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The Green's Function for Jacobi Special Functions

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The Green's function for the Jacobi differential equation is calculated hy means of the Sturm-Liouville method. As by -product we obtain Legendre, Gegenbauer and Tchebichef Green's functions.

Calcula-se a função de Green para a equação diferencial de Jacobi pelo método de Sturm-Liouville. Como casos particulares obtém-se também as funções de Green para as equações diferenciais de Legendre, Gegenbauer e Tchebichef.

1. INTRODUCTION

The present paper contains a sistematic calculation of the Sturm-Liouville expansion of the Green's function for the differential equation for a number of special functions; a common feature for all of them is that the corresponding differential equation can be derive from the Jacobi or hypergeometric differential equatiori. These include Legendre, Gegenbauer and Tchebichef functions.

In the first section we calculate the Green's functions for the Jacobi differential equation by the Sturm-Liouville methodand for define general spherical harmonics in order to simplify the expression for the Green's function.

In the second section we derive from the Jacobi Green's function particular cases, as Legendre, Gegenbauer and Tchebichef Green's function, and in the third section we present our discussions.

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2. JACOBI GREEN'S FUNCTION

The Jacobi differential operator[1] can be written in the following way

$$L_x = (1-x^2)\,\frac{d^2}{dx^2} - 2x\,\frac{d}{dx} - \left\{\frac{(\beta+\alpha)^2+(\beta-\alpha)^2-2(\beta-\alpha)(\beta+\alpha)x}{1-x^2} - \nu(\nu+1)\right\} \qquad (1)$$

where the convenient choice of combinations $\beta+\alpha$ and $\beta-\alpha$ for parameters of the differential operator is only introduced to simplify the notation.

In order to calculate the Green's function we first take out the weight functions $(1-x)^{\alpha}(1+x)^{\beta}$, and derive the Green's function for the following differential operator

$$L_x = (1-x^2)\,\frac{d^2}{dx^2} + \left[2(\beta-\alpha) - (2\beta+2\alpha+2)x\,\frac{d}{dx}\right] + n(n+2\alpha+2\beta+1) \qquad (2)$$

where $n = \nu-\alpha-\beta$.

The Green's function for this differential operator satisfies the following inhomogeneous differential equation

$$L_x\, g(x,\, x') = \delta(x - x') \qquad (3)$$

which is bounded on $0 \leq x < \infty$; here $\alpha > -1$ and $\beta > -1$ in order to make the weight function non-negative and integrable, but the formal relations are valid without this restriction.

The Sturm-Liouville method[2] consists to write the Green's function as the product of two linearly independent solutions of the corresponding homogeneous differential equation, $L_x\,\psi = 0$. These solutions for the Jacobi 'operator are $P_n^{(2\alpha,2\beta)}(x)$, regular at the origen and $Q_n^{(2\alpha,2\beta)}(x)$, regular at the infinity.

The solution of Eq. 3 is

$$g(x,x') = \exp[i\pi 2\alpha] 2^{-2(\alpha+\beta)} \frac{\Gamma(n+2\alpha+2\beta+1)}{\Gamma(n+2\alpha+1)\Gamma(n+2\beta+1)}$$

$$\cdot P_n^{(2\alpha,2\beta)}(x_<)\, Q_n^{(2\alpha,2\beta)}(x_<) \tag{4}$$

where $x_<$ and $x_>$ are the lesser and greater of X and x' respectively.

The Green's function for the L_x operator is easily obtained, incorporating the weight function in Eq. 4, and writing mv-a-0

$$G(x,x') = 2^{-2(\alpha+\beta)} \frac{\Gamma(\nu+\alpha+\beta+1)\Gamma(\nu-\alpha-\beta+1)}{\Gamma(\nu+\alpha-\beta+1)\Gamma(\nu+\beta-\alpha+1)} \cdot$$

$$\cdot\ (x-1)^{\alpha}(x+1)^{\beta}(x'-1)^{\alpha}(x'+1)^{\beta} P_{\nu-\alpha-\beta}^{(2\alpha,2\beta)}(x_<)\, Q_{\nu-\alpha-\beta}^{(2\alpha,2\beta)}(x_>)\ . \tag{5}$$

We simplify this expression, introducing general spherical harmonics, as used by Vilenkin[3] to analyse representations of the group $QU(2)$ of uni-modular quasi-unitary rnatrices of the second order. These functions are solutions of the homogeneous differential equation for the L_X operator and are defined in terms of Jacobi function by

$$B_\nu^{-(\alpha+\beta,\beta-\alpha)}(x) = 2^{-(\alpha+\beta)} \frac{\Gamma(\nu-\alpha-\beta+1)}{\Gamma(\nu+\alpha-\beta+1)} (x-1)^{\alpha}(x+1)^{\beta} P_{\nu-\alpha-\beta}^{(2\alpha,2\beta)}(x) \tag{6}$$

$$\kappa_\nu^{(\alpha+\beta,\beta-\alpha)}(x) = 2^{-(\alpha+\beta)} \frac{\Gamma(\nu+\alpha+\beta+1)}{\Gamma(\nu+\beta-\alpha+1)} (x-1)^{\alpha}(x+1)^{\beta} Q_{\nu-\alpha-\beta}^{(2\alpha,2\beta)}(x) \tag{7}$$

where we have used the Miller's normalization[4]. The Green's function for the L_X operator in terms of general spherical harrnonics is

$$G(x,x') = B_\nu^{(\alpha+\beta,\beta-\alpha)}(x_<)\,\kappa_\nu^{(\alpha+\beta,\beta-\alpha)}(x_>)\ . \tag{8}$$

3. PARTICULAR CASES

a) Legendre Green's function

The Green's function for the associated Legendre differential equation can be derived from the Jacobi Green's function if we put in Eq. 8 $\alpha=\beta=m/2$. Relations between Jacobi and Legendre functions can be easily obtained, expanding Jacobi functions in terms of hypergeometric functions[1]. These relations are

$$P_{\nu-m}^{(m,m)}(x) = 2^m \frac{\Gamma(\nu+1)}{\Gamma(\nu+m+1)} (x^2-1)^{-m/2} P_\nu^m(x) \tag{9}$$

$$Q_{\nu-m}^{(m,m)}(x) = \exp|im\pi] 2^m \frac{\Gamma(\nu+1)}{\Gamma(\nu-m+1)} (x^2-1)^{-m/2} Q_\nu^{-m}(x) \tag{10}$$

and the relations with general spherical harmonics are

$$B_\nu^{(m,0)}(x) = \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)} P_\nu^m(x) \tag{11}$$

$$K_\nu^{(m,0)}(x) = (-1)^m \frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)} Q_\nu^{-m}(x) \tag{12}$$

The associated Legendre Green's function is

$$G(x,x') = (-1)^m P_\nu^m(x_<) Q_\nu^{-m}(x_>) \tag{13}$$

that is the same as calculated by means of the isotropic harmonic oscillator Green's function[5]. This result can be also obtained, if we apply directly the Sturm-Liouville method to the Legendre associated differential equation, because P_ν^m and Q_ν^{-m} are two linearly independent solutions of the homogeneous differential equation, regular at the origin and at the infinity respectively.

b) Gegenbauer Green's function

Gegenbauer functions[1] are constant multiplies of Jacobi functions with $\alpha+\beta = \lambda-1/2$ and $\beta-\alpha=0$, with $\lambda > -1/2$ in order to have a real and integrable weight functions. Relations between general spherical harmonics and Gegenbauer functions can be easily obtained using relations between Gegenbauer and Legendre functions[1] and are

$$B_\nu^{(\lambda-1/2,0)}(x) = \frac{\Gamma(\nu-\lambda+3/2)}{\Gamma(\nu+\lambda+1/2)} (x^2-1)^{1/2(\lambda-1/2)} C^{\lambda-1/2}_{\nu-\lambda+1/2}(x) \qquad (14)$$

$$K_\nu^{(\lambda-1/2,0)}(x) = (x^2-1)^{1/2(\lambda-1/2)} D^{\lambda-1/2}_{\nu-\lambda+1/2}(x) \qquad (15)$$

where $C_a^\beta(x)$ and $D_\alpha^\beta(x)$ are the first and second Gegenbauer functionr respectively. The Gegenbauer Green's function is

$$G(x,x') = \frac{\Gamma(\nu-\lambda+3/2)}{\Gamma(\nu+\lambda+1/2)} (x^2-1)^{1/2(\lambda-1/2)} (x'^2-1)^{1/2(\lambda-1/2)}$$

$$\cdot\ C^{\lambda-1/2}_{\nu-\lambda+1/2}(x_<)\ D^{\lambda-1/2}_{\nu-\lambda+1/2}(x_>)\ . \qquad (16)$$

c) Tchebichef Green's function

There are two special Tchebichef[1] functions which are multíples of Jacobi functions; one with $\beta-\alpha=0$, $\alpha+\beta=-1/2$ for the first kind $T_n(x)$ function and other with $\beta-\alpha=0$, $\alpha+\beta=+1/2$ for the second kind $U_n(x)$ function.

For the first kind function we have one relation between $T_n(x)$ and $P_n^{(\alpha,\beta)}(x)$

$$P_{\nu+1/2}^{(-1/2,-1/2)}(x) = \frac{\Gamma(\nu+1)}{\Gamma(\nu+3/2)\Gamma(1/2)} T_{\nu+1/2}(x) \qquad (17)$$

and the relation with the first general spherical harmonic is

$$B_\nu^{(-1/2,0)}(x) = \frac{1}{\Gamma(1/2)}\ ,\ 2^{1/2}(x^2-1)^{-1/4}\ T_{\nu+1/2}(x)\ . \qquad (18)$$

The second solution for the first kind function can be defined in terms of the $Q_{\nu+1/2}^{(-1/2,-1/2)}(x)$, expanding in terms of hypergeometric functions[1]. If we call $V_{\nu+1/2}(x)$ the second solution the relation with $Q_{\nu+1/2}^{(-1/2,-1/2)}$ is

$$Q_{\nu+1/2}^{(-1/2,-1/2)}(x) = \frac{\Gamma(\nu+1)\ (3/2)}{\Gamma(\nu+1/2)} V_{\nu+1/2}(x) \qquad (19)$$

where $V_{\nu+1/2}(x)$ in terms of hypergeometric functions is given by

$$V_{\nu+1/2}(x) = 2^{1/2}(\nu+1/2)^{-1}\ {}_2F_1(-\nu-1/2,\nu+1/2;1/2;1/2-x/2) -$$

$$- 2(x-1)^{1/2}\ {}_2F_1(-\nu,\nu+1;3/2;1/2-x/2) \qquad (20)$$

The general spherical harmonic in terms $V_{\nu+1/2}(x)$ is

$$\kappa_\nu^{(-1/2,0)}(x) = 2^{1/2}\ \Gamma(3/2)(x^2-1)^{-1/4}V_{\nu+1/2}(x) \qquad (21)$$

The Green's function for the first kind Tchebichef function is

$$G(x,x') = (x^2-1)^{-1/4}(x'^2-1)^{-1/4}\ T_{\nu+1/2}(x_<)V_{\nu+1/2}(x_>) \qquad (22)$$

For the second kind $U_n(x)$, we can derive the relation between general spherical harmonics and Tchebichef functions in completely analogy with the first kind case

$$B_\nu^{(1/2,0)}(x) = \frac{\Gamma(\nu+1/2)}{\Gamma(3/2)\Gamma(\nu+3/2)}\ 2^{-1/2}(x^2-1)^{1/4}\ U_{\nu-1/2}(x) \qquad (23)$$

$$\kappa_\nu^{(1/2,0)}(x) = \frac{\Gamma(1/2)\Gamma(\nu+3/2)}{\Gamma(\nu+1/2)}\ 2^{-1/2}(x^2-1)^{1/4}\ \chi_{\nu-1/2}(x) \qquad (24)$$

where $\chi_{\nu-1/2}(x)$ is the second solution and in terms of hypergeometric functions we have

$$\chi_{\nu-1/2}(x) = - {}_2F_1(-\nu+1/2,\nu+3/2;3/2;1/2-x/2) +$$

$$+ 2^{-1/2}(\nu+1/2)^{-1}(x-1)^{-1/2} \quad {}_2F_1(-\nu,\nu+1;1/2;1/2-x/2) \qquad (25)$$

and the Green's function for the second kind function is

$$G(x,x') = (x^2-1)^{1/4} (x'^2-1)^{1/4} U_{\nu-1/2}(x_<) \bar{\chi}_{\nu-1/2}(x_>) . \qquad (26)$$

4. DISCUSSIONS

We have presented a global derivarion of the Green's function, bounded on the domain $0 \leq x < \infty$, for a number of special functions are also bounded in the domain $-1 \leq x \leq 1$, with the restriction that v rnust be an entire number, in order to have polynomial solutions for the corresponding homogeneous differential equation.

There are many problems in physics, where these differential equations appear. One example, is the quanturn mechanics Coulomb problem, in the momentum space, where the corresponding radial equation is a Gegnebauer differential equation[6], which gives the expression of the radial Coulomb Green's function in terms of the product of two independent Gegenbauer functions.

The quantum mechanics symmetric top[7], is another example, where we have a Jacobi differential equation. If we write the differential equation in Euler's angles, 0, ϕ and ψ and separate the variables, the differential equation in the θ variable is a Jacobi differential equation, and the Green's function is giving in terms of the general spherical harrnonics by Eq. (8), bounded in the domain $-1 \leq x \leq 1$.

This equation is also the differential equation for irreducible representations of the rotation group[8], and general spherical harmonics are the solutions of this differential equation in the domain $-1 \leq x \leq 1$.

It is interesting to note that these Green's functions here calculated can be also derived by means of the harmonic oscillator Green's function in the momentum space, and as a by-product we can derived integral representations for the Green's functions and some additions theorems for the general spherical harmonics. These calculations are object of another publication.

REFERENCES

1. Erdélyi, A., *Higher Transcendental Functions*, Vol. I, 2 (New York - McGraw - Hill) (1953).
2. Arfken, G., *Mathematical Methods* for *Physicists*, Academic Press - Edition 2ª, (1970).
3. Vilenkin, N.J., *Special Function and* the *Theory* of *Group Representations* - American Mathematical Society, (1968).
4. Willard Miller, Jr., *Lie Theory and Special Functions*, Academic Press Inc. (1968).
5. J.Bellandi Fº and E. Capelas de Oliveira, The Green's Function for the Associated Legendre Differential Equation, Revista Brasileira de Física, Vol. 9, nº 3 (1979).
6. E.A. Hylleraas Z. fur Physik, 75, 216 (1932).
7. L. Landau and E. Lifchitz, *Mécanique Quantique* (Editions Mir - 1966) pg. 449.
8. J.D. Talman. *Special Functions - A Group Theoretic Approach* - (W.A. Benjamin, Inc. - 1968) pg. 151.