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On the Propagation of Relativistic Nonlinear Electromagnetic Waves in a Cold Plasma

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A general theory of relativistic nonlinear eletromagnetic waves in a cold electron plasma is presented. Three modes of propagation are analysed: i) longitudinal waves, ii) transverse circularly polarized waves and iii) coupled transverse-longitudinal waves. The dispersion relations are obtained. In addition, a number of relativistic and nonlinear effects are discussed.

Uma teoria geral das ondas eletromagnéticas não - lineares e relativísticas num elétron plasma frio é apresentada. Três modos de propagação são analisados: i) ondas longitudinais, ii) ondas transversais polarizadas circularmente e iii) ondas transverso-longitudinais acopladas. As relações de dispersão são obtidas. Um certo número de efeitos relativísticos e não-lineares são discutidos.

1. INTRODUCTION

The plasma state is enormously rich in wave phenomena. Therefore, the topic of wave propagation in plasmas has been the subject of much literature. Most works on the propagation of waves[1] in plasmas, however, have been limited to the study of very small amplitude waves . The reason is that the equations can then be linearized

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in the wave quantities, thus neglecting quadratic and higher order terms in the perturbation variables; consequently, the mathematics involved is greatly simplified. Although the linear theory of plasma waves provides an explanation for many plasma phenomena, it is far from a complete theory. As the amplitude of plasma waves increases, the nonlinearity of the equations that describe the state of a plasma becomes important. If the wave amplitude is small, but finite, the perturbation method can be used to extend the linear theory into the nonlinear regime and thus serves as a basis for explanation of some nonlinear plasma phenomena. Nonetheless, the nonlinear perturbation technique is adequate only if the directed velocities of the plasma particles are much smaller then c (where c = the velocity of light in vaçuum). If the directed velocities of the plasma particles are near c, an exact (relativistic, nonlinear) theory is required to provide an accurate description of the plasma phenomena.

Comparatively recently, the subject of propagation of relativistic nonlinear waves in plasmas has received great attention[2-4]. Reasons for interest in this subject are the problemas connected with astrophysics and controlled thermonuclear fusion. In astrophysics, the motivation for the study derives from some theories[5,6] that predict pulsars to be sources of strong electromagnetic radiations. The interactions of strong waves with surrounding nebular plasmas are of great importance in understanding the physics of pulsars and the generation mechanism of relativistic cosmic rays in the vicinity of pulsars. On the other hand, in the field of controlled thermonuclear fusion, the possibility of using relativistic electron (or ion) beam[7] and high-intensity laser light[8] to heat plasmas to thermonuclear temperatures has aroused interest in the interaction of strong waves with plasmas.

There is a large number of relativistic and nonlinear effects that can occur when relativistic nonlinear waves interact with plasmas. In this work we shall direct our efforts on the study of some of these effects.

A common feature of relativistic nonlinear waves is the dependence of their dispersion relations on the wave amplitude. In the linear theory, the dispersion relations are independent of the wave amplitude. In the nonlinear theory, however, the wave amplitude plays an important role in determining the propagation characteristics of the waves.

On the basis of the linear theory, a wave can propagate in an unmagnetized plasma only if the wave frequency exceeds the plasma frequency. A distinguishing property of relativistic nonlinear waves is their ability to propagate in an overdense plasma (i.e., with the wave frequency less than the plasma frequency). This is because when the wave amplitude is sufficiently large, relativistic and nonlinear effects induced by the wave cause mass variations of the plasma particles. The increased effective mass reduces the effective plasma frequency and makes possible the propagation of waves in an overdense plasma.

According to the linear theory, a natural mode of the plasma is uncoupled if the plasma is free of external fields, hence a mode is either purely transverse or purely longitudinal. In the nonlinear theory, however, the coupling of modes is likely to occur and a mode can have both transverse and longitudinal field components.

As it is well known, the linear theory of eletromagnetic waves in an unmagnetized plasma indicates that the natural mode of the plasma is superluminous. It is the purpose of this work, therefore, to consider the exact theory of superluminous eletromagnetic waves in a cold electron plasma. The method of approach is to seek exact solutions of the equations governing the propagation of a travelling wave for which the space and time coordinates, in the laboratory frame S', enter only in the combination

$$\theta = t' - nz'/c \quad , \tag{1}$$

where the index of refraction n is a positive constant less than unity.

The general theory of relativistic nonlinear waves was initiated by Akhiezer and Polovin[9]. Since then the investigations have demonstrated that there are exact, periodic solutions that reduce to the monochromatic waves of the linear theory in the small-amplitude limit. However, in most cases, due to the complexity of the equations, the behaviour of the solutions can only be studied by approximation techniques. There remains much to be done in order to understand fully the characteristics of such solutions. This work concentrates upon extending our basic understanding of relativistic nonlinear waves. In particular, we shall investigate the general behaviour of three modes: i) purely longitudinal waves, ii) purely transverse, circularly polarized waves and iii) coupled transverse-longitudinal waves. It has been established[2-4] that the treatment of relativistic nonlinear waves can be simplified by the technique of Lorentz transformation. In this work we shall make use of this technique to simplify the analysis.

The problem is formulated in Sec. 2. The linear theory is derived in Sec. 3. The exact theory is investigated in Sec. 4. A summary is given in Sec. 5.

2. FORMULATION OF THE PROBLEM

The discussion is confined to the simplest relevant model: an infinite, uniform, collisionless, unmagnetized, cold, and stationary electron plasma. The positive ions (charge e, mass infinite) are at rest in the laboratory frame S' and constitute only a uniform, constant background charge of number density N'_0. In frame S' the wave propagates with velocity $c\hat{z}/n$, where the unit vector $\hat{z} = (0,0,1)$.

By a cold plasma we mean one in which the effects of the thermal motion of the plasma particles may be neglected compared with the directed motion induced by wave fields. In the presence of a strong wave, the plasma particles can acquire directed velocities comparable to c. Therefore, the assumption of a cold plasma is reasonable unless the plasma particles have relativistic thermal velocities as well. An immediate consequence of the cold plasma assumption is that we can describe the plasma by fluid equations. The dynamics of the plasma electrons are, therefore, characterized by the velocity v' and the number density N'.

The equations governing the problem in the laboratory frame S' (with quantities referred to this frame denoted by a prime) are the relativistic equation of motion

$$\left(\frac{\partial}{\partial t'} + \vec{v}'.\mathrm{grad}\right)(\gamma'\vec{v}') = -\frac{e}{m}\left(\vec{E}' + \frac{\vec{v}'}{c} \times \vec{B}'\right) , \tag{2}$$

the equation of continuity

$$\frac{\partial N'}{\partial t'} + \mathrm{div}\,(N'\,\vec{v}') = 0 , \tag{3}$$

and Maxwell's equations

$$\mathrm{div}\,\vec{E}' = 4\pi e\,(N'_0 - N') , \quad \mathrm{div}\,\vec{B}' = 0 , \tag{4),(5}$$

$$\mathrm{curl}\,\vec{E}' = -\frac{1}{c}\frac{\partial \vec{B}'}{\partial t'} , \quad \mathrm{curl}\,\vec{B}' = \frac{1}{c}\frac{\partial \vec{E}'}{\partial t'} - \frac{4\pi e N'\vec{v}'}{c} , \tag{6),(7}$$

where $-e$ is the electron charge, m the electron rest mass, and $\gamma' = (1 - v'^2/c^2)^{-1/2}$.

We are interested in periodic travelling wave solutions of Eqs. (2)-(7). Hence we can rewrite Eqs. (2)-(7) in terms of the phase θ given by (1), yielding a system of ordinary differential equations

$$(1 - nv'_z/c)\,\frac{d(\gamma'\vec{v}')}{d\theta} = -\frac{e}{m}\left(\vec{E}' + \frac{\vec{v}'}{c} \times \vec{B}'\right) , \tag{8}$$

$$\frac{dN'}{d\theta} - \frac{n}{c}\,\frac{d(N'v'_z)}{d\theta} = 0 , \tag{9}$$

$$-\frac{n}{c}\,\frac{dE'_z}{d\theta} = 4\pi e(N'_0 - N') , \quad \frac{dB'_z}{d\theta} = 0 \tag{10),(11}$$

and

$$n\hat{z} \times \frac{d\vec{E}'}{d\theta} = \frac{d\vec{B}'}{d\theta} , \quad -n\hat{z} \times \frac{d\vec{B}'}{d\theta} = \frac{d\vec{E}'}{d\theta} - 4\pi e N'\vec{v}' . \tag{12),(13}$$

Conditions describing the average charge and current densities can be obtained by taking the phase-average (i.e., average over one period in θ) of (10) and (13), which gives

$$\langle N' \rangle = N'_0 \quad , \quad \langle N'\vec{v}\,' \rangle = 0 \quad , \tag{14),(15}$$

where we have used angular brackets to denote phase-average. Eqs.(14) and (15) show that the charge and current densities are zero on average. In addition, (15) establishes that in the absence of external current densities the plasma is *stationary* in the sense that the average plasma particle flux is zero.

The behaviour of electron number density is obtained by integrating (9)

$$N' = \frac{N'_0}{1 - nv'_z/c} \quad , \tag{16}$$

where the constant of integration is chosen to satisfy (14) and (15). Note that N' is always positive, since the magnitude of v'_z is always less than the wave speed c/n for the superluminous waves; therefore, there is no possibility for wavebreaking to take place.

Upon integrating (12) we get

$$\vec{\mathcal{B}}\,' = n\hat{z} \times \vec{E}\,' + \vec{B}'_0 \quad , \tag{17}$$

where B'_0 is an arbitrary constant of integration and is taken to be zero. The physical interpretation of $B'_0 = 0$ is that the plasma has no magnetostatic field; or, in mathematical terms, the phase-averaged value of the magnetic field is zero. Hence (17) becomes

$$\vec{B}\,' = n\hat{z} \times \vec{E}\,' \quad , \tag{18}$$

which shows that

$$\hat{z} \,.\, \vec{B}\,' = 0 \quad , \quad \vec{E}\,' \,.\, \vec{B}\,' = 0 \quad . \tag{19}$$

Thus, in an unmagnetized plasma, the wave magnetic field is purely transverse to the direction of wave propagation and orthogonal to the wave electric field.

3. LINEAR THEORY

Before treating the exact solutions we shall first consider the linear solutions. For waves with infinitesimal amplitude, we take

$$\vec{v}' = \vec{v}'_p \quad , \qquad N' = N'_0 + N'_p \quad , \tag{20}$$

where $\vec{v}'_p$ and N'_p are small perturbations on the equilibrium state ($\vec{v}' = 0$, $N' = N'_0$). Putting (20) into (14) and (15) gives

$$<\vec{v}'_p> = <N'_p> = 0 \ . \tag{21}$$

To obtain the relevant equations we substitute (20) into (8) - (13) and drop the nonlinear terms yielding

$$\frac{d\vec{v}'_p}{d\theta} = -\frac{e}{m}\vec{E}' \quad , \tag{22}$$

$$\frac{dN'_p}{d\theta} - \frac{nN'}{c}\frac{dv'_{pz}}{d\theta} = 0 \quad , \tag{23}$$

$$\frac{n}{c}\frac{dE'_z}{d\theta} = 4\pi e N'_p \quad , \qquad \frac{dB'_z}{d\theta} = 0 \quad , \tag{24),(25}$$

and

$$n\hat{z} \times \frac{d\vec{E}'}{d\theta} = \frac{d\vec{B}'}{d\theta} \quad , \quad -n\hat{z} \times \frac{d\vec{B}'}{d\theta} = \frac{d\vec{E}'}{d\theta} - 4\pi e N'\vec{v}'_p \quad . \tag{26),(27}$$

Upon integrating (23) we have

$$N'_p = nN'_0 v'_{pz}/c \quad , \tag{28}$$

where the integration constant is chosen in accordance with (21). Taking the scalar product of $\hat{z}$ with (22) gives

$$\frac{dv'_{pz}}{d\theta} = -\frac{e}{m} E'_z \quad . \tag{29}$$

Substitution of (24) into the θ-derivative of (29) and using (28) gives

$$\frac{d^2 v'_{pz}}{d\theta^2} + \omega'^2_p v'_{pz} = 0 \quad , \qquad \omega'^2_p = \frac{4\pi N'_0 e^2}{m} \, , \qquad (30),(31)$$

where ω'_p is the electron plasma frequency measured in the laboratory frame. Thus linear solutions yield harmonic longitudinal waves oscillating at frequency

$$\omega' = \omega'_p \quad . \qquad (32)$$

Next we take the vector product of $\hat{z}$ with (22) making use of (18); then

$$\vec{B}' = -\frac{mn}{e}\,\hat{z} \times \frac{d\vec{v}'_p}{d\theta} \quad . \qquad (33)$$

Taking the vector product of $\hat{z}$ with (27) and using (18) gives

$$\frac{d\vec{B}'}{d\theta} = \frac{4\pi N'_0 e}{1/n - n}\;\hat{z} \times \vec{v}'_p \quad . \qquad (34)$$

Substitution of (34) into the θ-derivative of (33) therefore yields

$$\frac{d^2 v'_{p_\perp}}{d\theta^2} + (1 - n^2)^{-1}\;\omega'^2_p\, v'_{p_\perp} = 0 \quad , \qquad (35)$$

where $v'_{p_\perp} = v'_{px}, v'_{py}$. Hence, in addition to the longitudinal waves the linear solutions also yield harmonic transverse waves with frequency

$$\omega' = (1 - n^2)^{-1/2}\omega'_p \quad , \qquad (36)$$

which can be rewritten as

$$n^2 = 1 - \omega'^2_p/\omega'^2 \quad . \qquad (37)$$

(37) shows that n^2 is less than unity, therefore the transverse waves are superluminous. Furthermore, (37) shows that

$$\omega' > \omega'_p \tag{38}$$

is required for a wave to propagate. If $\omega' < \omega'_p$, the plasma becomes overdense and no wave propagation is posssible. Note that (35) admits both linearly polarized waves with solutions given by

$$\vec{v}'_p = v'_0 e^{i(\omega' t' - k' z')}\hat{x} \quad \text{or} \quad v'_p = \vec{v}'_0 e^{i(\omega' t' - k' z')}\hat{y} \ , \tag{39}$$

and circularly polarized waves with solutions given by

$$\vec{v}'_p = v'_0(\hat{x} \pm i\hat{y})e^{i(\omega' t' - k' z')} = v'_0(\cos \omega'\theta, \pm \sin \omega'\theta, 0), \tag{40}$$

where v'_0 is the maximum electron velocity, k' is the wave number = $= n\omega'/c$, $\hat{x} = (1,0,0)$, $\hat{y} = (0,1,0)$, and $\pm$ refers to the two possible senses of rotation.

For the remainder of this work we shall investigate the analogue, in the exact theory, of the three modes described above. It will be seen that the longitudinal and transverse circularly polarized modes remain uncoupled in exact theory. The purely transverse, linearly polarized waves, however, become coupled transverse-longitudinal waves in the exact theory. The longitudinal component arises because of the $\vec{v}' \times \vec{B}'$ force in (2), which vanishes in the linearized theory.

4. EXACT THEORY

4.1 The Basic Equations

We now seek exact (relativistic, nonlinear) solutions of (2)-(7). We shall apply the Lorentz transformation technique to simplify the analysis. For superluminous waves, the analysis can be done by refering to the frame of reference S (with quantities refered to this frame unprimed) that has velocity $(0,0,nc)$ with respect to the laboratory frame S'. In frame S, the space-time variable θ becomes the transformed time variable t (i.e., $t = (1-n^2)^{-1/2}\,\theta$). As a result, the wave disturbance has no spatial dependence. In frame S, the positive ions have constant number density N_0 and constant velocity $(0, 0, -nc)$.

In the absence of spatial dependence, div $\vec{B} = 0$ is satisfied; moreover it follows from Maxwell's equation (6) that $\vec{B}$ is constant. We take $\vec{B} = 0$ since we assume, in this work, that there is no magnetostatic field. The remaining governing equations (2) - (4) and (7) give, respectively

$$\frac{d(\gamma\vec{v})}{dt} = -\frac{e}{m}\vec{E} \quad , \quad N = \text{constant} \quad , \qquad (41),(42)$$

$$N = N_0 \quad , \quad \frac{d\vec{E}}{dt} - 4\pi e(N_0 nc\hat{z} + N\vec{v}) = 0 \ . \qquad (43),(44)$$

Substitution of (44) into the t-derivative of (41) and making use of (42) and (43) yields

$$\frac{d^2\vec{u}}{d\tau^2} + \vec{u}/\gamma = -n\hat{z} \qquad (45)$$

where

$$\tau = \omega_p t \quad , \quad \omega_p^2 = \frac{4\pi N_0 e^2}{m} \ ,$$
$$\vec{u} = \gamma\vec{v}/c \quad , \quad \gamma = (1 + u^2)^{1/2} \ . \qquad (46)$$

The electric field is given by

$$\vec{E} = -\frac{mc\omega_p}{e}\frac{d\vec{u}}{d\tau} \ . \qquad (47)$$

Any solution of (45) gives in frame S' a wave travelling with velocity $(0,0, c/n)$. The transformation between S and S' is given by the following relations:

$$u'_\perp = u_\perp \quad , \quad u'_z = \Gamma(u_z + n\gamma) \quad ; \qquad (48),(49)$$

$$E'_\perp = \Gamma\, E_\perp \quad , \quad E'_z = E_z \quad ; \qquad (50),(51)$$

$$B'_\perp = \Gamma\,\frac{n}{c}\,\hat{z} \times E \quad , \quad B'_z = 0 \quad ; \qquad (52),(53)$$

$$N_0' = N/\Gamma \quad , \quad \omega_p'^2 = \omega_p^2/\Gamma \quad ; \qquad (54),(55)$$

$$t = \Gamma(t' - nz'/c) \quad , \quad \tau = \Gamma^{3/2}\,\omega_p'\theta \equiv \Gamma^{3/2}\,\tau' \quad ; \qquad (56),(57)$$

where

$$\Gamma = (1 - n^2)^{-1/2} \quad . \qquad (58)$$

4.2. Longitudinal Waves

We first treat purely longitudinal waves with $E_x' = E_y' = 0$. If we write $\vec{u} = (u_x, u_y, u_z)$, for longitudinal waves only u_z component remains and (45) becomes

$$\frac{d^2u_z}{d\tau^2} + \frac{u_z}{(1 + u_z)^{1/2}} = -n \quad , \qquad (59)$$

with a first integral

$$\frac{1}{2}\left(\frac{du_z}{d\tau}\right)^2 = W - f(u_z) \quad , \qquad (60)$$

where W is a constant of integration, and

$$f(u_z) = (1 + u_z^2)^{1/2} + nu_z \quad . \qquad (61)$$

Analysis shows that $f(u_z)$ has a single minimum at

$$u_z = -n\Gamma \quad , \qquad (62)$$

for which the value of f is

$$W_{min} = 1/\Gamma \quad . \qquad (63)$$

Therefore W must exceed $1/\Gamma$ for a solution to exist.

A typical plot of f as a function of u_z is shown in Fig. 1. Evidently u_z oscillates between a minimum u and a maximum u. The values of u and u are found by setting $f = W$, yielding

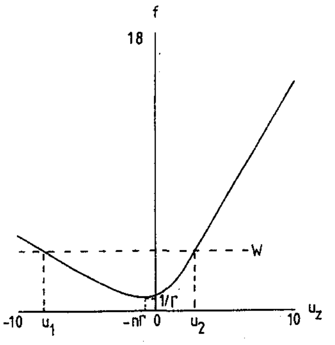

Fig.1 - Variation of f with u_z for $n = 0.5$.

$$u_{1,2} = - n\Gamma^2 W \pm \Gamma(\Gamma^2 W^2 - 1)^{1/2} \quad . \tag{64}$$

Thus the oscillation is periodic with period

$$P = \frac{2}{\omega_p} \int_{u_1}^{u_2} \frac{du_z}{\left[W - f(u_z)\right]^{1/2}} \quad . \tag{65}$$

Translated to frame S' the solution represents a purely longitudinal wave with period

$$P' = P/\Gamma = \frac{2\sqrt{2}}{\omega'_p} \int_0^{u'_0} \frac{du'_z}{(W' - \gamma')^{1/2}} = \frac{2\sqrt{2}}{\omega'_p c} \int_0^{v'_0} \frac{\gamma'^3}{(W' - \gamma')^{1/2}} \, dv'_z \quad , \tag{66}$$

u'_0 and v'_0 are the maximum values of u'_z and v'_z, respectively,

$$\gamma' = (1 + u'^2_z)^{1/2} = (1 - v'^2_z/c^2)^{-1/2} \quad ,$$

and

$$W' = \Gamma W = (1 + u'^2_0)^{1/2} = (1 - v'^2_0/c^2)^{-1/2} \quad .$$

Note that the dependence of the period of oscillation on the index of refraction n disappears upon transformation from S to S'.

Simple expressions for the period and the frequency can be obtained in two limiting cases. For small-amplitude waves ($u'_0 \ll 1$) we get

$$P' = \frac{2}{\omega'_p}\left(1 + \frac{3}{16}u_0'^2\right) \quad , \qquad \omega' = \left(1 - \frac{3}{16}u_0'^2\right)\omega'_p \ , \tag{67}$$

for large-amplitude waves ($u'_0 \gg 1$) we have

$$P' = \frac{4\sqrt{2}}{\omega'_p}\, u_0'^{1/2} \quad , \qquad \omega' = \frac{\sqrt{2}\,\pi}{4}\,\frac{\omega'_p}{u_0'^{1/2}} \ . \tag{68}$$

The general behaviour of ω'/ω'_p as a function of v'_0/c can be computed from (66), which is shown in Fig.2. This figure shows that for $v'_0 \simeq 0$, $\omega' = \omega'_p$, thus in agreement with (32) of the linear theory. As v'_0 increases ω' decreases and goes to zero as v'_0 approaches c.

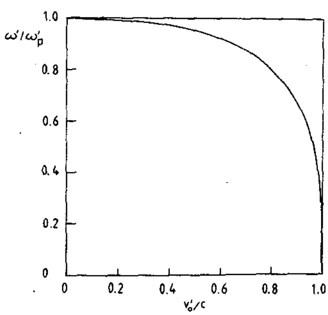

Fig.2 - Variation of ω'/ω'_p with v'_0/c.

4.3. Transverse Circularly Polarized Waves

For purely transverse waves with $E'_z = E_z = 0$, a solution exists for which γ is constant. Under these circumstances (41) shows that u_z and v_z are constant. Therefore, it follows from (49) that u'_z and v'_z are constant. From (15) we have

$$\langle N' v'_z \rangle = 0 \ , \tag{69}$$

which together with (16) indicate that

$$v'_z = u'_z = 0 \quad . \tag{70}$$

Combining (49) and (70) then gives

$$v_z = -nc\hat{z} \qquad \text{and} \qquad u_z = -n\gamma\hat{z} \; . \tag{71}$$

The transverse component of (45) is

$$\frac{d^2u_\perp}{d\tau^2} + u_\perp/\gamma = 0 \qquad , \qquad \gamma = (1 + u_x^2 + u_y^2 + u_z^2)^{1/2} \; . \tag{72}$$

Evidently (72) admits a solution in which γ is constant. The general solution of (71) and (72) is

$$\vec{u} = (u_0 \cos \omega t, \pm\; u_0 \sin \omega t, -n\gamma) \; , \tag{73}$$

where u_0 is an arbitrary constant, and

$$\omega^2 = \omega_p^2/\gamma \quad , \tag{74}$$

where ω_p^2/γ is the square of the proper plasma frequency which in the present case is constant.

The electric field is obtained from (47) and (73)

$$\vec{E} = \frac{mc\omega u_0}{e} (\sin \omega t, \pm\; \cos \omega t, 0) \; , \tag{75}$$

where ± signs refer to the two possible senses of rotation.

Transforming to frame S' the solution represents a purely transverse, circularly polarized wave with electric field

$$\vec{E} = \frac{mc\omega' u_0}{e} (\sin \omega'\theta, \pm \cos \omega'\theta, 0) \; , \tag{76}$$

where

$$\omega t = \omega'\theta = \omega'(t' - nz'/c), \; \omega' = \Gamma\omega \; . \tag{77}$$

The dispersion relation is obtained combining (74) and (77), yielding

$$n^2 = 1 - \frac{1}{(1 + R^2)^{1/2}} \frac{\omega'^2_p}{\omega'^2} , \qquad (78)$$

where

$$R^2 = u_0^2 , \; R = \frac{eE_{\max}}{mc\omega} = \frac{eE'_{\max}}{mc\omega'} , \qquad (79)$$

thus R is a dimensionless invariant parameter by which we characterize the wave amplitude. For very small-amplitude waves such that $R^2 \to 0$, (78) reduces to (37) of the linear theory. For large-amplitude waves such that $R^2 >> 1$, (78) becomes

$$n^2 = 1 - \frac{1}{R} \frac{\omega'^2}{\omega'^2} . \qquad (80)$$

Furthermore, it follows from (78) that the cut-off frequency is

$$\omega' = \frac{\omega'_p}{(1 + R^2)^{1/4}} , \qquad (81)$$

which shows that waves with frequencies lying in the range

$$\frac{\omega'_p}{(1 + R^2)^{1/4}} < \omega' < \omega'_p \qquad (82)$$

can also propagate in the plasma. This is in contrast to the condition (38) of the linear theory. Hence, wave propagation in an overdense plasma is possible for sufficiently large wave amplitudes.

4.4. Coupled Transverse-Longitudinal Waves

We now turn our attention to the coupled transverse-longitudinal waves[3] in which the wave electric field has components both transverse and longitudinal to the direction of wave propagation. The case examined here is that in which $u_x = 0$ always, and correspondingle $E'_x = E_x = 0$. The equations to be considered are therefore y and z components of (59)

$$\frac{d^2u_y}{d\tau^2} + u_y/\gamma = 0, \quad \frac{d^2u_z}{d\tau^2} + u_z/\gamma = -n, \quad \gamma = (1 + u_y^2 + u_z^2)^{1/2}. \tag{83}$$

A first integral of (83) is

$$\frac{1}{2}\left[\left(\frac{du_y}{d\tau}\right)^2 + \left(\frac{du_z}{d\tau}\right)^2\right] = W - \gamma - nu_z, \tag{84}$$

where W is a constant that must exceed $1/\Gamma$.

A solution of (83) specifies a path in the (u_y, u_z) plane, and (84) indicates that the path lies in the domain bounded by

$$\gamma + nu_z = W \quad , \tag{85}$$

which is the ellipse

$$u_y^2 + (u_z + n\Gamma^2 W)^2/\Gamma^2 = \Gamma^2 W^2 - 1 \quad . \tag{86}$$

The problems of finding paths in the (u_y, u_z) plane that represents solutions to (83) and (84) can be posed in terms of a second-order differential equation

$$\frac{d^2u_z}{du_y^2} = \frac{1}{2}\left[1 + \left(\frac{du_z}{du_y}\right)^2\right] \frac{u_y du_z/du_y - u_z - n\gamma}{\gamma(W - \gamma - nu_z)} \quad . \tag{87}$$

(See Eq.(91) of Ref.2, from which (87) is recovered in the limit $m_i \to \infty$).

When no wave is present the plasma is in its equilibrium state specified by

$$u_y = 0 \quad , \quad u_z = -n\Gamma \quad , \quad W = 1/\Gamma. \tag{88}$$

Solutions for which W only slightly exceeds $1/\Gamma$ describe the linear solutions as demonstrated below. Applying small perturbations to the equilibrium state

$$u_y = u_{yp} \quad , \quad u_z = - n\Gamma + u_{zp} \tag{89}$$

then (83), linearizing in u_{yp} and u_{zp}, gives

$$\frac{d^2 u_{yp}}{d\tau^2} + u_{yp}/\Gamma = 0 \; , \quad \frac{d^2 u_{zp}}{d\tau^2} + u_{zp}/\Gamma^3 = 0 \; . \tag{90}$$

whose solutions are

$$u_{yp} = a \cos(\tau/\Gamma^{1/2} + \alpha) \quad , \quad u_{zp} = b \cos(\tau/\Gamma^{3/2} + \beta) \; . \tag{91}$$

where a, α, b, and β are constants.

The angular frequencies of the oscillations in S,$\omega_p/\Gamma^{1/2}$ and $\omega_p/\Gamma^{3/2}$, determine the dispersion relations for transverse and longitudinal waves, respectively, in S'. Transforming to S' using (55) and (77), we get for the u_{zp} oscillation

$$\omega' = \omega'_p \quad , \tag{92}$$

and for the u_{yp} oscillation

$$\omega' = (1 - n^2)^{-1/2} \, \omega'_p \quad , \tag{93}$$

which recover the linear results, (32) and (36), respectively.

For the rest of this section, we shall consider the exact, periodic solutions of (83) which in the linear approximation yield solutions with b=0, hence reduce to purely transverse, linearly polarized waves. The associated exact solutions cannot be purely transverse, but have necessarily a longitudinal component.

It is convenient to introduce the parameter W through (84), and to consider solutions as paths in the (u_y,u_z) plane that satisfy (87).

All paths having the same value of W are confined to the interior of the ellipse (86). From any initial point (u_{y0},u_{z0}) within the ellipse, there is a unique path corresponding to any initial value of the slope $(du_z/du_y)_0$. In general, this path never reaches the ellipse, because the denominator in (87) tends to zero, and hence the curvature tends to infinity, as the boundary is approached. Exceptionally, however, a path may meet the ellipse by approaching it ultimately along the normal to the ellipse; for then the numerator in (87) also vanishes, and the curvature remains finite. In fact, (84) shows that, if $du_y/d\tau$ and $du_z/d\tau$ vanish simultaneously at some point on a particular path, then that point lies on the ellipse and, conversely, that any path meeting the ellipse has $du_y/d\tau$ and $du_z/d\tau$ zero at the point of contact. Analysis of (83) shows that the slope of the normal at a point (u_{y0}, u_{z0}) on the ellipse is given by

$$(du_z/du_y)_0 = (u_{z0} + n\gamma_0)/u_{y0} \quad . \tag{94}$$

The above discussion suggests a path represented by a single line, symmetric about the u_z axis, that joins the points $(\pm a, u_{z0})$ on the bounding ellipse. Such path corresponds to a periodic solution in which the longitudinal field component has twice the frequency of the remaining transverse field component. It might be conjectured that, for an arbitrary pair of values of n and W, there is just one such path. Computed solutions of (87) displaying paths that start from various different points on the ellipse indicate that such is indeed the case. Typical results are shown in Fig.3. The path in question starts from the

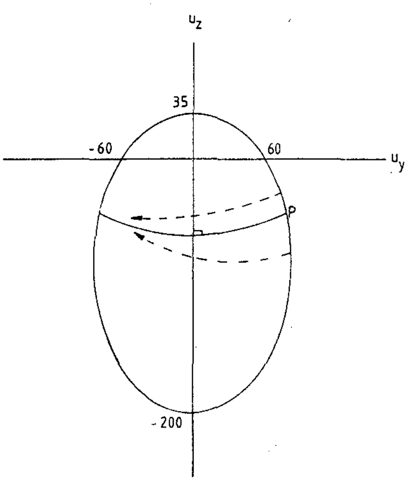

Fig.3 - Paths in (u_y, u_z) plane starting from bounding ellipse (n=0.7, W = 60).

point P and curves above the normal to the ellipse only to the small extent necessary to bring it parallel to the u_y axis at $u_y = 0$. For paths start above/below P, it is seen that u_z is still decreasing / increasing at $u_y = 0$. Such paths bend round before reaching the boundary of the ellipse in $u_y < 0$, and presumably, in general, spiral indefinitely without meeting the boundary again.

The technique of seeking the starting point P, that gives a computed path which crosses the u_z axis orthogonally, was applied to a range of n and W, and the results are presented in Figs. 4-5. In Fig. 4 (a), the product with ω_p of the period of oscilation P in S, is plotted against a (the maximum value of the u_y excursion) for different values of n. The corresponding results in the frame S' are displayed in Fig.4 (b), where the product of ω'_p with the period P' is plotted against a.

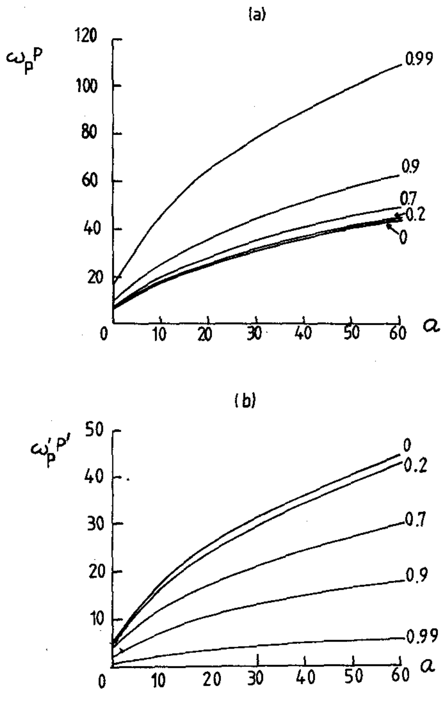

Fig.4 - Variation with a of (a) $\omega_p \times P$, (b) $\omega'_p \times P'$, for indicated values of n.

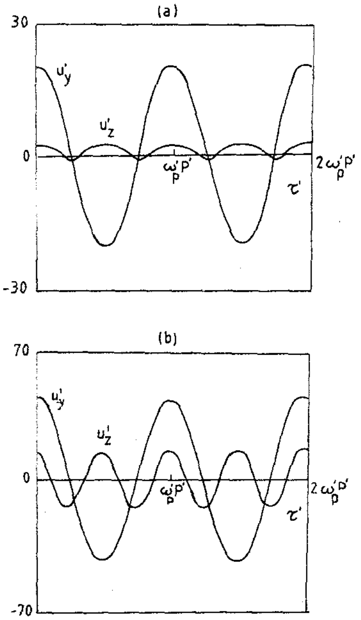

Fig.5 - $u_y'(\tau')$ and $u_z'(\tau')$ for (a) $n = 0.1$, (b) $n = 0.9$ ($W = 20$).

When the appropriate value of u_z, for a given u_y, is known at $\tau = 0$, the periodic solution can be computed directly from (83). Typical curves of u_y' and u_z' as function of τ' are shown in Fig. 5.

Approximate expressions of the dispersion relation can be obtained[3] in the cases $n \ll 1$ and $n \approx 1$. For the case $n \ll 1$, if the wave amplitude is small ($a \ll 1$)

$$n^2 = 1 - (1 - \frac{3}{8}a^2)(\omega_p'/\omega')^2 , \qquad (95)$$

and, if the wave amplitude is large ($a \gg 1$)

$$n^2 = 1 - \frac{2}{8a}\omega_p'^2/\omega'^2 . \qquad (96)$$

For the case $n \approx 1$

$$n^2 = 1 - \frac{(\omega_p'/\omega')^2}{(1 + \frac{1}{2}a^2)^{1/2}} , \qquad (97)$$

for arbitrary wave amplitudes.

5. SUMMARY

The exact theory of relativistic nonlinear electromagnetic waves propagating in a cold electron plasma was investigated.

The period of oscillation of the purely longitudinal waves depends on the wave amplitude, but does not depend on the wave velocity.

An exact, algebraic dispersion relation was derived for purely transverse, circularly polarized waves. These waves are monochromatic.

The coupled transverse-longitudinal waves were analyzed. For these waves, the wave fields have both transverse and longitudinal components. In the linear approximation, they reduce to purely transverse, linearly polarized waves. It was shown that, when the value of the wave velocity and the amplitude of the transverse component are given, there is in general just one such solution that is periodic. The longitudinal field component has twice the frequency of the transverse.

REFERENCES

1. T.H. Stix, *The Theory of Plasma Waves* (McGraw-Hill, New York, 1962).
2. P.C. Clemmow, J. Plasma Phys. *12*, 297 (1974).
3. A.C.-L. Chian and P.C.Clemmow, J.Plasma Phys. *14*, 505 (1975).
4. A.C.-L. Chian, Plasma Phys. *21*, 509 (1979).
5. F. Pacini, Nature *219*, 145 (1968).
6. J.E. Gunn and J.P. Ostriker, Phys. Rev. Lett. *22*, 728 (1969).
7. J. Jancarik and V.N. Tsytovich, Nucl. Fusion *13*, 472 (1970).
8. P. Kaw and J.M. Dawson, Phys Fluids *13*, 472 (1970).
9. A.I. Akhiezer and R.V. Polovin, Soviet Phys. JETP *3*, 696 (1956).