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Models for Colour-Confinement

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We look for observable effects arising from the saturation of colour confining forces, as suggested by recent theoretical analises. We try to put in evidence any effect due to a "plateau" in the confining potential of the $Q\bar{Q}$ system, by taking into account possible modifications in the mass spectrum and the leptonic widths of the narrow states. The case of the potential with a "plateau" is completely solved and compared to the pure linear case. The more important case of a Coulomb-plus -linear potential is also considered, using variational wave functions. We argue that there is no empirical evidence against the hypothesis of saturation.

Procuramos efeitos observáveis originados da saturação de forças que confinam a cor, como sugerido por análises teóricas recentes. Tentamos pôr em evidência qualquer efeito devido a um "plateau" no potencial confinante do sistema $Q\bar{Q}$, levando em conta possíveis modificações no espectro de massa e nas larguras leptônicas dos estados estreitos. O ca-

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so do potencial com um "plateau" é resolvido completamente e comparado ao caso linear puro. O caso, mais importante, de um potencial "Coulombiano mais linear", é também considerado, usando-se funções de onda variacionais. Sustentamos que não há evidência empírica contra a hipótese de saturação.

1. INTRODUCTION

Recently, in the framework of the massive Schwinger model in 1 + 1 dimensions, there appeared an analysis of the concept of confinement of colour[1], which leads, for heavy quarks, to a continuous quark-antiquark potential, with a linear rising part λr and a characteristic constant "plateau" V_0 located beyond some distance r_0, with $V_0 = \lambda r_0$.

The same type of potential resulted from a theoretical work on a 4-dimensional Euclidean lattice gauge theory[2].

In view of the relevance of those theoretical analyses, we performed a phenomenological study of the above potential, in the context of the quarkonium[3] (i.e., of the narrow states of the type $Q\bar{Q}$ with $Q = c,b,\ldots$, interpreted as n^3S_1 states).

Several treatments of the quarkonium have appeared recently. However, in contrast with the *colour confinement* potential, they all admit a linear rising part to infinity in order to guarantee the *quark* confinement.

Since the constituents are heavy, a non-relativistic treatment is licit for quarkonia.

Besides reproducing the masses of the narrow states of the $\psi(c\bar{c})$ family and of the $\Upsilon(b\bar{b})$ family, one requires that the correct leptonic widths of those states be also obtained, via the well-known Van Royen-Weisskopf formula[4]. A further constraint is that $\langle\beta^2\rangle \ll 1$, where β is the velocity of the constituent in units of the light velocity, in order to assure the validity of the non-relativistic approximation involved.

Our main hypothesis is to assume that the height of the "plateau" V_0 is given by the Zweig threshold for the decay of $Q\bar{Q}$ in pairs $Q\bar{q} + \bar{Q}q$, where q represents a light quark. It is a reasonable assumption, suggested by the phenomenology of the $Q\bar{Q}$ states.

The Schrödinger equation for the proposal potential admits an exact solution for S waves.

In section 2, the solution obtained is exhibited, together with expressions obtained for $|\psi_n(0)|^2$, the Bohr radius and $\langle\beta^2\rangle$. Section 3 is devoted to a quantitative discussion based on our assumption of taking the height of the plateau to be the Zweig threshold.

It is then concluded that the existence of the "plateau" does not affect the results of a pure linearly rising potential.

In section 4, we consider a coulombic correction at small distances (imputable to gluon exchange) which is currently included in phenomenological fittings of quarkonia. We conclude that the presence of the "plateau" does not damage the sucessful results which have been obtained through the use of the more conventional potential which rises indefinitely.

2. S-WAVE SOLUTIONS

The colour-confinement potential[1,2] may be written as

$$V(r) = \lambda r\ \theta(r-r_0) + V_0\ \theta(r_0-r), \qquad (1)$$

where $\theta(r)$ is the Heaviside step-function.

Continuity at r_0 gives

$$V_0 = \lambda r_0 \ . \qquad (2)$$

The Schrödinger equation for the potential (1) admits exact solutions for S-waves in terms of Airy functions. The energy eigenvalues are obtained from a transcedental equation derived by "matching" the logarithmic derivatives of the wave function at both sides of r_0.

Since the derivation is elementary, we simply exhibit the results obtained.

Let

$$u(r) = r\,\psi(r) \quad , \tag{3}$$

with $u(0) = u(\infty) = 0$.

One gets, for $E < V_0$

$$u(r) = N_1\left[Ai(\rho(r)) - \frac{Ai(\rho(0))}{Bi(\rho(0))}\,Bi(\rho(r))\right] \quad ,$$

$$u(r) = N_2\, e^{-kr} \tag{4}$$

In (4), $Ai(\rho)$ and $Bi(\rho)$ are Airy functions in the dimensionless variable

$$\rho(r) = (2\mu\lambda)^{1/3}\,(r - \frac{E}{\lambda}) \quad , \tag{5}$$

where μ is the reduced mass of the system and

$$k^2 = 2\mu\,(V_0 - E) \tag{6}$$

The normalization constant N_1 and N_2 are given by

$$N_1 = N_2\,\frac{Bi(\rho(0))e^{-a}}{V(\eta,\omega)} \quad ,$$
$$N_2 = \left[\frac{a}{2\pi r_0}\;\frac{e^{2a}}{1 + \dfrac{2a}{\pi\rho_0 V^2(\eta,\omega)}}\right] \quad , \tag{7}$$

where

$$a = kr_0 \quad , \quad \rho_0 = (2\mu\lambda)^{1/3}\,r_0 \tag{8}$$

and

$$V(\eta,\omega) = Ai(\rho(r_0))Bi(\rho(0)) - Ai(\rho(0))Bi(\rho(r_0)) \quad . \tag{9}$$

The "matching" condition at r_0 leads to the equation

$$\eta + \frac{Ai'(\eta^2) - \dfrac{Ai(\eta^2-\omega^2)}{Bi(\eta^2-\omega^2)} Bi'(\eta^2)}{Ai(\eta^2) - \dfrac{Ai(\eta^2-\omega^2)}{Bi(\eta^2-\omega^2)} Bi(\eta^2)} = 0 \quad . \tag{10}$$

with $\eta < \omega$, where

$$\eta = \frac{a}{\rho_0} \quad \text{and} \quad \omega = \frac{\sigma^{1/2}}{\rho_0} = \frac{(2\mu V_0 r_0)^{1/2}}{\rho_0} , \tag{11}$$

Notice that the energy eigenvalues, in view of eqs.(6), (8) and (11), are given by

$$E = V_0 - \frac{\rho_0^2}{2\mu\, r_0^2} \eta^2 , \tag{12}$$

η being a solution of eq. (10).

For the modulus square of the wave-function at the origin, one has, in terms of already defined quantities,

$$|\psi(0)|^2 = \frac{2\mu\lambda}{4\pi} \left[1 - \left(1 + \frac{2\eta}{\pi^2\, V^2(\eta,\omega)} \right)^{1} \right] \quad . \tag{13}$$

The value of $<\beta^2>$ is given by

$$<\beta^2> = \frac{V_0}{6\mu} \left[\frac{\omega^2-\eta^2}{\rho_0} - \frac{1}{1 + \dfrac{2\eta}{\pi^2\, V^2(\eta,\omega)}} \right] \quad . \tag{14}$$

Finally, the "Bohr radius" may be expressed as

$$<r> = \frac{\omega^2-\eta^2}{2\eta\omega^4} (1 + 2\eta\rho_0) r_0 + <\beta^2> \left[\frac{4\mu}{\lambda} - \frac{3\mu r_0}{\eta\rho_0 V_0} (1 + 2\eta\, V_0) \right] \quad . \tag{15}$$

Before closing this section, we quote the result for the energy levels in the pure linear case

$$E_n = \left(\frac{\lambda^2}{2\mu}\right)^{1/3} \quad |a_n| \ , \tag{16}$$

where a_n is the nth zero of $Ai(z)$.

3. QUANTITATIVE CONSIDERATIONS

As it was mentioned before, in the present treatment we take V_0 as being the Zweig threshold for the decays of $Q\bar{Q} \rightarrow Q\bar{q} + \bar{Q}q$. For the ψ-family, this corresponds to decays in $D\bar{D}$ pseudoscalars where $M_D = 1.86$ GeV. As we have for the mass M of the narrow states of the $Q\bar{Q}$ system

$$M = 2(2\mu) + E < 2\ M_D \ ,$$

we then assume

$$V_0 = 2\ M_D - 2(2\mu) \ . \tag{17}$$

If we take for 2μ and λ, the values usually considered in the current phenomenological analysis of the ψ-family, we get, from eqs.(2) and (17), in general, rather large values for r_0. Hence, one may expect that the influence of "plateau" be small and that energy values very near to those corresponding to the pure linear case should be obtained.

In fact, consider a fitting by a pure linear potential without "plateau". We get from eq. (10) the following relations:

$$\left(\frac{\lambda^2}{2\mu}\right)^{1/3} (4.08795 - 2.33811) = M_{\psi'} - M_{\psi} = 0.591 \text{ GeV} \tag{18}$$

and

$$M = 2(2\mu) + \left(\frac{\lambda^2}{2\mu}\right)^{1/3} 2.33811 = 3.095 \text{ GeV} \ . \tag{19}$$

In such a way we determine the values of the parameters λ and 2μ:

$$\lambda = 0.21 \text{ GeV}^2 \quad ; \quad 2\mu = 1.15 \text{ GeV} \tag{20}$$

Through the use of eq.(17), once 2μ is known, we can find V_0. And then using eq. (2), we find r_0 also. Then:

$$V_0 = 1.44 \text{ GeV} \quad ; \quad r_0 = 6.86 \text{ GeV} \tag{21}$$

Now that the parameters of the "plateau" have been determined we have to consider how it influences the energy eigenvalues, which are given by eq. (10). First of all, notice that by imposing the condition:

$$Ai(\eta^2-\omega^2) = 0 \ , \tag{22}$$

eq.(10) turns out into the identity below

$$\frac{Ai'(\eta^2)}{Ai(\eta^2)} + \eta = 0 \ . \tag{23}$$

Using eqs.(8), (2) we can reduce eq.(12). which gives the energy eigenvalues for the potential with the "plateau", to a more suitable form, namely:

$$E = \left(\frac{\lambda^2}{2\mu}\right)^{1/3} (\omega^2-\eta^2) \ . \tag{24}$$

It is evident that eq. (24) is identical to eq.(16), since the roots of eq. (10) have been reduced to those of eq.(22).

So the occurrence of a "plateau" does not affect the energy levels of the narrow states (i.e. located below threshold) at all.

We can, as so easily, investigate the possible effects of the "plateau" on the leptonic widths, or, equivalently, on the wave functions at the origin of the narrow states. As discussed before, the "plateau" tends to reduce $|\psi(0)|^2$ by a known factor, explicitly given in eq.(13). The relevant quantity here, the function $V(\eta,\omega)$, in view of eq.(22), simply reduces to

$$V(\eta,\omega) = Ai(\eta^2)\ Bi(\eta^2-\omega^2) \ , \tag{25}$$

which can be readily calculated since we know η^2 and ω^2.

Calling $V_n(\eta,\omega)$ the value that this function takes for the nth radial excitation, we obtain

$$|V_1(\eta,\omega)| = 1.4\ 10^{-2}$$

$$|V_2(\eta,\omega)| = 0.11 \ . \tag{26}$$

These values of $V(\eta,\omega)$ are so small that there is no noticeable change in $|\psi(0)|^2$. In fact, for the ψ the correction is completely negligible, while for the ψ' it amounts at most to a reduction of about 30%. In any event, this effect of 30% is difficult to check with experimental data because of the uncertainty in the mixing between the states $2\ ^3S_1$ and $3\ ^3D_1$, which is not calculable in the framework of the non-relativistic approximation.

Corrections to $|\psi_2(0)|^2$ are larger than to $|\psi_1(0)|^2$ by the simple reason that the $2\ ^3S_1$ state is more spread in space than the $1\ ^3S_1$ -state, so that it senses better the presence of the "plateau"(*)

The suggestive result obtained in this section tells us that the "plateau" does not substantially affect the properties of the bound sates. The more realistic case of a Coulomb-plus-linear potential will be considered in the next section.

4. INFLUENCE OF THE "PLATEAU" ON A COULOMB-PLUS-LINEAR POTENTIAL MODEL

It is important to establish if the saturation of the linear confining forces will affect in any circumstance the quarkonium phenomenology, which is sucessfully provided by the potential below

$$U_\infty(r) = -\frac{\gamma}{r} + \lambda r, \tag{27}$$

where, as usual

$$\gamma = \frac{4}{3}\alpha_s . \tag{28}$$

Our aim is to find which are the possible shifts in the energy levels of the bound states and the modifications on the wave functions at the origin, when the potential eq. (27) is replaced by

$$U(r) = -\frac{\gamma}{r} + V(r) , \tag{29}$$

(*) A calculation with the linear potential, furnishes the mean radii:

$$\langle r_1\rangle = 2.5\ \text{GeV}^{-1} \quad ; \quad \langle r_2\rangle = 4.4\ \text{GeV}^{-1}.$$

where $V(r)$ is given in eq.(1). The potential $U_\infty(r)$ is considered as a limit case of $U(r)$, for $r_0 \to \infty$.

In this section we shall adopt the convention of using the subscript ∞ to designate quantities obtained from the potential $U_\infty(r)$ (eq.(27)).

Unfortunately, the potential $U_\infty(r)$ does not admit closed analytical solutions and the problem is generally handled with numerical integration of the Schrödinger equation.

A possible issue to this problem is to use trial wave functions by means of a variational principle. It has the advantage, besides being easier than numerical integration, of rendering more transparent the effects of the "plateau".

It seems that the presence of the "plateau" is possibly relevant only for charm. Heavier flavours have a so small mean radius that certainly could not feel any effect (for the upsilon one finds $\langle r \rangle = 0.98$ GeV^{-1}, $r_0 = 5.4$ GeV^{-1}), on the other side, lighter mesons are all above the Zweig threshold.

So we have to consider only the first two radial excitations of S-waves. We take harmonic oscillator wave functions as the trial functions

$$\begin{aligned} \psi_1(r) &= N_1 \, e^{-\frac{\Omega r^2}{2}} , \\ \psi_2(r) &= N_2 \, e^{-\frac{\Omega r^2}{2}} (r-c) , \end{aligned} \tag{30}$$

where N_1, N_2 are normalization constants. The wave function of $1S$-state being nodeless, while the wave function of the $2S$-state has one node.

The wave functions are normalized as usual by

$$4\pi \, N_n^2 \int_0^\infty |\psi_n(0)|^2 \, r^2 \, dr = 1 \; . \tag{31}$$

Such that

$$4\pi\ N_1^2 = \frac{4}{\sqrt{\pi}}\ \Omega^{3/2}\ ,$$

$$4\pi\ N_2^2 = \frac{1}{B_2 c^2 + B_1 c + B_0} \qquad (32)$$

and

$$B_2 = \frac{\sqrt{\pi}}{4\Omega^{3/2}}\ ;\quad B_1 = -\frac{1}{\Omega^2}\ ;\quad B_0 = \frac{3\sqrt{\pi}}{8\Omega^{5/2}}\ . \qquad (32a)$$

The quantity Ω is determined by requiring that it corresponds to a minimum of the ground state energy

$$<H> = <T> + <U>\ , \qquad (33)$$

where $<T>$ and $<U>$ are the expectation values of the kinetic and potential energies, respectively.

The minimization condition reads:

$$\frac{\partial <H>_1}{\partial \Omega} = 0 \quad ; \quad \frac{\partial^2 <H>_1}{\partial \Omega^2} > 0\ . \qquad (34)$$

Since Ω has been found, the position of the zero of the $2S$-wave functions is determined by a similar argument

$$\frac{\partial <H>_2}{\partial c} = 0 \quad ; \quad \frac{\partial^2 <H>_2}{\partial c^2} \quad 0\ . \qquad (35)$$

A straightforward but lengthy calculation gives the following expressions for the energy eigenvalues, in the case of the standard potential eq. (27)

$$<H>_{\infty,1} = \frac{3}{2}\ \frac{\Omega}{2\mu} + \frac{2\lambda}{\sqrt{\pi}}\frac{1}{\Omega} - \frac{2\gamma}{\sqrt{\pi}}\ \Omega^{1/2}\ . \qquad (36)$$

and for the $n = 2$ state,

$$<H>_{\infty,2} = 4\pi\ N_2^2 (A_2 c^2 + A_1 c + A_0) \qquad (37)$$

where

$$A_2 = \frac{3\sqrt{\pi}}{8m\Omega^{1/2}} - \frac{\gamma}{2\Omega} + \frac{\lambda}{2\Omega^2} ,$$

$$A_1 = -\frac{1}{m\Omega} + \frac{\gamma\sqrt{\pi}}{2\Omega^{3/2}} - \frac{3\lambda\sqrt{\pi}}{4\Omega^{5/2}} , \qquad (38)$$

$$A_0 = \frac{7\sqrt{\pi}}{16m\Omega^{3/2}} - \frac{\gamma}{2\Omega^2} + \frac{\lambda}{\Omega^3} ,$$

$4\pi N_2^2$ being given in eq. (32).

The equation determining the value of Ω, which can be obtained from eq. (34), is

$$\frac{3}{2}\frac{1}{2\mu} - \frac{\lambda}{\sqrt{\pi}\,\Omega^{3/2}} - \frac{\gamma}{\sqrt{\pi}\,\Omega^{1/2}} = 0 . \qquad (39)$$

The parameter c is found through eq. (35), which reduces to

$$(A_2B_1 - B_2A_1)c^2 + 2(A_2B_0 - B_2A_0)c + (A_1B_0 - B_1A_0) = 0 \qquad (40)$$

where the A's and B's are given in eqs. (38) and (32), respectively.

The wave functions at the origin can, also, be found. They are given by the integral relation[5],(*).

$$|\psi(0)|^2 = \frac{2\mu}{4\pi} < \frac{dU(r)}{dr} > . \qquad (41)$$

In the case of the standard potential we get the simple expressions:

$$|\psi_1(0)|^2_\infty = \frac{2\mu}{4\pi}\left[\lambda + 2\gamma\Omega\right] ,$$

$$|\psi_2(0)|^2_\infty = \frac{2\mu}{4\pi}\left[\lambda + 4\pi N_2^2\,\gamma\left(\frac{\sqrt{\pi}\,c^2}{2\Omega^{1/2}} - \frac{c}{\Omega} + \frac{\sqrt{\pi}}{4\Omega^{3/2}}\right)\right] . \qquad (42)$$

(*) The procedure of taking the limit $r \to 0$ in eq. (30) is not reliable, since in this way $|\psi(0)|^2$ should depend to much on the shape of the trial function.

In the sames lines we determine the mean radius

$$
\begin{aligned}
\langle r_1 \rangle_\infty &= \frac{2}{\sqrt{\pi}\,\Omega^{1/2}} \\
\langle r_2 \rangle_\infty &= 4\pi\, N_2^2 \left\{ \frac{c^2}{2\Omega^2} - \frac{3\sqrt{\pi}}{4\Omega^{5/2}}\, c + \frac{1}{\Omega^3} \right\}
\end{aligned}
\tag{43}
$$

where $4\pi\, N_2^2$ is given in eq. (32).

Let us now test the accuracy of the variational method. A good fitting of charmonium, through the potential $U_\infty(r)$, is obtained in Ref. (5), with the following values of the parameters:

$$2\mu = 1.37 \text{ GeV} \quad ; \quad \lambda = 0.17 \text{ GeV}^2$$

$$\alpha_s = 0.38 \longrightarrow \gamma = 0.51 \tag{44}$$

The corresponding values of Ω and c which can be obtained from eqs. (39) and (40), respectively, are:

$$\Omega = 0.3037 \text{ GeV}^2 \quad ; \quad c = 1.9961 \text{ GeV}^{-1} \tag{45}$$

Calculating the energy eigenvalues, wave functions at the origin and mean raddi, we get:

	Variational	Exact (Ref.(5))
$M_1^{(\infty)}$ (GeV)	3.1035	3.097
$M_2^{(\infty)}$ (GeV)	3.868	3.686
$\lvert\psi_1(0)\rvert^2_\infty$ (GeV^3)	5.3 10^{-2}	6.3 10^{-2}
$\lvert\psi_2(0)\rvert^2_\infty$ (GeV^3)	8.7 10^{-2}	4.4 10^{-2}
$\langle r_1\rangle_\infty$ (GeV^{-1})	2.0	2.0
$\langle r_2\rangle_\infty$ (GeV^{-1})	2.6	5.0

(46)

In (46) a comparison with the exact results are also provided. The $1S$--state is well reproduced. While the efficiency of the variational approach considered here is poorer for the $2S$-state(*). Anyway this degree of accuracy proves to be good enough to our present purposes.

4.1. The Effect of the "Plateau"

The use of the trial functions exposed before rather simplifies the comparison between the two potentials.

We shall work in detail the case of the energy eigenvalues of the $1S$-state, as an example. This quantity is given by eq.(33) where in $\langle U\rangle$ we have to insert the potential with the "plateau". Quite simply we get:

$$\langle H\rangle_1 = \langle H\rangle_{1,\infty} + 4\pi N_1^2 \int_{r_0}^{\infty} dr\, e^{-\Omega r^2} (V_0 - \lambda r) r^2 \tag{47}$$

if we adopt the convention of designating by ΔQ the shift due to the plateau, in the quantity Q:

$$Q = Q_\infty + \Delta Q \tag{48}$$

we write

$$\begin{aligned}\Delta\langle H\rangle_1 &= 4\pi N_1^2 \lambda \int_{r_0}^{\infty} dr \quad e^{-\Omega r^2}\, r^2 (r_0 - r) \\ &= V_0\, erfc\, (\sqrt{\Omega}\, r_0) - \frac{\lambda}{\Omega^{1/2}} \frac{2}{\sqrt{\pi}} e^{-\Omega r_0^2} ,\end{aligned} \tag{49}$$

where the continuity condition, eq. (2), has been used. And similarly,

$$\Delta \left(\frac{\partial \langle H\rangle_1}{\partial \Omega} \right) = \frac{2}{\sqrt{\pi}} e^{-\Omega r_0^2} \left(\frac{\lambda}{\Omega^{3/2}} + \frac{V_0\, r_0}{\Omega^{1/2}} \right) . \tag{50}$$

Notice that the additional terms arising from the "plateau" are exponentially damped, confirming our expectation that they are negligible for large enough values of r_0.

(*) This drawback can be eliminated by considering, as trial function of the $2S$-state, a superposition of several gaussians, such that the shape of the wave function could be better reproduced.

On the same lines we get, for the $2S$-state:

$$\Delta\langle H\rangle_2 = 4\pi\, N_2^2\, \lambda \int_{r_0}^{\infty} dr\, e^{-\Omega r^2}\, r^2 (r_0 - r)(c - r)^2 \,,$$

$$\Delta\left(\frac{\partial\langle H\rangle_2}{\partial c}\right) = 8\pi\, N_2^2\, \lambda \int_{r_0}^{\infty} dr\, e^{-\Omega r^2}\, r^2 (r_0 - r)(c - r) \qquad (51)$$

The corrections on the wave functions at the origin aregiven below, respectively for the $1S$ and $2S$-states:

$$\Delta|\psi_1(0)|^2 = -\frac{2\mu}{4\pi}\,(4\pi\, N_1^2\, \lambda) \int_{r_0}^{\infty} dr\, r^2\, e^{-\Omega r^2} \,,$$

$$\Delta|\psi_2(0)|^2 = -\frac{2\mu}{4\pi}\,(4\pi\, N_2^2\, \lambda) \int_{r_0}^{\infty} dr\, r^2 (r-c)^2\, e^{-\Omega r^2}. \qquad (52)$$

From the values given in (44), we calculate V_0 and r_0

$$V_0 = 1 \text{ GeV} \quad ; \quad r_0 = 5.9 \text{ GeV}^{-1} \,. \qquad (53)$$

Now we can prove that the values of Ω and c determined before, i.e. in the case of the indefinitely rising potential, are not changed by the "plateau". In fact, the additional term which appears in eqs. (39) and (40) is completely negligible, if evaluated with the previous values (45).

For instance:

$$\Delta\left(\frac{\partial\langle H\rangle_1}{\partial\Omega}\right) \simeq 1.7\ 10^{-4} \text{ GeV}^{-1} \,. \qquad (54)$$

The shifts in the energy eigenvalues are estimated to be

$$\Delta\langle H\rangle_1 \simeq -\,4.6\ 10^{-6} \text{ GeV}.$$

$$\Delta\langle H\rangle_2 \simeq -\,1.7\ 10^{-4} \text{ GeV}. \qquad (55)$$

The corrections to the wave functions at the origin are also extremely small

$$\Delta|\psi_1(0)|^2 \simeq -\ 2.10^{-6}\ \text{GeV}^3\ ,$$
$$\Delta|\psi_2(0)|^2 \simeq -\ 5\ 10^{-5}\ \text{GeV}^3\ . \qquad (56)$$

A similar calculation for the upsilon family precludes a study of the "plateau" by heavier systems, the corrections being much smaller than for the psion family.

We could ask for a more accurate treatment of the $2S$-state, since the variational wave functions introduced here are not faithful enough. Nevertheless, as remarked before, a more detailed analysis of the $2S$-state would not be decisive in disentangling the "plateau" because of the uncertainties associated to the $\psi'(3686)/\psi''(3772)$ mixing. Finally, we remark that hyperfine corrections are not interesting for this sake, because they are sensible only to the small-distance behaviour of the potential.

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