Revista Brasileira de Física, Vol. 10, Nº 1, **1980**

Lagrangian Solutions to the Three-Body Problem with Forces r^{-p} (p integer)+

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Recebido em 18 de Agosto de 1979

The exact solutions to the three-body problem in Celestial Mechanics, due to Lagrange (triangular solutions) and Euler (collinear solutions), are generalized to the case of forces r^{-p} (p being an integer). The stability of the system is also investigated in a local sense (small variations about steady motion) for triangular and collinear solutions and conditions restricting the values of p for which there are stable oscillatory modes of vibration are obtained. Furthermore, for the solutioris under consideration, Bohr or Bohr-Sommerfeld quantization is performed and compared, for some cases of interest, with the WKB approximation, derived from an Hamiltonian of the system obtained by reducing it to a one-body problern under the action of a central force at the system's center of mass.

As soluções exatas do problema de três corpos na Mecânica Celestial, devidas a Lagrange (soluções triangulares) e Euler (soluções colineares), são generalizadas para o caso de forças r^{-p} (p inteiro). A estabilidade do sistema é também investigada no sentido local (pequenas variações em torno da solução) para soluções triangulares e colineares, e são obtidas condições sobre os valores de p para que correspondam a modos de vibração oscilatórios estáveis. Além disso, para as soluções em questão, as quantizações de Bohr e de Bohr-Sommerfeld são efetuadas e comparadas, para alguns casos de interesse, à aproximação WKB, deduzida de um

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+ Work done with partial support by FINEP (Contract 522/CT).

++ With a fellowship of FAPESP, São Paulo.

hamiltoniano do sistema obtido reduzindo-o a um problema de 1 corpo sob a ação de uma força central no centro de massa do sistema.

1. INTRODUCTION

It is well-known that the three-body problem in Celestial Mechanics cannot be solved, in general, in a closed form. The differential equations of motion of the system make the problem of the 18^{th} order. This system of equations can he redcced to the 8^{th} order by means of the integrals corresponding to conservation of the center of mass, energy and angular momentum. Lagrange[1] proved that it is possible to reduce the problem to the 7^{th} order and no further reduction has been obtained.

Lagrange alsu showed[1] that, if the triangle fornied by the three celestial bodies always remains similar to itself, then the three-body admits exact plane[2] solutions and the bodies describe conic sections, similãr to each other, with the center of mass being at one focus.

If the bodies are collinear, there are also exact solutions, which were first derived by Euler[4,5].

We recail that these classical solutions, due to Lagrange aad Euler[6], have astronomical significance: a) the system composed by the sun, Jupiter and a Trojan astercid forms an equilnteral triangle and was found to be a realization of the Lagrangian solution; b) the "gegenschein", that is, the patch of light (already observed by last century astronomers) due to a concentration of meteors on the line through the earth and the sun (collinear solution)[8]

In the present note, it is shown in section 2, that Lagrange and Euler solutions can be generalized to the case of forcesproportional to r^{-p} (p integer).

In view of the above examples of realization of those solutions in nature (for *newtonian* **forces** i.e $p = 2$), it is reasonable to suppose that also in the case $p \neq 2$, these solutions may have a physical significance, even in the microscopical domain. This consideration motivated us to study the stability (section 3) and the quantization (settion 4) of those solutions, Section 5 is devoted to some final remarks.

2. GENERALIZED LAGRANGE AND EULER SOLUTIONS

We assume that the two-body forces between the bodies j and k are central and proportional to r_{jk}^{-p}, so that the total force acting on the body k is given by

$$\underset{\sim}{F}_k = \sum_{j\neq k} C_{jk} \frac{\underset{\sim}{r}_j-\underset{\sim}{r}_k}{r_{jk}^{p+1}} \quad , \qquad (j,k = 1,2,3)\,. \tag{2.1}$$

For $p = 2$ and $C_{jk} = G\, m_j\, m_k$, we have the classical case studied by Lagrange and Euler.

First, we note that it is easy to extend to any value of p, $p \neq 3$, Carathēodory's proof[3] that the plane formed by the bodies is fixed in space[9].

For each type of solution (triangular or rollinear) we shall consider two cases:

a) *Circular* (or "*steady motion*"): the mutual distances between the bodies are constants and the bodies rotate about their center of mass with an angular velocity ω.

b) *non-circular* (or "*periodic*"): the ratios between mutuai distances remain constant and the orbits of the bodies vary according to the value of p.

2.1. Lagrange type solutions

a) *Circular* case

The equations of motion, referred to the center of mass are given, in Cartesian coordinates by

$$\ddot{x}_k = \sum_{j\neq k} m_k^{-1}\, C_{jk}\, \frac{x_j - x_k}{r_{jk}^{p+1}}$$

$$(j,k = 1,2,3).$$

$$\ddot{y}_k = \sum_{j\neq k} m_k^{-1}\, C_{jk}\, \frac{y_j - y_k}{r_{jk}^{p+1}} \qquad (2.1.1)$$

If (ξ_k,η_k) rotates with angular velocity ω with respect to the (x_k,y_k) coordinates and if $z_k = \xi_k + i\eta_k$, then it follows from (2.1.1) that

$$-z_k = \sum_{j\neq k} \omega^{-2}\, m_k^{-1}\, C_{jk}\, \frac{z_j - z_k}{r_{jk}^{p+1}} \quad . \qquad (2.1.2)$$

Defining now

$$\rho_1 = \omega^{-2}\, C_{23}\, r_{23}^{-(p+1)}$$

$$\rho_2 = \omega^{-2}\, C_{31}\, r_{31}^{-(p+1)} \qquad (2.1.3)$$

$$\rho_3 = \omega^{-2}\, C_{12}\, r_{12}^{-(p+1)}$$

we get, from (2.1.2), that

$$\begin{cases} (m_1 - \rho_2 - \rho_3) z_1 + \rho_3 z_2 + \rho_2 z_3 = 0 \\ \rho_2 z_1 + \rho_1 z_2 + (m_3 - \rho_2 - \rho_1) z_3 = 0 \\ m_1 z_1 + m_2 z_2 + m_3 z_3 = 0 \end{cases} \qquad (2.1.3)$$

The last equation expresses that the center of mass is at rest. Eqs. (2.1.3) are consistent if

$$\rho_1 = m_2 m_3 M^{-1} \quad , \quad \rho_2 = m_1 m_3 M^{-1} \quad , \quad \rho_3 = m_1 m_2 M^{-1} \qquad (2.1.4)$$

where M is the total mass $M = m_1 + m_2 + m_3$.

It follows that, for $p \neq -1$,

$$r_{jk} = \left(\frac{C_{jk}\, M}{\omega^2 m_j m_k} \right)^{\frac{1}{p+1}} \qquad j \neq k;\ j,k = 1,2,3 \qquad (2.1.5)$$

For $p = -1$, that is, for *harmonic forces* we have

$$\rho_1 = \omega^{-2} C_{23}, \quad \rho_2 = \omega^{-2} C_{31}, \quad \rho_3 = \omega^{-2} C_{12} \tag{2.1.6}$$

and $C_{jk} = \omega^2 M^{-1} m_j m_k$: a solution exists for arbitrary r_{jk}.

We notice that, if $C_{jk} = C$ and $m_j = m$ or if $C_{jk} = G m_j m_k$, then from (2.1.5), one sees that $r_{12} = r_{23} = r_{31}$ and the bodies form an equilateral triangle.

b) *Non-circular* case

Let us suppose $C_{jk} = C$ and $m_j = m$ (equal masses case). Let (x_j, y_j) be the coordinates at $t = t_0$, r_{jk} the initial mutual distances and (ξ_j, ξ_j) the coordinates at t (with mutual distances ρr_{jk}, ρ being the proportionality factor). We can write

$$\begin{aligned} \xi_j &= \rho(x_j \cos\theta - y_j \sin\theta) \\ \eta_j &= \rho(x_j \sin\theta - y_j \cos\theta) \end{aligned} \tag{2.1.7}$$

where θ is the rotation angle.

Since

$$\begin{aligned} \ddot{\xi}_k &= -\frac{C}{m} \sum_{j\neq k} \frac{\xi_k - \xi_j}{r_{jk}^{p+1}} \\ \ddot{\eta}_k &= -\frac{C}{m} \sum_{j\neq k} \frac{\eta_k - \eta_j}{r_{jk}^{p+1}} \end{aligned} \tag{2.1.8}$$

we are lead, from (2.1.7) and (2.1.8), to

$$\begin{aligned} \ddot{\rho} - \frac{y_k}{x_k}\frac{\dot{\psi}}{\rho} - \frac{\psi^2}{\rho^3} &= \frac{C}{m\rho^p x_k} \sum_{j\neq k} \frac{x_j - x_k}{r_{jk}^{p+1}} \\ \ddot{\rho} + \frac{x_k}{y_k}\frac{\dot{\psi}}{\rho} - \frac{\psi^2}{\rho^3} &= \frac{C}{m\rho^p y_k} \sum_{j\neq k} \frac{y_j - y_k}{r_{jk}^{p+1}} \end{aligned} \tag{2.1.9}$$

$$\psi = \rho^2\dot{\theta} \quad .$$

Equations (2.1.9) are necessary conditions for the existence of solutions. They are consistent if

$$\dot{\psi} = 0 \tag{2.1.10}$$

and

$$\sum_{j\neq k} \frac{x_k - x_j}{r_{jk}^{p+1}} = \alpha\, x_k$$
$$\sum_{j\neq k} \frac{y_k - y_j}{r_{jk}^{p+1}} = \alpha\, y_k \tag{2.1.11}$$

Equations (2.1.11) are verified if

$$r_{jk} = a \quad \text{and} \quad a = \frac{3}{a^{p+1}} . \tag{2.1.12}$$

And from (2.1 .9) and (2.1 .11), we get:

$$\ddot{\rho} - \frac{c_0^2}{\rho^3} = - \frac{3C}{ma^{p+1}} \frac{1}{\rho^p}$$
$$\dot{\theta} = \frac{c_0}{\rho^2} \tag{2.1.13}$$

The bodies, at any instant, form an equilateral triangle. The ratios between mutual disrances remain equal and constant (for p -2, see ref.8).

2.2. Euler type solutions

a) *Circular case*

A similar treatment lead us to the equations

$$- m_1\, \xi_1 = \omega^{-2} \left[\frac{C_{12}}{(\xi_2-\xi_1)^p} + \frac{C_{31}}{(\xi_3-\xi_1)^p} \right]$$
$$m_3\, \xi_3 = \omega^{-2} \left[\frac{C_{31}}{(\xi_3-\xi_1)^p} + \frac{C_{23}}{(\xi_3-\xi_2)^p} \right] \tag{2.2.1}$$
$$m_1\, \xi_1 + m_2\, \xi_2 + m_3\, \xi_3 = 0$$

Let us define

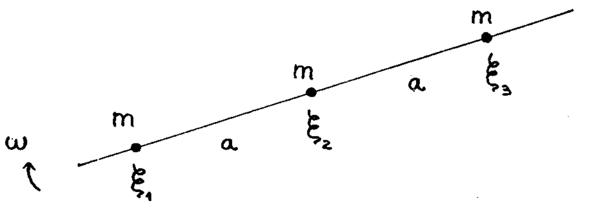

Fig.1 - Three-body configuration corresponding to Euler solution with equal masses, in the circular case.

$$
\begin{aligned}
a &= \xi_2 - \xi_1 \\
a\rho &= \xi_3 - \xi_2 \qquad (2.2.2) \\
a(1+\rho) &= \xi_3 - \xi_1 \ .
\end{aligned}
$$

We then obtain from (2.2.1), under the assurnption that $C_{jk} = C$ and $m_3 = m$

$$\frac{2+\rho}{1+2\rho} = \frac{1+(1+\rho)^{-p}}{(1+\rho)^{-p}+\rho^{-p}} \qquad (2.2.3)$$

Notice that $\rho = 1$ verifies (2.2.3) for any p and corresponds to

$$\xi_2 - \xi_1 = a, \quad \xi_3 - \xi_2 = a \ , \ \xi_3 - \xi_1 = 2a \ .$$

Thus, if $C_{jk} = C$ and $m_3 = m$, there are solutions for any p, corresponding to the situation of Fig.1. If $p = -1$, (2.2.3) becomes an identity and so one has a solution for arbitrary ξ_j.

b) *Non-circular case*

As in section 2.1 b (non-circular Lagrange solutions), wecan derive

$$
\begin{aligned}
\sum_{j\neq k} \frac{x_k - x_j}{r_{jk}^{p+1}} &= \alpha\, x_k \\
& \qquad\qquad (2.2.4) \\
\sum_{j\neq k} \frac{y_k - y_j}{r_{jk}^{p+1}} &= \alpha\, y_k
\end{aligned}
$$

In this case, one must have

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} = \frac{y_3}{x_3}$$

and (2.2.4) is verified if

$$r_{12} = r_{23} = a \ ; \quad r_{13} = 2a \ ; \quad \alpha = \frac{1+2^{-p}}{a^{p+1}} \ .$$

Furthermore, we have

$$\ddot{\rho} - \frac{c_0^2}{\rho^3} = - \frac{(1 + 2^{-p})C}{m\, a^{p+1}} \ \frac{1}{\rho^p}$$

$$\dot{\theta} = c_0 \ \rho^{-2} \ . \tag{2.2.5}$$

If $p = -1$, $\alpha = 3\, a^{-(p+1)}$, which coincides with the value of α for the triangular case (2.1.12).

In the non-circular case (triangular or collinear) it is possible, for p = -5, -3, -1, 0, 2, 3, 4, 5 and 7, to solve (by circular or elliptic functions) the integral that gives the orbits of the bodies[10].

If p=2, the orbits are conics with the center of mass at one focus. For p = -1, the orbits are still conics but the center is at the center of mass.

3. STABILITY OF THE SOLUTIONS

3.1. Stability of Lagrange type solutions

For $C_{jk} = G\, m_j\, m_k$, we shall consider small vibrations about steady motion, in the circular case.

Let us take the origin at m_3 and, as coordinates, the distances r_{13}, r_{23} and the angles α_1 and α_2 as in Fig. 2. As auxiliary variables we have: r_{12}, θ_1, θ_2 and θ_3. The ignorable coordinate does not appear explicitly.

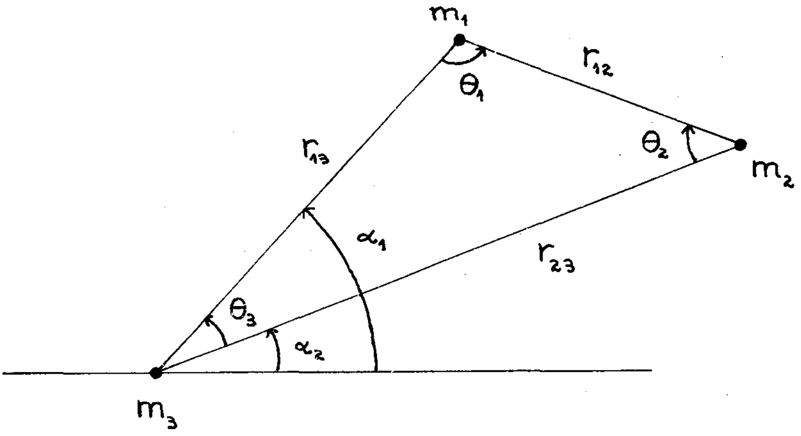

Fig.2 - Coordinate system and auxiliary variablasfor Lagrange's steady motion.

Writing the equations of motion in terms of those coordinates and taking solutions near Lagrange's steady motion, we can derive[9] a secular determinant from which results the condition $(p\neq 3)$:

$$K^2\left(M+\frac{K^2}{3-p}\right)\left[K^4+(3-p)MK^2+\frac{3}{4}\left(m_1m_2+m_1m_3+m_2m_3\right)(p+1)^2\right] \qquad (3.1.1)$$

where K stands for the frequency'of oscillation and M is the total mass.

The roots of (3.1.1) are

(a) $K^2=0$, which corresponds to the additional ignorable coordinate.

(b) $K^2=-(3-p)$, for which there are stable oscillatory modes if

$$p<3 \qquad (3.1.2)$$

(c) $K^2=-(3-p)\frac{M}{2} \pm \left[(3-p)^2M^2-3(p+1)^2(m_1m_2+m_2m_3+m_1m_3)\right]^{1/2}$,

which corresponds to stable modes if

$$M^2(3-p)^2>3(p+1)^2\,(m_1m_2+m_2m_3+m_1m_3)\,. \qquad (3.1.3)$$

For the case $p=3$, we verify that there are no stable modes.

Thus, (3.1.2) and (3.1.3) are the conditions of stability for Lagrange type circular solutions.

3.2. Stability of Euler type solutions

Let us suppose $C_{jk} = m^2G$ and take the coordinates r_{13}, r_{23}, α_1 and a as in Lagrange's case. The Euler type solutions correspond to the values

$$r_{13} = a, \quad r_{23} = 2a, \quad \alpha_1 = a = wt .$$

As in subsection 3.1, we look for solutions near the steady motion. From the equations of motion, we obtain, to the first order, four equations in the small variables, which lead to a secular determinant[3] whose roots are given by

(a) $K = 0$.

(b) $K^2 = (p-3)\omega^2$.

(c) $K^2 = -\frac{A}{2} \pm \frac{1}{2}(A^2 - 4B)^{1/2}$

where

$$A = \frac{\omega^2}{1 + 2^{-p}} (5 - 3p + 2^{-p+1})$$

$$B = -\frac{2\omega^4}{(1+2^{-p})^2} \left[1 + 2^{-(p+1)} + 3p(1-2^{-(p+1)}) - 2^{-(2p+1)}\right]$$

Thus, the conditions for stable modes are

from b) :

$$p < 3 \; ; \qquad (3.2.1)$$

from c) :

$$p < -3 \; . \qquad (3.2.2)$$

We conclude that only if $p < -3$, there is stability for Euier type circular solutions, in the equal masses case.

4. SEMI-CLASSICAL QUANTIZATION OF THE SOLUTIONS

In this section, we perform the quantization of the generalized solutions, applying first Bohr quantization to the simplest caseof circular solutions and Bohr-Sommerfeld quantization for the non-circular ones, for both Lagrangian and Eulerian type cases.

4.1. Bohr's quantization

a) *Lagrange circular solutions*

Let us suppose $C_{jk} = C$ and $m_j = m$. The total energy E is given, in this case, for $p \neq 1$, by

$$E = \frac{1}{2}\,\frac{p-3}{p-1}\,m\,\omega^2\,a^2 \tag{4.1.1}$$

with

$$\omega^2 = \frac{3C}{ma^{p+1}} \tag{4.1.2}$$

The angular momentum w.r.t the center of mass is

$$L = m\omega a^2 = n\hbar\ , \quad n = 1,\ 2,\ .. \tag{4.1.3}$$

It follows, fcr $p\neq 1$ and $p\neq 3$, that

$$E_n = \frac{1}{2}\,\frac{p-3}{p-1}\,(3C)^{\frac{2}{3-p}}\left(\frac{n^2\hbar^2}{m}\right)^{\frac{1-p}{3-p}}\ . \tag{4.1.4}$$

If $p=3$, we have $E=0$. For $p=1$, corresponding to the logarithmic two-body potential

$$V(r_{jk}) = C\,\ln\frac{r_{jk}}{r_0}\ , \quad \text{we get}$$

$$E_n = \frac{3C}{2}\left[1 + \ln\left(\frac{n^2\,\hbar^2}{3mCr_0^2}\right)\right]\ . \tag{4.1.5}$$

b) *Euler* type *circular solutions*

For $p\neq 1$ and $p\neq 3$, we obtain in a similar way, the following expression for the quantized energies (equal masses case).

$$E_n = 2^{\frac{2(1-p)}{p-3}} \frac{p-3}{p-1} \left[(1+2^{-p})C\right]^{\frac{2}{3-p}} \left(\frac{n^2\hbar^2}{m}\right)^{\frac{p-1}{p-3}} \tag{4.1.6}$$

Again, if $p=3$, we have $E=0$ and for $p=1$, we are lead to

$$E_n = \frac{3C}{2}\left[1 + \ln\left(\frac{n^2\hbar^2}{6mCr_c^2}\right)\right] \quad . \tag{4.1.7}$$

We see, from (4.1.4) and (4.1.6), that for $p=2$ one gets a Balmer-like spectrum. Notice also that, if $p=-1$ (harmonic forces), (4.1.4) and (4.1.6) give rise to identical expressions.

4.2. Bohr-Sommerfeld quantization of non-circular solutions

An important feature of the non-circular solutions is that the corresponding equations of motion can be derived from an Hamiltonian $H(q_\rho,p_\rho)$, where the canonical variable is $q_\rho = \rho$ (ρ is the already defined ratio of the mutual distances between the bodies) and the conjugate momentum $p_\rho = ma^2\dot{\rho}$. Indeed, from the expressions for the kinetic and potential energies of the system, we obtain

$$H(\rho,p_\rho) = \frac{1}{2}\frac{p_\rho^2}{ma^2} + \frac{1}{2}\, m\, c_0^2\, a^2\, \rho^{-2} - \begin{cases} \dfrac{\alpha C a^2}{p-1}\,\rho^{1-p} & , \text{ if } p\neq 1 \\ -\alpha C a^2 \ln\rho & , \text{ if } p=1 \, , \end{cases} \tag{4.2.1}$$

with

$$\alpha = \begin{cases} a^{-(p+1)} & \text{(triangular case)} \\ (1+2^{-p})\,a^{-(p+1)} & \text{(collinear case)} . \end{cases}$$

Therefore, for those solutions, the system of three bodies is equivalent to one body under the action of a central force at the system's center of mass. In order to accomplish the Bohr-Sommerfeld quantization, we introduce the action variable $J_\rho = \oint p_\rho \, d_\rho$, where p_ρ is obtained by means of (4.2.1) through the equation $H(\rho, p_\rho) = E$. The action integral can then be solved by themethod of residues, for the cases $p=2$, $p=-1$ and $p = 3$.

a) *Lagrange type solutions*

We quote the following results: i) $p = 2$,

$$E_n = -\frac{9mC^2}{2\hbar^2}\frac{1}{n^2} \quad , n = 1, 2, \ldots \tag{4.2.2}$$

a result which coincides with that obtained from (4.1.4) for $p = 2$;

ii) for $p = -1$,

$$E_n = \left(\frac{3C}{m}\right)^{1/2} \hbar n \quad , \qquad n = 1, 2, \ldots \tag{4.2.3}$$

Again, the result coincides with (4.1.4) for this value of p.

b) *Euler* type *solutions*

We obtain the following results:

i) for $p = 2$,

$$E_n = -\frac{25}{16}\frac{mC^2}{\hbar^2}\frac{1}{2n^2} \quad . \tag{4.2.4}$$

ii) for $p = -1$,

$$E_n = \left(\frac{3C}{m}\right)^{1/2} \hbar n \quad . \tag{4.2.5}$$

4.3. The WKB approximation

Starting from the Hamiltonian (4.2.1), which describes the solutions under consideration, we can write the corresponding Schrödinger equation, whose eigenvalues are the quantized energies of the system. For instance, for the Newtonian case (p=2), in the unequal masses case, one can readily derive, from well-known quantum-mechanical results, the expression:

$$E_n = - \frac{G^2 (m_1 m_2 + m_2 m_3 + m_3 m_1)^3}{(m_1 + m_2 + m_3)\hbar^2} \frac{1}{2n^2}, \quad n = 1, 2, \ldots \tag{4.3.1}$$

valid for the Lagrangian solution.

The same formula can be obtained be means of Bohr or Bohr-Sommerfeld quantization conditions.

We may apply the WKB approximation to get energy expressions in some significant cases (for instance, the logarithmic potential case p=1, and the linear potential case p=0).

With the Hamiltonian (4.1.1), we have

$$J_\rho = \oint p_\rho \, d\rho = a \oint \left[2m(E-V_{eff})\right]^{1/2} d\rho \tag{4.3.2}$$

where

$$V_{eff}(\rho) = \frac{ma^2 c_0^2}{2\rho^2} - \begin{cases} \dfrac{\alpha C a^2}{(p-1)\rho^{p-1}} \\ -\alpha C a^2 \ln \rho \, . \end{cases} \tag{4.3.3}$$

In the WKB approximation for the three-dimensional problem, we have

$$J_\rho = 2\pi\hbar \left(n - \frac{1}{4}\right), \quad n = 1, 2, \ldots \tag{4.3.4}$$

Considering the case of S-waves (c_0=0), the results for *Lagrange* **type** *solutions* are:

i) $p = 1$, $V_{\text{eff}}(r) = 3C \ln \frac{r}{r_0}$;

$$E_n = 3C \ln \left[(n - \tfrac{1}{4}) \left(\frac{2\pi}{mC}\right)^{1/2} \frac{\hbar}{r_0} \right] . \tag{4.3.5}$$

ii) $p = 0$, $V_{\text{eff}} = 3Cr$;

$$E_n = \frac{1}{2} \left[9 \frac{C \hbar \pi}{\sqrt{m}} (n - \tfrac{1}{4}) \right]^{2/3} . \tag{4.3.6}$$

For $n = 1$, (4.3.6) gives

$$E_1 = 2.3202 \left(\frac{9C^2\hbar^2}{2m} \right)^{1/3} .$$

(notice that the exact factor is 2.3381).

For the *Euler type solutions*, we obtain:

i) $p = 1$, $V_{\text{eff}} = \frac{3C}{2} \ln \frac{r}{r_0}$,

$$E_n = \frac{3C}{2} \ln \left[(n - \tfrac{1}{4}) \left(\frac{2\pi}{mC}\right)^{1/2} \frac{\hbar}{r_0} \right] . \tag{4.3.7}$$

ii) $p = 0$, $V_{\text{eff}} = 2\, Cr$;

$$E_n = \frac{1}{2} \left[6 \frac{C\hbar\pi}{\sqrt{m}} (n - \tfrac{1}{4}) \right]^{2/3} . \tag{4.3.8}$$

The above results are to be compared with those obtained in section (4.1) for $p = 0$ and $p = 1$.

5. FINAL REMARKS

We have shown that it is possible to generalize Lagrange and Euler type exact solutions to the three-body problem for forces proportional to r^{-p} (p integer).

The conditions of stability of the circular solutions have also been established for arbitrary values of p.

For $p=-1$ (harmonic forces), there are triangular and collinear solutions for arbitrary r_{jk}. This occurs because, as is well known, the problem is always separable. Equilateral triangular solutions are allways stable, even if the masses are different. Collinear solutions with equal masses are not stable. The quantized energies (equal masses) are identical in Lagrange's and Euler's case (circular and non-circular, see eqs. 4.2.3 and 4.2.5).

For Newtonian forces ($p=2$) with equal masses, the solutions are not stable (see eqs. 3.1.3 and 3.2.2). In both cases, a Balmer-like formula is obtained, even if the masses are distinct (see for instance, eq. 4.?.1).

We notice that it was shown[9] that particular solutions also exist for the case of four-bodies with forces proportional to r^{-p}, which are analogous to the Lagrange and Euler solutions here discussed for the 3-body problem.

Finally, we think it would be interesting to speculate about the possible existence of microscopical systems realizing the Lagrangian or Eulerian solutions in that domain. However, no attempt in this direction was made here.

REFERENCES AND NOTES

1. Lagrange, J.L. "Essai sur le Problème des Trois Corps" (1772), Oeuvres, vol. VI, Paris.
2. A proof that the movement in this case occurs in a fixed plane was given by Lagrange, and in a more elementary form by Carathéodory (see ref. 3).
3. Carathéodory, C. "Uber die Strengen Lösungen des Dreikörperproblems", Sit. Bayr. Akad. Wiss. 257, 1933.
4. Euler, L. "De motu rectilineo trium corporum se mutuo attrahentium", Novi Comm. Acad. Sci. Imp. Petrop. *11*, 144 (1767).
5. Siegel, C.L.: *Vorlesungen* Über *Himmelsmechanik*, Springer-Verlag, Berlin, 1956.
6. Some authors refer to both of them as Lagrange solutions. Following

H. Pollard (ref.7), we prefer to distinguish between Euler's (collinear) and Lagrange's (triangular) case.

7. Pollard H.: Mathematical Introduction to Celestial Mechanics, Prentice-Hall Inc., New Jersey, 1966.

8. Moulton, F.R. : Celestial Mechanics, Macmillan Comp., London, 1947.

9. R. de Azeredo Campos, "Soluções Lagrangianas do Problema de 3 Corpos com Forças proporcionais a r^{-p}", Tese de Mestrado, IFT, São Paulo, 1979.

10. Whittaker, E.T. : Analytical Dynamics, Dover Publ., New York, 1944.