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The Method of Contour Rotations and the Three Particle Amplitudes

JOSE R. BRINATI
Instituto de Física, UFRJ, 27970 Rio de Janeiro, RJ

and

GERHARD W. BUND
Instituto de Física Teórica, São Paulo, SP

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The application of the method of contour rotations to the solution of the Faddeev-Lovelace equations and the calculation of the break-up and stripping amplitudes in a system of three distinct particles is reviewed. A relationship between the masses of the particles is obtained, which permits the break-up amplitude to be calculated from a single iteration of the final integral equation.

Estuda-se a aplicação do método de rotações de contorno à solução das equações de Faddeev-Lovelace e ao cálculo das amplitudes de break-up e stripping num sistema de três partículas distintas. Obtém-se uma relação entre as massas das partículas que permite o cálculo da amplitude de break-up através de uma única iteração da equação integral final.

1. INTRODUCTION

In a recent article[1] we presented the numerical betweenan exact calculation and the corresponding DWBA for a (d,p) stripping reaction from a heavy target, leading to a resonance in the final state. In this caiculation the exact break-up amplitude was obtained through the Lovelace amplitudes, which satisfy coupled equations of the Faddeev type.

As is well known the Faddeev equations give a mathematically correct formulation of the three body problem of non-relativistic particles. However their solution presents difficulties for real energies and momenta, due to the singularities which occur in the kernels of these equations.

Severa1 methods[2-5] have been proposed to circumvent these difficulties. If the potential may be represented by separable terms of simple analytic form, the method of contour' rotation[2] has been applied with success, particularly in the case of identical particles[6], where the method becomes straightforward.

Here we give a review of this method, which we also used in our work[1], with emphasis on its application to the calculation of the break-up amplitude in the case of three distinct particles.

We shall not derive the coupled equations and the expressions for the amplitude which can be found elsewhere[8], and shall further restrict our study mainly to separable s-wave interactions, mentioning only briefly the generalization to interactions acting in other partia1 waves.

In section 2 we summarize the coupled equations for the Lovelace amplitudes in terms of which the break-up amplitude is written and section 3 is devoted to the study of the singularities of the corresponding kernels. In section 4 the analytic continuation of the coupled equations along a straight line in the fourth quadrant is performed and in section 5 the equations which give the Lovelace amplitudes for real momenta in terms of the corresponding ones for complex momenta are discussed.

2. THE FADDEEV-LOVELACE EQUATIONS AND THE BREAK-UP AMPLITUDE

The system with which we shall work consists of three spinless particles. We shall use the center of momentum frame; the momenta of the particles in this frame shall be designated by $\vec{p}_\alpha$, where α labels the particles, α = 1,2 or 3. The pair of particles obtained by excluding the particle α from the system shall also be denoted by α; thus $\vec{q}_\alpha$ shall represent the relative momentum of the pair α. The mass of particle α, the reduced mass of the pair α and the reduced mass of particle α with respect to the c.m. of the pair α shall be designated by m_α, μ and μ^α respectively.

The two-body interactions are of the separable type

$$V_\alpha = |g_\alpha\rangle \, \lambda_\alpha \, \langle g_\alpha| \quad . \tag{1}$$

For simplicity we shall assuma s-wave interactions, that is, $\langle\vec{q}_\alpha|g_\alpha\rangle$ is a function $g_\alpha(q_\alpha^2)$, depending solely on the magnitude of $\vec{q}_\alpha$.

We shall also need to assume that g_α is an analytic function of q_α^2, except for singularities on the real axis such that $q^2 \leq -b_\alpha^2$. The singularities of g_α lying closest to the origin are thus $q_\alpha = \pm i b_\alpha$. One example is the Yamaguchi potential given by

$$g_\alpha(q_\alpha^2) = N_\alpha (q_\alpha^2 + b_\alpha^2)^{-1} \quad , \tag{2}$$

which has poles at $q_\alpha = \pm i b_\alpha$.

The two-body T-matrix corresponding to the potential (1) is written

$$t_\alpha(w_\alpha) = |g_\alpha\rangle \, \tau_\alpha(w_\alpha) \, \langle g_\alpha| \quad , \tag{3}$$

where w_α is the energy in the system of the pair α and τ_α is given[8] by

$$\tau_\alpha^{-1}(w_\alpha) = \lambda_\alpha^{-1} - \langle g_\alpha|(w_\alpha - H_{0\alpha})^{-1} g_\alpha\rangle \quad , \tag{4}$$

$H_{0\alpha}$ being the kinetic energy operator in this system.

The tunction $\tau_\alpha(w_\alpha)$ is analytic in the cut w_a-plane except for a pole at the energy $-\varepsilon_\alpha$, corresponding to a bound-state. The functions g_a and λ_a may also be made to depend[7] on w_α, in which case they also have to be analytic in the cut w_α-plane. For instance in ref.1 one obtains a narrow two body resonance by choosing λ_2 proportional to $1 - E/(w_\alpha-\varepsilon_r^0)$.

The exact amplitude for the break-up, in which partirles 2 and 3 form a bound pair in the entrance channel, is given[8] by

$$T_{01}(\vec{p}_2,\vec{q}_2,\vec{p}_1^{\,0}) = \sum_{\alpha=1}^{3} g_\alpha(q_\alpha^2)\ \tau_\alpha(E - p_\alpha^2/2\mu^\alpha) X_{\alpha 1}(\vec{p}_\alpha,\vec{p}_1^{\,0}), \tag{5}$$

where $X_{\alpha 1}$ are the Lovelace amplitudes which obey the equations[8]

$$X_{\alpha 1}(\vec{p}_\alpha,\vec{p}_1^{\,0}) = Z_{\alpha 1}(\vec{p}_\alpha,\vec{p}_1^{\,0},E) + $$

$$+ \sum_{\beta=1}^{3} \int d\vec{p}_\beta^{\,\prime}\ Z_{\alpha\beta}(\vec{p}_\alpha,\vec{p}_\beta^{\,\prime},E)\tau_\beta(E+i\eta-p_\beta^{\prime 2}/2\mu^\beta)X_{\beta 1}(\vec{p}_\beta^{\,\prime},\vec{p}_1^{\,0}) \ . \tag{6}$$

In eqs. (5) and (6) $\vec{p}_1^{\,0}$ is the momentum of the incoming particle, $\vec{p}_\alpha$ and $\vec{q}_\alpha$, a = 1,2,3, are the momenta corresponding to the parricles ir: the exi: channel and E the total energy of the three particle system. Further in eq.(6), the function $Z_{\alpha\beta}$ is given[8] by the expression

$$Z_{\alpha\beta}(\vec{p}_\alpha,\vec{p}_\beta^{\,\prime},E) = (1-\delta_{\alpha\beta}) \cdot \frac{g_\alpha(q_\alpha^2)\ g_\beta(q_\beta^{\prime 2})}{E + i\eta - \dfrac{p_\alpha^2}{2\mu^\alpha} - \dfrac{q_\alpha^2}{2\mu_\alpha}}, \tag{7}$$

where the momenta $\vec{q}$ and $\vec{q}_\alpha^{\,\prime}$ are given by the relatlons

$$\left.\begin{aligned} \vec{q}_\alpha &= \mp \vec{p}_\beta^{\,\prime} \mp \frac{\mu_\alpha}{m_\gamma}\vec{p}_\alpha \\ \vec{q}_\beta &= \pm \vec{p}_\alpha \pm \frac{\mu_\beta}{m_\gamma}\vec{p}_\beta \end{aligned}\right\} \quad \gamma \neq \alpha\ ,\ \gamma \neq \beta \tag{8}$$

In eq.(8) the upper(lower) sign apply when $\alpha\beta\gamma$ is cyclic (anticyclic)

3. THE INTEGRAL EQUATION FOR COMPLEX MOMENTA

The Faddeev-Lovelace equations shall not be studied in the form given by eq. (6) but rather through their partial-wave decomposition

$$X^{\ell}_{\alpha 1}(p_\alpha,p^0_1) = Z^{\ell}_{\alpha 1}(p_\alpha,p^0_1,E) + \frac{4\pi}{2\ell+1}\sum_{\beta=1}^{3}\int_0^\infty dp'_\beta\, p'^2_\beta\, Z^{\ell}_{\alpha\beta}(p_\alpha,p'_\beta,E)\,\tau_\beta(E+i\eta-p'^2_\beta/2\mu^\beta)\;. \; X^{\ell}_{\beta 1}(p'_\beta,p^0_1)\;, \tag{9}$$

where

$$X^{\ell}_{\alpha 1}(p_\alpha,p^0_1) = (\ell+\tfrac{1}{2})\int_{-1}^{+1} d\cos\theta_{\alpha 1}\; X_{\alpha 1}(\vec{p}_\alpha,\vec{p}^{\,0}_1)P_\ell(\cos\theta_{\alpha 1})\;. \tag{10}$$

Here $\theta_{\alpha 1}$ denotes the angle formed by the momenta $\vec{p}_\alpha$ and $\vec{p}^{\,0}_1$. The function $Z^{\ell}_{\alpha\beta}$ in eq.(9) is given by an equation analogous to eq.(10). In what follows in this section we shall make an analysis of the kernel $p'^2_\beta\;.\;\tau_\beta\;.\;Z^{L}_{\alpha\beta}$ of the integral equations (9), in the complex p'_β-plane.

3.1. Analiticity of τ_β

The mapping of the singularities of the function $\tau_\beta(w_\beta)$ into tha complex p'_β-plane are given through the transformation

$$p'_\beta = ((E+i\eta-w_\beta)2\mu^\beta)^{1/2}\;. \tag{11}$$

Tnis transformation maps the cut, which coincides with the positive w_β half-axis, into two curves in the first and third quadrants of the p'_β--plane as plotted in fig. 1.

A pole at the energy $w_\beta = -\varepsilon_\beta$, associated to a bound-state does correspond in the p'_β-plane to the points $\pm(2\mu^\beta(E+i\eta+\varepsilon_\beta))^{1/2}$ (B^+ and B^- in fig.1). Resonance poles at energies $E_R - i\Gamma/2$ correspond to points R^+ and R^- in fig.1 across the cut in the second sheet of the variable p'_β.

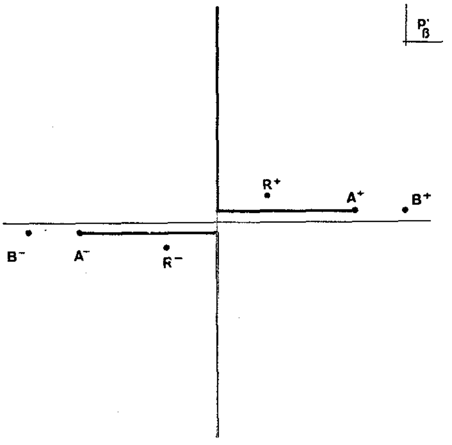

Fig.1 - Singularities of $\tau_\beta(E+i\eta-p_\beta'^2/2\mu^\beta)$ in the complex p'_β -plane. The curves starting at $A^\pm = \pm\sqrt{2\mu^\beta(E+i\eta)}$ are the cuts, the points B^+ and B^- correspond to the bound-stete pole, the points R^+ and R'' located in the second sheet correspond to a resonance.

3.2. Analiticity of $Z^\ell_{\alpha\beta}$

From the definition (7) of $Z_{\alpha\beta}$ and from eq.(8), the function $Z^\ell_{\alpha\beta}$ is written

$$Z^\ell_{\alpha\beta}(p_\alpha,p'_\beta,E) = (1-\delta_{\alpha\beta})(\ell+\frac{1}{2})\int_{-1}^{+1}\frac{F_{\alpha\beta}(p_\alpha,p'_\beta,\rho)}{R_{\alpha\beta}(p_\alpha,p'_\beta)-\rho}\,P_\ell(\rho)\,d\rho, \tag{12}$$

where

$$F_{\alpha\beta}(p_\alpha,p'_\beta,\rho) = \frac{m_\gamma}{p_\alpha p'_\beta}\,g_\alpha\Big(p_\beta'^2+\frac{\mu_\alpha^2}{m_\gamma^2}p_\alpha^2+2\frac{\mu_\alpha}{m_\gamma}p_\alpha p'_\beta\rho\Big)\,.$$
$$\cdot\, g_\beta\Big(p_\alpha^2+\frac{\mu_\beta^2}{m_\gamma^2}p_\beta'^2+2\frac{\mu_\beta}{m_\gamma}p_\alpha p'_\beta\rho\Big) \tag{13}$$

and

$$R_{\alpha\beta}(p_\alpha,p'_\beta) = \frac{m_\gamma}{p'_\beta p_\alpha}(E+i\eta) - \frac{m_\gamma\,p_\alpha}{2\mu_\beta p'_\beta} - \frac{m_\gamma\,p'_\beta}{2\mu_\alpha p_\alpha}\,. \tag{14}$$

The singularities of $Z^{\ell}_{\alpha\beta}$ in the p'_{β}-plane are determined by the singularities of g_{α} and g_{β} and by the zeros of $R_{\alpha\beta}-\rho$. We consider first the singularities of $g_{\alpha}(q^2_{\alpha})$. Thus the location of a pole of g_{α} at $q^2_{\alpha} = -b^2$ will be transformed, through replacement of q_{α} by eq.(8), into

$$p'^2_{\beta} + \frac{\mu^2_{\alpha}}{m^2_{\gamma}} p^2_{\alpha} + 2\frac{\mu_{\alpha}}{m_{\gamma}} p_{\alpha} p'_{\beta} \rho = -b^2_{\alpha}, \quad -1 \leq \rho \leq 1 . \tag{15}$$

In the p'_{β}-plane, eq.(15) describes two symmetrical arcs BC and $B'C'$ of a circle centered at the origin, as depicted in fig.2c, where the point B is given here by $B = -\mu_{\alpha}p_{\alpha}/m_{\gamma} - ib_{\alpha}$. The largest angle by which the positive half-axis may be rotated clockwise around the origin, without meeting any point of the curve BC, is arctg $(m_{\gamma}b_{\alpha}/\mu_{\alpha}p_{\alpha})$. In addition, as $q^2_{\alpha} = -b^2_{\alpha}$ is the singularity of g_{α} lying closest to the origin, it is easily seen that no other singularity associated with g is crossed.

By making an analogous study for g_{β}, one obtains that the largest angle $\phi_{\alpha\beta}$ of rotation of the positive half-axis, such that no singularities of $Z^{\ell}_{\alpha\beta}$ originating from singularities of either g_{α} or g_{β} are met, is

$$\phi_{\alpha\beta}(p_{\alpha}) = \mathrm{Min}(\mathrm{arctg}\,(\frac{m_{\gamma}b_{\alpha}}{\mu_{\alpha}p_{\alpha}}),\ \mathrm{arctg}(\frac{b_{\beta}}{p_{\alpha}})) . \tag{16}$$

Next consider the singularities generated by the zeros of $R_{\alpha\beta}-\rho$ in eq.(12). From eq. (14) the equation for these singularities is

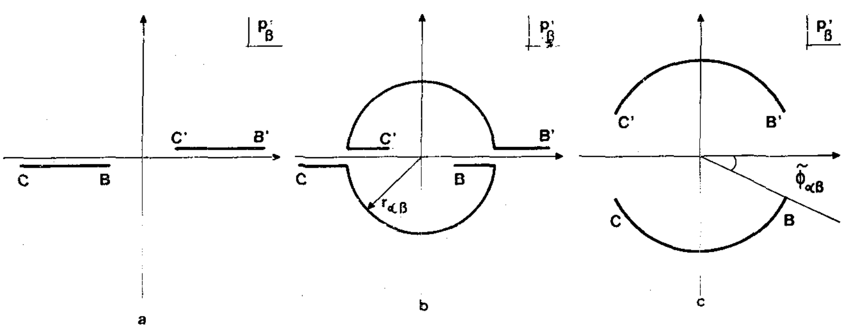

Fig.2 - BC and B'C' are the curves of singularity of $Z^{\ell}_{\alpha\beta}(p_{\alpha},p'_{\beta},E)$ in the p'_{β} - plane given by eq. (17); figs.a, b and c correspond to the cases $p_{\alpha} < Q_{\alpha\beta}$, $Q_{\alpha\beta} < p_{\alpha} < Q^{\alpha}$ and $p_{\alpha} > Q^{\alpha}$ respectively. The end points B(B') and C(C') correspond in eq. (17), respectively to $\rho = -1$ and $\rho = +1$.

$$m_\gamma(E+i\eta) - \frac{m_\gamma}{2\mu_\beta} p_\alpha^2 - \frac{m_\gamma}{2\mu_\alpha} p_\beta'^2 - p_\alpha p_\beta' \rho = 0 \ , \tag{17}$$

where $-1 \leq \rho \leq +1$. In this case one has a dependence of the curves of singularities on the total energy E, As we are studying the break-up reactions we have $E>0$. The momenta $\vec{p}_\alpha$ and $\vec{q}_\alpha$ corresponding to this channel satisfy

$$E = p_\alpha^2/2\mu^\alpha + q_\alpha^2/2\mu_\alpha \ ; \tag{18}$$

p_α thus meets the on-shell condition if

$$p_\alpha \leq \sqrt{2\mu^\alpha E} \equiv Q^\alpha \ . \tag{19}$$

The curves of singularities given by eq.(17) with p_α subject to condition (19), fall into the following two separate cases:

a) $p_\alpha < Q_{\alpha\beta}$

where

$$Q_{\alpha\beta} = \sqrt{2\mu_\beta E} \ . \tag{20}$$

In this case the curves of singularities are two line segments, displaced from the real axis by a distance of the order of magnitude of η, localized in the first and third quadrants in the p_β'-plane as shown in fig.2a.

b) $Q_{\alpha\beta} < p_\alpha < Q^\alpha$

The corresponding singularities are as shown in fig.2b, represented by two half circles of radius

$$r_{\alpha\beta}(p_\alpha,E) = \left(\frac{\mu_\alpha}{\mu_\beta}\left(p_\alpha^2 - Q_{\alpha\beta}^2\right)\right)^{1/2} \ , \tag{21}$$

with center at the origin; each half-circle being connected to two line segments parallel to the real axis, displaced from the later by a distance proportional to η. We introduce for later use the quantity $y^{\alpha\beta}(p_\alpha,E)$ which gives the position of the point B in fig.2b,

$$y^{\alpha\beta}(p_\alpha,E) = \frac{\mu_\alpha}{m_\gamma} p_\alpha - \left[2\mu_\alpha(E + i\eta - p_\alpha^2/2\mu^\alpha)\right]^{1/2} \tag{22}$$

Finally we consider momenta $p_a > Q^\alpha$, corresponding, for a = 1, to the range of momenta appropriate to the entrance and scattering channels and, for a $\neq$ 1, to the rearrangement channels. The curve of singularities corresponding to eq. (17) are two symmetrically disposed arcs of circle which do not touch the real axis as shown in fig.2c. The maximum angle $\tilde{\phi}_\beta$ by which the real axis may be rotated without crossing this curve is given by

$$\tilde{\phi}_{\alpha\beta}(p_\alpha,E) = \text{arctg}\left[\left\{\frac{(m_\gamma^2(p_\alpha^2 - 2\mu^\alpha E)}{\mu_\alpha \mu^\alpha p_\alpha^2}\right\}^{1/2}\right] . \tag{23}$$

We may isolate in $Z^\ell_{\alpha\beta}$ the singularity corresponding tu eq.(17), by casting eq.(12) into the form

$$Z^\ell_{\alpha\beta}(p_\alpha,p'_\beta,E) = (1-\delta_{\alpha\beta})(\ell+\tfrac{1}{2}) \left\{ \int_{-1}^{+1} d\rho\, P_\ell(\rho)\left[F_{\alpha\beta}(p_\alpha,p'_\beta,\rho) - \right.\right.$$
$$\left. - F_{\alpha\beta}(p_\alpha,p'_\beta,R_{\alpha\beta}(p_\alpha,p'_\beta))\right]\left[R_{\alpha\beta}(p_\alpha,p'_\beta) - \rho\right]^{-1} +$$
$$\left. + 2F_{\alpha\beta}(p_\alpha,p'_\beta,R_{\alpha\beta}(p_\alpha,p'_\beta))\, Q_\ell(R_{\alpha\beta}(p_\alpha,p'_\beta))\right\} , \tag{24}$$

where Q_R is the Legendre function of the second kind[9], defined by

$$Q_\ell(z) = \frac{1}{2}\int_{-1}^{+1} d\rho\, \frac{P_\ell(\rho)}{z-\rho} , \tag{25}$$

whicn has a branch cut for real values of z in the interval $-1 \leq z \leq +1$. By making $z = R_{\alpha\beta}(p_\alpha,p'_\beta)$ this branch cut is mapped into the curves inthe p'_β-plane depicted in fig.2.

We shall need the analytic continuation $Q^{(-)}_\ell(z)$ of $Q_\ell(z)$ across this cut into the lower half of the complex z-plane[9],

$$Q^{(-)}_\ell(z) = Q_\ell(z) - i\pi\, P_\ell(z) , \tag{26}$$

where P_R are the Legendre polinomials. We also define for future usethe function $Z^{(-)\ell}_{\alpha\beta}$, with Q_R in eq.(24) replaced by $Q^{(-)}_\ell$,

$$Z^{(-)\ell}_{\alpha\beta}(p_\alpha,p'_\beta,E) = Z^{\ell}_{\alpha\beta}(p_\alpha,p'_\beta,E) - i\pi(2\ell+1)\; F_{\alpha\beta}(p_\alpha,p'_\beta,R_{\alpha\beta}(p_\alpha,p'_\beta))\,.$$

$$\cdot\, P_\ell(R_{\alpha\beta}(p_\alpha,p'_\beta))(1-\delta_{\alpha\beta})\;. \tag{27}$$

Before proceeding we would like to point out that the present study may be extended to other than s-wave interactions. Indeed, in an appropriate representation, one obtains coupled equations of the type (9), the kernels being of the same form as those of eqs. (12)-(14) multiplied by angular momentum coupling coefficients. Thus, as far as our discussion is concerned, the effect of the introduction of the additional angular momentum quantum numbers is the increase of the number of coupled equations. For details see for instance eqs.(11) and (22) in ref.3.

As a final observation we remark that the function $F_{\alpha\beta}(p_\alpha, p'_\beta, R_{\alpha\beta})$, which enters into the decomposifion (24) of $Z^{\ell}_{\alpha\beta}$, is analytic in the fourth quadrant of the p'_β-plane. The singularities of $F_{\alpha\beta}$ are obtained by replacing ρ by $R_{\alpha\beta}(p_\alpha,p'_\beta)$ in eq. (15) for the singular; ties of g_α and in tha analogous equation for those of g_β. One obtains fixed singularities in the first and third quadrants given by

$$\left.\begin{aligned} p'^2_\beta &= 2\mu^\beta(E+i\eta) + \frac{\mu^\beta}{\mu_\beta}\, b^2_\beta \\ p^2_\alpha &= 2\mu^\alpha(E+i\eta) + \frac{\mu^\alpha}{\mu_\alpha}\, b^2_\alpha \end{aligned}\right\} \tag{28}$$

The above singularities are only apparent singularities of $Z^{\ell}_{\alpha\beta}$, as they are cancelled by analogous singularities in the first term in eq. (24). They are also nof relevant to $Z^{(-)\ell}_{\alpha\beta}$, as they are situated outside of the domain of the variables p and p'_β where this function shall be utilized.

4. THE AMALYTIC CONTINUATION QF THE INTEGRAL EQUATIONS

In section 3 we determined the location of the singularities of the kernel of the integral equations (9) in the complex p'_β-plane for real values of p_α. We found singularities close to the real axis at a distance proportional to η; these singularities prevent a straighforward numerical solution of the coupled integral equations. In particular, it has been shown[10] that the kernel is not square integrable in the limit in which η is set equal to zero.

One way which avoids these difficulties is achieved by performing the analytical continuation of the integral equations (9) such that the variables p_α and p'_β are both defined along a half-line in the fourth quadrant, starting at the origin and making an angle ϕ with the positive half axis.

The procedure becomes feasible because the singularities assoeiated with τ_β are fixed, τ_β being analytic in the fourth quadrant and those of $Z^\ell_{\alpha\beta}$ keep moving downward, as the point p_α is shifted downward in the fourth quadrant. The curves of singularities which correspond to p_α in the fourth quadrant have the property that they nevercross the half line which starts at the origin and passes through p_α. This Dehaviour is illustrated in fig.3.

Thus the singularities of the kernel do not pinch the path of integration, as this path is displaced frorn the real axistohalf line in the fourth quadrant heginning at the origin.

This rotation may be carried on towards larger values of ϕ as long as the path of integration stays inside the domain where the domain where the functions $X^\ell_{\beta 1}$ are analytic. It has been shown[2,10], that

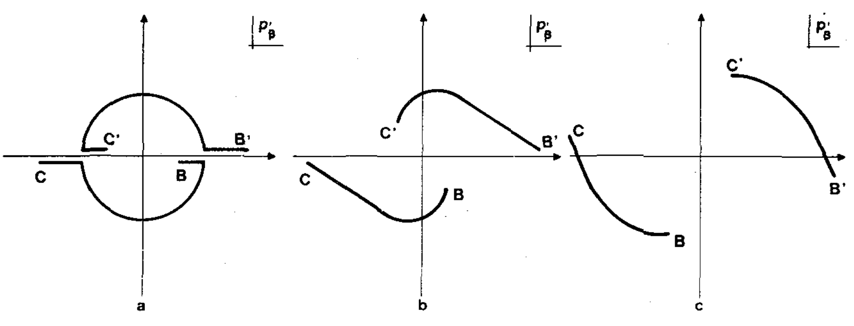

Fig.3 - Branch cuts of $Q_\ell(R_{\alpha\beta}(p_\alpha,p'_\beta))$ in the p'_β -plane for complex p_α, $p_\alpha = pe^{-i\phi}$ for increasing values of ϕ. The points B(B') and C(C') correspond respectively to $\rho = -1$ and $\rho = +1$ in eq. (17).

the amplitudes $X^{\ell}_{\beta 1}$ are analytic functions of p'_{β} in any domain bounded by the real positive axis and a half line in the fourth quadrant, provided that the inhomogeneous terms $Z^{\ell}_{\alpha 1}(p_{\alpha},p^0_1,E)$ are analytic functions of p_{α} in this domain. The region of analiticity of this function was obtained already in section 3.2, in view of the symmetry $Z^{\ell}_{\alpha\beta}(p_{\alpha},\ p'_{\beta},\ E) = Z^{\ell}_{\beta\alpha}(p'_{\beta},p_{\alpha},E)$.

Thus, by starting from eq.(9) and by performing the contour rotation discussed above, one obtains the following set of coupled equations for X^{ℓ}_{α}, with complex momenta

$$X^{\ell}_{\alpha 1}(p_{\alpha}e^{-i\phi},p^0_1) = Z^{\ell}_{\alpha 1}(p_{\alpha}\ e^{-i\phi},p^0_1,E)$$

$$+ \frac{4\pi}{2\ell+1} \sum_{\beta=1}^{3} \int_0^{\infty} dp'_{\beta}\ p'^2_{\beta}\ e^{-i3\phi}\ Z^{\ell}_{\alpha\beta}(p_{\alpha}\ e^{-i\phi},\ p'_{\beta}\ e^{-i\phi},\ E)\ .$$

$$.\ \tau_{\beta}(E - p'^2_{\beta}\ e^{-i2\phi}/2\mu^{\beta})\ X^{\ell}_{\beta 1}(p'_{\beta}\ e^{-i\phi},\ p^0_1)\ , \tag{29}$$

where ϕ is any positive quantity which satisfies the inequality

$$\phi < \mathrm{Min}(\phi_{1\beta}(p^0_1),\ \ \tilde{\phi}_{1\beta}(p^0_1,\ E)),\ \beta = 2,3\ , \tag{30}$$

where $\phi_{1\beta}$ and $\tilde{\phi}_{1\beta}$ are given by eqs. (16) and (23) respectively. For angles ϕ larger than those sbeying (30), the path of integration would cross curves of singularities of $X^{\ell}_{\beta 1}$

5. THE CALCULATION OF TME AMPLITUDES ON-SHELL

The solution of the integral equations (29) gives us the amplitudes $X^{\ell}_{\alpha 1}(p_{\alpha}\ e^{-i\phi},p^0_1)$. Once these amplitudes are obtained, one returns to eq.(9) in order to obtain the amplitudes $X^{\ell}_{01}(p_{\alpha},p^0_1)$, with real momenta p . The path of integration corresponding to the Integral on the right hand side of eq.(9) is now deformed into a curve in the fourth quadrant along which the functions $X^{\ell}_{\beta 1}(p'_{\beta},p^0_1)$ are known by solving eqs.(29).

In order to be able to perform tnese contour rotations one needs to determine the singularities of the funceions $Z^{\ell}_{\alpha\beta}\, \tau_\beta\, p'^2_\beta$ in the p'_β -plane for fixed p_a. This analysis was already performed in section 3. From this. study one concludes that, in order to avcid the curves of singularities of $Z^{\ell}_{\alpha\beta}$ associated with g_α and g_β, the angle ϕ in eqs. (29) has to satisfy, besides (30), also the conditions

$$\phi < \phi_{\alpha\beta}(p_\alpha), \quad \beta \neq \alpha, \quad \alpha = 1,2,3\ ;\ \beta = 1,2,3\ , \tag{31}$$

where $\phi_{\alpha\beta}$ is given by eq.(16).

With regard to the cut of the functicn $Q_a(R_{\alpha\beta})$, given by eq. (17), we found different types of curves, depending on the value of p_α. Thus for $p_\alpha < Q_{\alpha\beta}$ the function $Q_\ell(R_{\alpha\beta})$ is analyric in the fourth quadrant and the path may be deformed into the half line defined by the angle ϕ.

For $Q_{\alpha\beta} < p_\alpha < Q^\alpha$, the singularities shown in fig.2b seem to forbid the deformation of the contour of integration into the half line niaking an angle ϕ with the real axis. Nevertheless one may deform the path of integration, without changing the value of the integral on the right hand side of eq.(9), if one makes a detour arouod the ramification point and crosses twice the cut of the function $Q_\ell(R_{\alpha\beta})$ as shown in fig. 4a. provided that in the section BCD of the path, Q_ℓ is replaced by its analytic continuation $Q^{(-)}_R$ (cf.eq.(26)) across the cut.

That $Q^{(-)}_\ell$ s indeed the required analytic continuation canbe verified by performing the mapping of the p'_β -plane into the $R_{\alpha\beta}$ -plane through eq.(14). The cut given by eq.(17) is mapped into the line segment with end points $R_{a\beta} = +1$ and $R_{a\beta} = -1$ and the path ABCDE in fig. 4a is mapped into the curve A'B'C'D'E' in fig. 4b thrcugh this transformation. Thus the anãlytic continuation is as given by eq.(26), namely from above the cut into the lower half of the $R_{\alpha\beta}$-plane.

In what follows we shall use the kernels

$$K^{\ell}_{\alpha\beta}(p_\alpha,p'_\beta) = \frac{4\pi}{2\ell+1}\, p'^2_\beta\, \tau_\beta(E+i\eta-p'^2_\beta/2\mu^\beta)\, Z^{\ell}_{\alpha\beta}(p_\alpha,p'_\beta,E)\ , \tag{32}$$

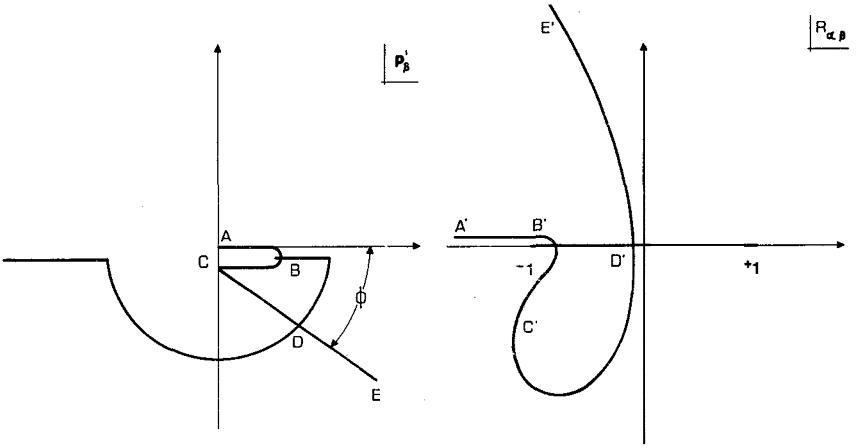

Fig.4 - In fig. a the curve ABCDE is the deformed contour of integration appropriate to the case $Q_{\alpha\beta} < p_\alpha < Q^\alpha$. Fig. b represents the mapping of fig. a into the complex $R_{\alpha\beta}$ -plane through eq. (14); the letters in fig. a correspond to the dashed ones in fig. b.

$$K_{\alpha\beta}^{(-)\ell}(p_\alpha,p'_\beta) = \frac{4\pi}{2\ell+1}\, p'^2_\beta\ \tau_\beta(E+i\eta-p'^2_\beta/2\mu^\beta)Z_{\alpha\beta}^{(-)\ell}(p_\alpha,p'_\beta,E) \ , \tag{33}$$

where $Z_{\alpha\beta}^{(-)\ell}$ is defined by eq. (27), and also the discontinuity of the across the cut given by

$$x_{\alpha\beta}(p_\alpha,p'_\beta) = K_{\alpha\beta}^{\ell}(p_\alpha,p'_\beta) - K_{\alpha\beta}^{(-)\ell}(p\ ,p'_\beta) \ . \tag{34}$$

Thus by performing the contour rotation discussed above, eq.(9) becomes for $Q_{\alpha\beta} < p_\alpha < Q^\alpha$

$$X_{\alpha 1}^{\ell}(p_\alpha,p_1^0) = \tilde{X}_{\alpha 1}^{\ell}(p_\alpha,p_1^0) + \sum_{\beta=1}^{3}\int_0^{y^{\alpha\beta}(p_\alpha,E)} dp'_\beta\ \ x_{\alpha\beta}(p_\alpha,p'_\beta)X_{\beta 1}^{\ell}(p'_\beta,p_1^0) \ , \tag{35}$$

where

$$\tilde{X}_{\alpha 1}^{\ell}(p_\alpha,p_1^0) = Z_{\alpha 1}^{\ell}(p_\alpha,p_1^0,E) +$$

$$+ \sum_{\beta=1}^{3}\Bigg[\int_0^{r_{\alpha\beta}(p_\alpha,E)} dp'_\beta\ K_{\alpha\beta}^{(-)\ell}(p_\alpha,p'_\beta e^{-i\phi})\ X_{\beta 1}^{\ell}(p'_\beta e^{-i\phi},p_1^0) +$$

$$\left. \int_{r_{\alpha\beta}(p_\alpha,E)}^{\infty} dp_\beta^0 \, K_{\alpha\beta}^{\ell}(p_\alpha, p_\beta' e^{-i\phi}) X_{\beta 1}^{\ell}(p_\beta' \, e^{-i\phi}, p_1^0) \right] \tag{36}$$

Here $r_{a\beta}$ and y^{aB} are given by eqs. (21) and (22) respectively. The secorid term in eq. (35) is the contribution from the section ABC of the path of integration in fig. 4a.

For the cases $p_a < Q_{\alpha\beta}$ and $p_\alpha > Q^\alpha$, (cf. section 3.2), eqs. (35) and (36) are valid, provided one sets $r_{\alpha\beta}$ and $y^{\alpha\beta}$ equal to zero. In the case $p_\alpha > Q^\alpha$, where $X_{\alpha\beta}^{\ell}$ is the partial wave amplitudes of the elastic or rearrangement channels, one has the additional condition on ϕ

$$\phi < \tilde{\phi}_{\alpha\beta}(p_\alpha, E) \ . \tag{37}$$

It should be observed that for $Q_{\alpha\beta} < p_\alpha < Q^\alpha$, eq. (35) is still an integral equation, since the second term requires the knowledge of $X_{\beta 1}^{\ell}$ for real momenta $p_\beta' < y_{\alpha\beta}(p_\alpha, E)$. Hetherington and Schick [2] have shown that eq. (35) may be solved stepwise. For this purpose they introduce the parameterization

$$p_\alpha = \lambda \, Q^\alpha \tag{38}$$

and make use of the relationship

$$y^{\alpha\beta}(\lambda Q^\alpha, E) < \lambda \, Q^\beta \ . \tag{39}$$

Thus if one wishes to calculate $X_{\alpha 1}^{R}$ for $p_\alpha = \lambda_n Q^\alpha$, according to eq. (35) one needs to know X_{a1}^{ℓ} for momenta $p_a' = \lambda Q^\beta$ such that

$$\lambda \leq y^{\alpha\beta}(\lambda_0 Q^\alpha, E) . (Q^\beta)^{-1} = \lambda_1 \ . \tag{40}$$

Now by making use of eq. (39) one gets that $\lambda_1 < \lambda_0$. This last inequality allows the solution of eq. (35) by increasing λ by small amounts.

Eq. (35) may also be conveniently solved by making iterations in the amplitude $\tilde{X}_{\alpha 1}^{\ell}$. In particular, if the masses of the particles satisfy the inequality given below, one single iteration of eq. (35) is sufficient for all values of the momenta p_α, $\alpha = 1,2,3$. This means that in the right hand side of eq. (35) $X_{\beta 1}^{\ell}$ may be replaced simply by $\tilde{X}_{\beta 1}^{\ell}(p_{B}'$

$p_1^0)$. By studying the iterated equation one finds that this is indeed the case if the condition

$$y^{\beta\gamma}(p'_\beta,E) \lesssim 0 \ , \qquad (41)$$

holds for all $p'_\beta < y^{\alpha\beta}(p_\alpha,E)$. From our study of the function $y^{\alpha\beta}$ in section 3.2, it will be seen that the condition (41) is, for $p < Q^\alpha$, equivalent to

$$y^{\alpha\beta}(Q^\alpha,E) \lesssim (2\mu_\gamma E)^{1/2} \ , \quad \gamma \neq \beta \ . \qquad (42)$$

Inserting the explicit expression (22) for $y^{\alpha\beta}$, one obtains that eq.(42) is satisfied provided that the masses of the particles obey the relationship

$$(m_{\alpha_1} + m_{\alpha_2} + m_{\alpha_3})m_{\alpha_3} \gtrsim m_{\alpha_1} m_{\alpha_2} \ , \qquad (43)$$

where the masses have been ordered according to their increasing magnitude

$$m_{\alpha_1} \gtrsim m_{\alpha_2} \gtrsim m_{\alpha_3} \qquad (44)$$

In fig.5, where the variables are $Y_2 = m_{\alpha_2}/m_{\alpha_1}$ and $Y_3 = m_{\alpha_3}/m_{\alpha_1}$ which span the triangle OAC, the hatched area is the domain of validity of eq. (43). The case of two equal masses corresponds to the segments OC and CB. Thus, if two particles have the same mass, one concludes that one single iteration of eq.(35) is enough provided that the ratio batween the mass of the third particles and the mass of the equal particles is larger than 0.4142.

In the region OAB more than one iteration of eq.(35) may be necessary, depending on the value of tne on-sheli momenta.

Finally, we summarize the steps which are required in the calculation of the break-up amplitude:

1) The coupled integral equation (29) are solved, the upper bound on ϕ being given by eqs. (30) and (31);

2) The break-up amplitudes on-shell are calculated through eqs. (35) and (36) by applying one of the methods outlined in this section.

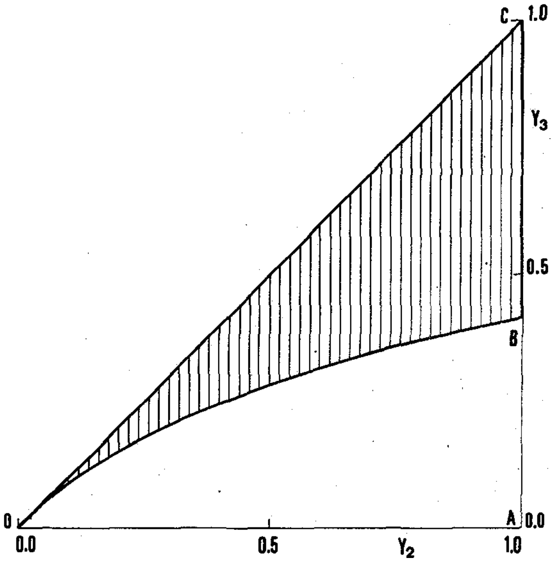

Fig.5 - The hatched area is the domain where the inequality (43) between *the masses is satisfied. The* variable *are* $Y_2 = m_{\alpha_2}/m_{\alpha_1}$ *and* $Y_3 = m_{\alpha_3}/m_{\alpha_1}$. The triangle *OAC* corresponds to $Y_3 \leq Y_2 \leq 1$ according to (44).

In the calculation of the scattering or stripping amplitudes the condition (37) on ϕ has also to be considered, however step 2) is simplified, as the second term of eq.(35) is absent in this case.

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