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Polarisation and Magnetisation of Electronic Matter

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The present paper studies the behaviour of a system of spin-electrons in a weak external electric or magnetic field. Already in the case of a single free electron classical and quantum theory lead to different results concerning the Lorentz transformation of the magnetic moment (Thomas factor of spin-orbit coupling). The separation of the current into a convection and a spin part can be performed in a covariant way. While the convection current is responsible for the diamagnetism of a system, the spin current accounts for paramagnetic behaviour. After a Lorentz transformation of a diamagnetic system paraelectric components appear, while a paramagnetic system, after transformation, exhibits diaelectric properties, $\varepsilon<1$. Similarly, a paraelectric substance ($\varepsilon > 1$) after a Lorentz transformation shows diamagnetic components, while a diaelectric system would acquire paramagnetic behaviour. Quantum electrodynamics leads to the result, that Dirac's electron vacuum behaves like a paramagnetic medium. It follows from this result, that the electron vacuum in a weak external electric field represents a diaelectric system.

Neste trabalho se estuda o comportamento de um sistema de eletrons com spin num campo externo elétrico ou magnético fraco. Já no caso de um só eletron livre as teorias clássica e quântica conduzem a resultados diferentes relativos a uma transformação de Lorentz do momento magnético (fator de Thomas do acoplamento spin-orbita). A separação da corrente numa parte convectiva e uma parte de spin pode ser efetuada de maneira covariante. Enquanto a corrente convectiva responde pelo diamagnetismo dum sistema, a corrente de spin intervém no comportamento paramagnético. Depois duma transformação dum sistema diamagnético aparecem componentes paraelétricos, enquanto que um sistema paramagnético, depois

da transformação apresenta propriedades diaelétricas, $\varepsilon < 1$. Analogamente, uma substância paraelétrica ($\varepsilon > 1$) apresenta componentes diamagnéticas. Um sistema diaelétrico exibirá um comportamento paramagnético. A eletrodinâmica quântica conduz ao resultado que o vácuo eletrônico de Dirac se comporta como um meio paramagnético. Daí resulta que o vácuo eletrônico, num campo elétrico externo fraco, representa um sistema diaelétrico.

1. THE CLASSICAL SPIN PARTICLE

A classical spin particle is defined by a velocity four-vector U^i and by a spin angular momentum tensor S^{ik}, obeying the relation

$$S^{ik}.U_k = 0 \quad (1)$$

In the rest system of a free particle we have

$$\overset{0}{U}{}^i = (1,0,0,0) \; : \; \overset{0}{S}{}^{0k} = \overset{0}{S}{}^{k0} = 0$$

Further, we have to assume, that in the rest system the magnetic moment of the spin particle is given by

$$\overset{0}{M}{}^{ik} = \frac{e.g}{2mc} . \overset{0}{S}{}^{ik} \quad (2)$$

where g is an arbitrary numerical constant.

Since the magnetic moment M^{ik} of a system is defined as an integral over a tensorial magnetic momentum density, M^{ik} cannot be a tensor itself. Relation (2) can, therefore, only be satisfied rigorously in one particular system of reference, which in our case we choose to be the rest system. In current literature[1] this fact is, however, generally disregarded and (2) is taken to be a true tensor relation. If, however, we restrict our attention to slow motions, neglecting terms of the order $(v/c)^2$, we may use (2) as if it was a tensor relation.

The equations of motion, which determine the time development of U^i, S^{ik} and M^{ik} have to be compatible with the relativistic transformation laws of these quantities. For the case of free particle, which we consider here, their particular form, however, is irrelevant.

Performing a Lorentz transformation, so that our particle movez with velocity $\vec{v}$ we find

$$U^{i} = \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} , \frac{\frac{\vec{v}}{c}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) \tag{3}$$

$$\vec{S} = (S^{23}, S^{31}, S^{12}) = \frac{\vec{\overset{o}{S}}}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{\frac{\vec{v}}{c} \cdot (\frac{\vec{v}}{c} \cdot \vec{\overset{o}{S}})}{1 - \frac{v^2}{c^2} (1 + \sqrt{1 - \frac{v^2}{c^2}})}$$

$$\vec{D} = (S^{01}, S^{02}, S^{03}) = - \frac{\frac{\vec{v}}{c} \wedge \vec{\overset{o}{S}}}{\sqrt{1 \; \frac{v^2}{c^2}}} \tag{4}$$

$$\vec{M} = (M^{23}, M^{31}, M^{12}) = \vec{\overset{o}{M}} - \frac{\frac{\vec{v}}{c} \cdot (\frac{\vec{v}}{c} \cdot \vec{\overset{o}{M}})}{1 + \sqrt{1 - \frac{v^2}{c^2}}} \tag{5}$$

$$\vec{M} = (-M^{01}, -M^{02}, -M^{03}) = \frac{\vec{v}}{c} \wedge \vec{\overset{o}{M}}$$

We may, therefore, say that the moving spin particle carries a magnetic moment $\vec{M}$ and an electric dipole moment $\vec{P} = \frac{\vec{v}}{c} \wedge \vec{\overset{o}{M}}$ with it.

2. THE DIRAC ELECTRON

On order to compare the above given formulae with the quantum theoretical expressions we write Dirac's equation

$$p_0 \psi = (\vec{\alpha} . \vec{p} + \beta . mc) \psi \tag{6}$$

and, introducing two two-component spinors u and v for the four-spinor

$$\psi = \binom{u}{v} \tag{7}$$

we find

$$(p_0 - mc) . u = \frac{\hbar}{i} \ \vec{\sigma}. \text{ grad } v$$
$$(p_0 + mc) . v = \frac{\hbar}{i} \ \vec{\sigma}. \text{ grad } u \qquad (8)$$

where, now, $\vec{\sigma}$ denotes Pauli's two by two matrices.

For the current vector we obtain according to Gordon

$$j^0 = e \ . \ \tilde{\psi}\psi = j^0_c - \text{div } \vec{P}$$
$$\vec{j} \ = e \ . \ \tilde{\psi}\vec{\alpha}\psi = \vec{j}_c + \frac{1}{c} \ \frac{\partial \vec{P}}{\partial t} + \text{rot } \vec{M} \qquad (9)$$

with

$$j^0_c = \frac{e\hbar}{2imc} \ . \ \left(\frac{1}{c} \frac{\partial \tilde{\psi}}{\partial t} \beta\psi - \tilde{\psi}\beta \frac{1}{c} \frac{\partial \psi}{\partial t}\right) \qquad (10)$$

$$\vec{j}_c = \frac{e\hbar}{2imc} \ . \ (\tilde{\psi}\beta \ .\text{grad } \tilde{\psi} - \text{grad } \tilde{\psi}.\beta\psi)$$

$$\vec{M} = \frac{e\hbar}{2mc} \ (\tilde{\psi} \ . \beta\vec{\sigma} \ \ . \ \psi) \qquad (11)$$

$$\vec{P} = \frac{e\hbar}{2imc} \ (\tilde{\psi} \ . \ \vec{\gamma} \ . \ \psi)$$

Replacing ψ by the two-component spinors (7) the current expressions become

$$j^0 = e.(\tilde{u}.u + \tilde{v}.v)$$
$$\vec{j} \ = e.(\tilde{u}.\vec{\sigma}.v + \tilde{v}.\vec{\sigma}.u) \qquad (12)$$

$$j^0_c = \frac{e\hbar}{imc} \left(\frac{1}{c} \frac{\partial \tilde{u}}{\partial t} u - \tilde{u} \frac{1}{c} \frac{\partial u}{\partial t} - \frac{1}{c} \frac{\partial \tilde{v}}{\partial t} v + \tilde{v} \frac{1}{c} \frac{\partial v}{\partial t} \right)$$
$$\vec{j}_c = \frac{e\hbar}{imc} \ (\tilde{u}.\text{grad } u - \text{grad } \tilde{u}.u - \tilde{v}.\text{grad } v + \text{grad } \tilde{v}.v) \qquad (13)$$

and the magnetization and polarization density tensor (11) takes the form

$$\vec{M} = \frac{e\hbar}{2mc}\,(\tilde{u}.\vec{\sigma}.u - \tilde{v}.\vec{\sigma}.v)$$
$$\vec{P} = \frac{e\hbar}{2imc}\,(\tilde{u}.\vec{\sigma}.v - \tilde{v}.\vec{\sigma}.u) \tag{14}$$

Gordons decomposition (9) of the electron current separates in a covariant way the convection (or orbital) terms (10), (13) from the spin terms (11), (14).

For an electron at rest we find from (8)

$$u = \overset{\circ}{u} \quad , \quad v = 0 \tag{15}$$

with the normalization

$$\int \tilde{\overset{\circ}{u}}.\overset{\circ}{u}.d\overset{\circ}{\tau} = 1$$

In a moving system of reference we have

$$u = Ch\,\frac{\nu}{2}\,.\overset{\circ}{u} \quad , \quad v = \vec{\sigma}\,.\frac{\vec{v}}{c}\,.Sh\,\frac{\nu}{2}\,.\overset{\circ}{u} \quad , \quad Th\,\nu = \frac{|\vec{v}|}{c}$$

and the normalization becomes

$$\int(\tilde{u}.u + \tilde{v}.v)\,.\sqrt{1 - v^2/c^2}\,.d\tau = 1 \tag{16}$$

With (12), (13), (14) and (15) we obtain for an electron at rest

$$\overset{\circ}{j}{}^0 = \overset{\circ}{j}{}^0_c = e.\tilde{\overset{\circ}{u}}.\overset{\circ}{u} \; ; \; \vec{\overset{\circ}{j}} = \vec{\overset{\circ}{j}}_c = 0$$
$$\vec{M} = \frac{e\hbar}{2mc}\,.(\tilde{\overset{\circ}{u}}.\vec{\sigma}.\overset{\circ}{u}) \quad , \quad \vec{\overset{\circ}{P}} = 0 \tag{17}$$

in complete agreement with (1) and (2) for a classical spin particle.

3. THE CURRENT OF A MOVING ELECTRON

We shall now evaluate expressions (12), (13) and (14) for a moving electron, using equations (8) to eliminate the small spinors v by the spinor u. For a stationary state, i.e. if p_0 takes a well defined numerical value, we find

$$\tilde{v} = - \frac{\hbar}{i(p_0+mc)} . (\text{grad } \tilde{u}.\vec{\sigma}) \quad , \quad v = \frac{\hbar}{i(p_0+mc)} . (\vec{\sigma}. \text{ grad } u) \qquad (18)$$

Introducing (18) in (12) one obtains

$$\begin{aligned} \vec{j} &= \frac{e\hbar}{i(p_0+mc)} \{ \tilde{u}.\vec{\sigma}. (\vec{\sigma}.\text{grad } u) - (\text{grad } \tilde{u}.\vec{\sigma}) .\vec{\sigma}.u \} \\ &= \frac{e\hbar}{i(p_0+mc)} \{ \tilde{u}.\text{grad } u - \text{grad } \tilde{u}.u\} + \frac{e\hbar}{p_0+mc} \text{rot}(\tilde{u}.\vec{\sigma}.u) \end{aligned} \qquad (19)$$

which agrees with the corresponding expressions in (13) and (14).

According to (19) a moving electron carries a magnetic moment density

$$\vec{M} = \frac{e\hbar}{p_0+mc} (\tilde{u}.\vec{\sigma}.u) \qquad (20)$$

just as classical spin particle carries it according to (3) and (5).

Examining, however, the charge density in (12) we find

$$\begin{aligned} j^0 &= e. \{ \tilde{u}.u + \frac{\hbar^2}{(p_0+mc)^2} . (\text{grad } \tilde{u}.\vec{\sigma}) . (\vec{\sigma}. \text{ grad } u) \} \\ &= e. \{ \tilde{u}.u - \frac{\hbar^2}{2(p_0+mc)^2} . (\tilde{u}.\Delta u + \Delta\tilde{u}.u) \} \\ &+ \text{div } \frac{e\hbar^2}{2(p_0+mc)^2} \{ \text{grad } (\tilde{u}.u) - \tilde{u}.(i\vec{\sigma} \wedge \text{ grad } u) - (\text{grad } \tilde{u} \wedge i\vec{\sigma}) .u\} \end{aligned} \qquad (21)$$

$$\begin{aligned} j^0_c &= e. \frac{p_0}{mc} . \{ \tilde{u}.u - \frac{\hbar^2}{(p_0+mc)^2} . (\text{grad } \tilde{u}.\vec{\sigma}) .(\vec{\sigma}. \text{ grad } u) \} \\ &= e. \frac{p_0}{mc} . \{ \tilde{u}.u + \frac{\hbar^2}{2(p_0+mc)^2} . (\tilde{u}.\Delta u + \Delta\tilde{u}.u) \} \end{aligned}$$

$$- \operatorname{div} \frac{e\hbar^2 p_0}{2(p_0+mc)^2 mc} \{\text{grad}\ (\tilde{u}.u) - \tilde{u}.(i\vec{\sigma} \wedge \text{grad}\ u)$$

$$- (\text{grad}\ \tilde{u} \wedge i\vec{\sigma}).u\} \tag{22}$$

The electric dipole moment density

$$\vec{P}' = - \frac{e\hbar}{2(p_0+mc)} \{\text{grad}\ (\tilde{u}.u) - \tilde{u}.(i\vec{\sigma} \wedge \text{grad}\ u) - (\text{grad}\ \tilde{u} \wedge i\vec{\sigma}).u\}$$

$$= - \frac{e\hbar}{(p_0+mc)} \{\frac{\hbar}{2}\ \text{grad}\ (\tilde{u}.u) - \tilde{u}.(\vec{p} \wedge \vec{\sigma}).u\} \tag{23}$$

contained in (21) is by no means the Lorentz transformed component of (17) but differs from it, at small energies, $p_0 \simeq mc$, by approximately a factor 1/2 and by an additional term which gives rise to contact interactions between the electron and external charges. The factor 1/2 is well known as the Thomas factor of the spin orbit coupling, the additional term is known as the Darwin term.

The convection charge density, which in classical theory does not contain any polarization density, contains, according to (22) a polarization density

$$\vec{P}'' = - \frac{p_0}{mc} . \vec{P}' \tag{24}$$

Contrarily to j^i, j^i_c and $\vec{M}$, $\vec{P}'$ and $\vec{P}''$ do not represent tensor components of a Lorentz transformation.

One verifies easily, that the expression for the tensorial quantity $\vec{P}$ resulting from (14) satisfies the relation

$$\vec{P} = \vec{P}' - \vec{P}''$$

4. THE ORIGIN OF THE THOMAS FACTOR

The above given derivation shows that the strange behaviour of the electric dipole moment represents an intrinsic property of the free Dirac electron and has its origin in the characteristic Lorentz transformation laws of relativistic spinors. Contrarily to current views it has no analoge in classical theory in which no spinors appear[2].

The formulae used above follow rigorously from Dirac's equation and can be immediately generalized in the case of stationary external electric and magnetic fields.

Instead of eliminating the small spinor components v by (18), maintaining the normalization (16), we can also use the unitary transformation introduced by Foldy and Wouthuysen

$$T = \begin{pmatrix} (p_0 + mc) & \vec{\sigma}.\vec{p} \\ -\vec{\sigma}.\vec{p} & (p_0 + mc) \end{pmatrix} . \frac{1}{\sqrt{2p_0(p_0+mc)}} \tag{25}$$

which transform

$$T . \begin{pmatrix} u \\ \dfrac{\vec{\sigma}.p}{p_0+mc}\, u \end{pmatrix} = \begin{pmatrix} w \\ 0 \end{pmatrix} \tag{26}$$

and maintains the normalization

$$\int \tilde{w}.w. \sqrt{1 - v^2/c^2} \, . \, d\tau = 1$$

This transformation is of advantage, if we want to perform a perturbation proceedure starting from Pauli's two-component equations. It has, however, the disadvantage, that, except in the case of the free electron, (26) cannot be realized exactly in a finite number of steps. For a sufficiently weak external field, it leads to the same results as the ones obtained above.

In order to find the potential energy of the charge distribution $e.\tilde{w}.w$ in a very weak electric field of potential Φ we have to form

$$(\tilde{w},\ 0).T.e\Phi.\ \tilde{T}.\ \binom{w}{0} = e.\Phi.\tilde{w}.w + \text{grad}\ \Phi\ .\ \frac{e\hbar}{2mc(p_0+mc)}\ .\tilde{w}.\ \{\frac{1}{i}\vec{p} + \vec{\sigma}\ \vec{p}\}\ w \tag{27}$$

which is essentially equivalent to (21) and (23). The transformation T is closely related to the relativistic spinor transformations and the polarization terms, in (27) proportional to grad Φ, arise from the non--commutativity of $\vec{p}$ and Φ, evidenciating again the quantum origin of the phenomenon[3].

5. POLARIZATION AND MAGNETIZATION

If we place a material substance, which contains positive and negative electric charges in an external electric field, the charges become displaced relative to eachother and the substance becomes polarized. If the substance is isotropic, the polarization vector will be parallel to the applied field. We use to call, such a substance a dielectric and attribute to it a dielectric constant $\varepsilon>1$. For the purpose of the present paper, we shall call such a substance *paraelectric*. Material substance in which the polarization is antiparallel to the applied field, $\varepsilon<1$, which we would have to call *diaelectric* are not known in nature. Only in the case of timedependent external field, in the domain of anomaleous dispersion, we find situations in which $\varepsilon<1$.

If, however, we place a substance in an external magnetic field, the substance may either be diamagnetic, $\mu<1$, or paramagnetic, $\mu>1$. Diamagnetism is a direct consequence of Faradays induction law. The application of a magnetic field induces in the substance a circular current which, in turn produces a magnetic field which is opposite to the applied external field. Diamagnetism is, therefore, a general property of matter containing charges. It may, however, be overcompensated by the alinement of intrinsic magnetic dipoles (or circular currents) giving rise to paramagnetism.

In the case of free electrons, the convenction currents (10) and (13) are responsible for the diamagnetism, while the magnetization terms (11) and (14) will be responsible for paramagnetism. In general, the magnetization of a substance will result from the superposition of a diamagnetic and a paramagnetic part, e.g. in the case of the Alfen -de Haas - effect.

There does not exist any a priori relation between the polarization and the magnetization of a material substance. It will be, in general, paraelectric and dia-, or paramagnetic. We will find below, however, an important exception from the mentioned rule.

Let us, now, perform a Lorentz transformation with a polarized or magnetized substance. In general, this implies that we consider, now, a substance which moves relatively to an observer. Since $(\vec{E},\vec{B})$ and $(\vec{P},-\vec{M})$ represent two tensors, a paraelectric substance will show, when in motion a magnetization which is opposite to the magnetic field due to the same transformation. A diamagnetic substance will, by the Lorentz transformation acquire an electric moment parallel to the electric field and a paramagnetic substance, if moving will have an electric moment which is opposed to the electric field due to the transformation, as if it was diaelectric. It seems, therefore, that diaelectric behaviour is only possible in a moving substance, in presence of an external magnetic field.

The restriction to a moving substance, however, becomes meaningless if we apply our considerations to a polarizable and magnetizable vacuum, as it is suggested by Dirac's theory of the electron.

6. VACUUM POLARIZATION

According to relativity, a stationary magnetic field can be considered to be the difference between two electric fields in relative motion, the electric parts of which compensate. Conversely, an electric field can be represented as the difference between two magnetic fields in relative motion.

If polarization and magnetization form the components of a tensor and if vacuum, in a magnetic field behaves like a paramagnetic substance, in an electric field it will behave like a diaelectric and vice versa.

After what we have said in (21) and (24) the electric polarization of a single electron is not necessarily a tensor quantity, though its divergence forms part of a four vector. In the case of a neutral system, such as vacuum, we may, however, suppose, that the total polarization and magnetization have tensor character, though this assumption, cannot yet, at present, be deduced from Dirac's theory.

On the other hand we know, that vacuum, in a magnetic field behaves like a paramagnetic medium. The fact that the magnetic moment of an electron, due to vacuum effects, is slightly larger than the magnetic moment resulting from Dirac's original theory is due to that fact. If our above assumption is correct, it follows, therefore, that, in an electric field, vacuum is a diaelectric medium.

The possibility, that vacuum represents a diaelectric medium has, for the first time, been discussed by P. Pitanga[4].

Diaelectricity of the vacuum does not help us in any way to understand the necessity of mass renormalization in quantum electrodynamics. It is, however, likely to facilitate the physical interpretation of charge renormalization. The concept of bare charges of elementary particles is certainly meaningless. Nevertheless this concept is a good approximation and most results of atomic physics have been obtained by admitting only bare charges to be present. And vacuum polarization and other vacuum effects have been obtained by perturbation calculations, assuming that those effects are small ones.

If vacuum was a paraelectric medium, as is generally assumed, a charge would be diminuished by attracting from the vacuum charges of opposite sign and pushing away to infinity charges of the same sign. This would represent a large change of the initial system and would contradict the way in which the effect becomes, actually, calculated. If we assume, however, that the vacuum is diaelectric, a charge placed in it attracts charges of the same sign, thus increasing its charge and accumulating in the neighbourhood charges of opposite sign, compensating the increase at the centre.

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