$

Using (6.15) we rewrite (6.14) as

$$G_{\mu\nu} = \begin{bmatrix} s_{\mu\nu 4}\ \omega_4 & -\vec{s}_{\mu\nu}\cdot\vec{\beta} \\ \vec{r}_{\mu\nu}\cdot\vec{\beta} & r_{\mu\nu 4}\ \beta_4 \end{bmatrix} = G_{\mu\nu}\ (s;r\ )$$

Taking The Hermitian conjugate we get

$$G^{\dagger}_{\mu\nu} = \begin{bmatrix} r^{*}_{\mu\nu 4}\ \beta_4 & -r^{*}_{\mu\nu}\cdot\vec{\beta} \\ \vec{s}^{*}_{\mu\nu}\cdot\vec{\beta} & s^{*}_{\mu\nu 4}\ \beta_4 \end{bmatrix} = G^{*}_{\mu\nu}\ (r;s)$$

Considering, as before, that the fields $\underline{r}$ and $\underline{s}$ are Hermitian with respect to the world indices, we find

$$G^{\dagger}_{\mu\nu}(s;r) = G_{\nu\mu}(r;s)$$

This symmetry condition here generalizes the Hermitian symmetry condition (1.5) for quartenions (rnatrices which are linear combination of the ω_i). In the limit s $\to r$ we recover the quartenion symmetry condition $G^{\dagger}_{\mu\nu} = G_{\nu\mu}$.

We mention that derivatives (or covariant derivatives) may be defined by the operator $D_\mu = \mathbb{1}\ \partial_\mu$ ($\mathbb{D}_\mu = \mathbb{1} \odot \mathcal{D}_\mu$ for covariant derivatives). Thus, according to the Einstein definition of covariant derivative in the nonsymmetric theory we may write

$$\mathbb{D}_\mu \odot G_{\alpha\beta} = \partial_\mu\ G_{\alpha\mu} - \Gamma^{\lambda}_{\alpha\mu} \odot G_{\lambda\beta} - \Gamma^{\lambda}_{\mu\beta} \odot G_{\alpha\lambda} = (\mathcal{D}_\mu)^{\rho\sigma}{}_{\alpha\beta} \odot G_{\rho\sigma}$$

where

$$(\mathcal{D}_\mu)^{\rho\sigma}{}_{\alpha\beta} = \mathbb{1}\ \partial_\mu\ \delta^{\rho}_{\alpha}\ \delta^{\sigma}_{\beta} - \Gamma^{\lambda}_{\alpha\mu}\ \delta^{\rho}_{\lambda}\ \delta^{\sigma}_{\beta} - \Gamma^{\lambda}_{\mu\nu}\ \delta^{\rho}_{\alpha}\ \delta^{\sigma}_{\lambda}$$

The definition of the operator $\mathcal{D}_\mu$ depends on the space where $\mathcal{D}_\mu$ operates. In general we can write $\nabla_{\mu} = (\mathcal{D}_\mu)^A{}_B$ where the indices A, B are to be taken as world indices, spinor indices, isotopic spinor indices, etc, depending of the object which is considered in this derivative. In the above example $A \rightarrow \rho,\sigma$; $B \rightarrow \alpha,\beta$. In the foilowing we consider two possible applications of this formalism.

(i) Consider the situation where

$$\vec{h}_\mu^{(\alpha)} = 0 \quad , \; h_{\mu 4}^{(\alpha)} = \sqrt{\phi}\, k_{\mu 4}^{(\alpha)} \quad , \; \phi^* = \phi$$

with $k_{\mu 4}^{(\alpha)}$ and $\vec{k}_\mu^{(\alpha)}$ describing a guaternion of the unified matrix theory of Borchsenius. Then, we have for (6.14) and (6.15):

$$G_{\mu\nu} = \begin{pmatrix} s_{\mu\nu 4}\ \beta_4 & -\sqrt{\phi}\ k_{\nu 4}^{(\alpha)}\ k_\mu^{*(\beta)} \cdot \vec{\beta}\ \eta_{\alpha\beta} \\ \sqrt{\phi}\ k_{\mu 4}^{*(\alpha)}\ \vec{k}_\nu^{(\beta)} \cdot \vec{\beta}\ \eta_{\alpha\beta} & \phi\ k_{\mu 4}^{*(\alpha)}\ k_{\nu 4}^{(\beta)}\ \eta_{\alpha\beta}\ \beta_4 \end{pmatrix}.$$

with

$$s_{\mu\nu 4} = 1/2\ h_{\mu\nu} = 1/2\ (g_{\mu\nu} + iK\ F_{\mu\nu}) = (k_{\mu 4}^{*(\alpha)}\ k_{\nu 4}^{(\beta)} + \vec{k}_\mu^{*(\alpha)} \cdot \vec{k}_\nu^{(\beta)})\,\eta_{\alpha\beta}$$

$$\vec{s}_{\mu\nu} = \sqrt{\phi}\ k_{\nu 4}^{(\beta)}\ \vec{k}_\mu^{*(\alpha)}\,\eta_{\alpha\beta} = iK\mu\ \vec{f}_{\mu\nu}$$

$$r_{\mu\nu 4} = \phi\ k_{\mu 4}^{*(\alpha)}\ k_\nu^{(\beta)}\ \eta_{\alpha\beta} \quad , \quad \vec{r}_{\mu\nu} = \sqrt{\phi}\ k_{\mu 4}^{*(\alpha)}\ \vec{k}_\nu^{(\beta)}\ \eta_{\alpha\beta}$$

Taking

$$k_\mu^{(\alpha)} = \frac{1}{\sqrt{2}}\, m_\mu^{(\alpha)} \quad , \quad \vec{k}_\mu^{(\alpha)} = \frac{1}{\sqrt{2}}\, \vec{n}_\mu^{(\alpha)} \quad ,$$

$$m_\mu^{(\alpha)} = r_\mu^{(\alpha)} + i s_\mu^{(\alpha)} \quad , \; \vec{n}_\mu^{(\alpha)} = \vec{\theta}_\mu^{(\alpha)} + i \vec{\pi}_\mu^{(\alpha)}$$

we get

$$g_{\mu\nu} = (r_\mu^{(\alpha)}\ r_\nu^{(\beta)} + s_\mu^{(\alpha)}\ s_\nu^{(\beta)} + \vec{\theta}_\mu^{(\alpha)}\cdot\vec{\theta}_\nu^{(\beta)} + \vec{\pi}_\mu^{(\alpha)}\cdot\vec{\pi}_\nu^{(\beta)})\eta_{\alpha\beta}$$

$$K\ F_{\mu\nu} = (r_\mu^{(\beta)}\ s_\nu^{(\alpha)} - s_\mu^{(\alpha)}\ r_\nu^{(\beta)} + \vec{\pi}_\nu^{(\beta)}\cdot\vec{\theta}_\mu^{(\alpha)} - \vec{\pi}_\mu^{(\beta)}\cdot\vec{\theta}_\nu^{(\alpha)})\eta_{\alpha\beta}$$

$$\mu K\ f_{\mu\nu} = \frac{\phi}{4}\ (s_{[\nu}^{(\beta)}\ \vec{\theta}_{\mu]}^{(\alpha)} - r_{[\nu}^{(\beta)}\ \vec{\pi}_{\mu]}^{(\alpha)})\eta_{\alpha\beta}$$

$$0 = (r_\nu^{(\beta)}\ \vec{\theta}_\mu^{(\alpha)} + s_\nu^{(\beta)}\ \vec{\pi}_\mu^{(\alpha)})\eta_{\alpha\beta}$$

$$\mathrm{Re}\ r_{\mu\nu 4} = \frac{\phi}{2}\ (r_\mu^{(\alpha)}\ r_\nu^{(\beta)} + s_\mu^{(\alpha)} s_\nu^{(\beta)})\eta_{\alpha\beta}, \quad \mathrm{Im}\ r_{\mu\nu 4} = \frac{\phi}{2}\ (r_\mu^{(\alpha)} s_\nu^{(\beta)} - r_\nu^{(\alpha)} s_\mu^{(\beta)})\eta_{\alpha\beta}$$

$$\mathrm{Re}\ \vec{r}_{\mu\nu} = \frac{\sqrt{\phi}}{2}\ (r_\mu^{(\alpha)}\vec{\theta}_\nu^{(\beta)} + s_\mu^{(\alpha)}\vec{\pi}_\nu^{(\beta)})\eta_{\alpha\beta}, \quad \mathrm{Im}\ \vec{r}_{\mu\nu} = \frac{\sqrt{\phi}}{2}(r_\mu^{(\alpha)}\vec{\pi}_\nu^{(\beta)} - s_\mu^{(\alpha)}\vec{\theta}_\nu^{(\beta)})\eta_{\alpha\beta}$$

The only simple possibility for interpreting these formulas is to take $\vec{\theta}_\mu^{(\alpha)} = 0$, $\vec{\pi}_\mu^{(\alpha)} = 0$ (there is no Yang-Mills field) which gives for $g_{\mu\nu}$ and $F_{\mu\nu}$ the values of the Moffat theory. For $r_{\mu\nu 4}$ we find

$$\mathrm{Re}\ r_{\mu\nu 4} = \frac{\phi}{2}\ g_{\mu\nu} \quad \text{(conformal to the Moffat metric)},$$

$$\mathrm{Im}\ r_{\mu\nu 4} = \frac{\phi}{2}\ K\ F_{\mu\nu} \quad \text{("conformal" to } F_{\mu\nu}\text{)} .$$

Combining these results we get

$$r_{\mu\nu 4} = \frac{\phi}{2}\ h_{\mu\nu} \quad \text{(conformal to the metric } h_{\mu\nu}\text{)} .$$

The octonion $G_{\mu\nu}$ in this case takes the simple form

$$\begin{pmatrix} 1/2\ (g_{\mu\nu} + iK\ F_{\mu\nu})\beta_4 & 0 \\ 0 & \phi/2\ (g_{\mu\nu} + iK\ F_{\mu\nu})\beta_4 \end{pmatrix}$$

(ii) Taking $\vec{h}_\mu^{(\alpha)} = \sqrt{\phi}\ \vec{k}_\mu^{(\alpha)}$, $h_{\mu 4}^{(\alpha)} = \sqrt{\phi}\ k_{\mu 4}^{(\alpha)}$, $\phi^* = \phi$, with $k_{\mu 4}^{(\alpha)}$, $\vec{k}_\mu^{(\alpha)}$ describing a quartenion similarly to the case (i). We find in this case

$$G_{\mu\nu} = \begin{pmatrix} 1/2\ (g_{\mu\nu} + iK\ F_{\mu\nu})\beta_4 & -i\ \sqrt{\phi}\ K\ \mu \vec{f}_{\mu\nu}\cdot\vec{\beta} \\ i\ \sqrt{\phi}\ K\ \mu\ \vec{f}_{\mu\nu}\cdot\vec{\beta} & 1/2\ \ (g\ \ \ + iKF_{\mu\nu})\beta_4 \end{pmatrix}$$

where

$$s_{\mu\nu4} = 1/2\ h_{\mu\nu} = (k_{\mu4}^{(\alpha)}\ k_{\nu4}^{(\beta)} + \vec{k}_{\mu}^{(\alpha)} \cdot \vec{k}_{\nu}^{(\beta)})\eta_{\alpha\beta}$$

$$\text{Im}\ \vec{s}_{\mu\nu} = \sqrt{\phi}\ K\ \mu\ \vec{f}_{\mu\nu}$$

with the explicit value for $\text{Im}\ \vec{s}_{\mu\nu}$ given by Eq.(2.19). For $r_{\mu\nu4}$ and $\vec{r}_{\mu\nu}$ we have

$$r_{\mu\nu4} = \phi\ s_{\mu\nu4} = \frac{1}{2}\ h_{\mu\nu}\ \phi$$

$$\vec{r}_{\mu\nu} = i\ \sqrt{\phi}\ K\ \mu\ \vec{f}_{\mu\nu}$$

again $K\mu\vec{f}_{\mu\nu}$ in this last equation is given by (2.19). In the limit $\phi\to 1$ the octionion $G_{\mu\nu}$ of this example tends to the quaternion of the Borchsenius theory.

6. CONCLUSION

The vierbein formalism associated to the matrix extension of the Moffat-Boal unified theory is determined. The Hermitian symmetry condition of the non-symmetric tensor $h_{\mu\nu}$ of the Moffat-Boal theory is extended to the matrix formalism by the condition $G^{\dagger}_{\mu\nu} = G_{\nu\mu}$. In the limit where a fundamental constant $K = L^2/e$ tends to zero only gravitation remainsdescribed in terms of vierbeins, and the EMYM (Einstein-Maxwell Yang-Mills) theory is obtained from the generalized field equation of the matrix theory.

The generalized γ-*matrices* which correspond to the extended matrix-tetrads are given. The anticommutation relations of these γ-*matri-*

ces are determined and generalize the usual anticommutation relations used in general relativity. The introduction of generalized *γ-matrices* suggests the possibility of considering a modified equation of motion for a system with spin 1/2 and isotopic spin 1/2 in the extended matrix theory, similarly as the Dirac equation is modified in presence of gravitation in general relativity.

The vierbein associated to the exact spherically symmetric solution of the unified field equation without Yang-Mills field is derived. Some algebraic properties of, the *metric* of the unified matrix formulation are discussed and the *orthogonallity* conditions of the matrix-tetrad are determined. A generalization of the quaternion-matrix theory of Borchsenius involving octonions is proposed.

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