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The Green's Function for the Associated Legendre Differential Equation

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The Green's function for the associated Legendre differential equation is calculated by means of the isotropic harmonic oscillator Green's function.

Calcula-se a função de Green para a equação diferencial associada de Legendre usando a função de Green para um oscilador harmônico isotrópico.

The present paper contains a derivation of the Green's function for the associated Legendre differential equation.

In quantum mechanics we know that the radial part of the Schrödinger equation for the Coulomb problem in momentum space is a associated Legendre differential equation[1]. It is well known that there is a connection between the radial Schrödinger equations for the Coulomb problem and the isotropic harmonic oscillator[2]. There is also a relationship between the corresponding radial Green's functions[3].

We use this relationship to calculate the Green's function for the associated Legendre differential equation.

The associated Legendre differential equation is

$$\left[(1-z^2)\frac{d^2}{dz^2} - 2z\frac{d}{dz} + \ell(\ell+1) - \frac{m^2}{1-z^2}\right]\psi(z) = 0 \tag{1}$$

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for

$$0 \leq z \leq \infty .$$

write

$$\psi(z) = \left[(1-z)/(1+z)\right]^{m/2}\phi(z)$$

and get for $\phi(z)$

$$\left[(1-z^2)\frac{d^2}{dz^2} - 2(m+z)\frac{d}{dz} + \ell(\ell+1)\right]\phi(z) = 0 \qquad (2)$$

We calculate the Green's function $g(z,z')$ for the eq. (2), which satisfies the inhomogeneous differential equation

$$\left[(1-z^2)\frac{d^2}{dz^2} - 2(m+z)\frac{d}{dz} + \ell(\ell+1)\right]g(z,z') = \delta(z-z') \qquad (3)$$

Fourier transforming Eq.(3), we get in momentum space

$$\left[\frac{d^2}{dp^2} + \frac{2}{p}\frac{d}{dp} + 1 - \frac{2mi}{p} - \frac{\ell(\ell+1)}{p^2}\right]F(p,p') = -1/p^2 \quad \delta(p-p') \qquad (4)$$

This differential equationis similar to the radial equation for the tridimensional Coulomb problem in configuration space. As showed by Bellandi and Caetano this Green's function can be easily calculated by means of the radial Green's function for the isotropic harmonic oscillator and we can write for $F(p,p')$ the following integral representation.

$$F(p,p') = -(pp')^{-1/2}\int_0^\infty dv \ \coth^{-2m}(v/2)$$

$$\exp\left[-i(p+p')\cosh v\right]I_{2\ell+1}\left(2i(pp')^{1/2}\sinh v\right) \qquad (5)$$

where $I_{2\ell+1}$ is the modified Bessel function.

The Green's Function $g(z,z')$ is given by

$$g(z,z') = \frac{1}{2\pi}\int_0^\infty dp \int_0^\infty dp' \exp\left[ipz\right]F(p,p')\exp\left[-ip'z'\right] \qquad (6)$$

Transforming the integral in Eq.(5) in an integral on the complex plane and using the following integrals[4]

$$\int_0^\infty J_{2\gamma}(at)\exp[-b^2t^2]dt = (\pi)^{1/2}/2b\ \exp[-a^2/8b^2]I_\gamma(a^2/8b^2) \qquad (7)$$

$$(\pi/2)^{1/2}\int_0^\infty du(u)^{-1/2}\exp[-ux]I_{\ell+1/2}(u) = Q_\ell(x) \qquad (8)$$

to integrate Eq.(6) in p and p' we can obtain an integral representation for $g(z,z')$

$$g(z,z') = [(z-1)(z'+1)]^{m/2}\ [(z'-1)(z+1)]^{-m/2}\ \frac{1}{2\pi i}\int_{\infty-\pi i}^{\infty+\pi i} dy$$

$$\exp[-my]\ Q_\ell(zz'-t\cosh y) \qquad (9)$$

where $Q_\ell(x)$ is the Legendre function of the second kind and

$$t = [(z^2-1)(z'^2-1)]^{1/2}\ .$$

Introducing Eq.(8) in Eq.(9) we have

$$g(z,z') = [(z-1)(z'+1)]^{m/2}\ [(z'-1)(z+1)]^{-m/2}\ (\pi/2)^{1/2}$$

$$\cdot\int_0^\infty du\ u^{-1/2}\exp[-uzz']I_{\ell+1/2}(u) \qquad (10)$$

$$\cdot\frac{1}{2\pi i}\int_{\infty-\pi i}^{\infty+i\pi}\exp[-my+tu\cosh y]$$

The second integral is an integral representation for the modified Bessel function $-I_m(tu)$ [4]. The integral in the u variable is easy to perform[5] and the final expression for $g(z,z')$ is

$$g(z.z') = (-1)^m\ [(z-1)/(z+1)]^{m/2}\ \ [(z'-1)/(z'+1)]^{-m/2}$$

$$\cdot\ P_\ell^m(z'_<)\ Q_\ell^{-m}\ (z_>) \qquad (11)$$

Therefore the Green's function for Eq.(1), the associated Legendre differential equation is

$$G(z,z') = (-1)^m\ P_\ell^m(z_<)Q_\ell^{-m}(z_>)$$

where $z_<$ ($z_>$) is the smaller (larger) of z and z'.

REFERENCES

1. E.A. Hylleras: Z.für Physik *74*, 216 (1932).

2. D. Bergmann and Y. Frishiman: Journ. Math. Phys. *6*, 1855 (1965).

3. J. Bellandi Filho and E.S. Caetano Neto: Lett.Nuovo Cimento *16*, 331 (1976).

4. G.N. Watson: *A Treatise on the Theory of Bessel Functions*, Combidge at the University Press, 1966 chaps 6 and 13.

5. A. Erdely: *Tables of Integral Transforms*, vol.*1* (New York, N.Y.1954) p.196.