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Noether's Theorem in Superspace

J. W. FURTADO VALLE

Physics Department, Syracuse University, Syracuse, New York 13210

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A generalization of Noether's theorem (*NT*) for action functionals of superfields defined over a superspace (*SS*) is formally derived.

Deduz-se formalmente uma generalização do teorema de Noether para funcionais de ação definidos num superespaço.

1. INTRODUCTION

Supersymmetry is a spin-containing relativistic symmetry scheme which, it is hoped, nature obeys to a certain extent.[1] The basic idea was put forward by Ramond[2] in the context of dual models and then generalized to four dimensions by Wess and Zumino.[3] We assume that the description of a physical system is completely given by a set of functions (super-fields) $\phi(z^A)$, where z^A stands for the local coordinates of *SS*, called the base space. *SS* is a Z_2 - "graded differentiable manifold."[4] It seems to be the natural arena for supersymmetric theories, rather than the usual Minkowski space,[5] to which all actions can be ultimately reduced.

In carrying through the formal proof of *NT* for *SS* actions, the main tool that is called for is the definition of a determinant of *SS* matrices, given in reference 6.

2. NT IN SS

"If an action functional depending on superfields over a SS is invariant under a supergroup[7] of SS coordinate transformations, then there exist supercurrents obeying generalized continuity equations".

Let $z^A = (z^{\alpha}, z^{a})$ denote local SS coordinates, z^{α} are bose--type (even elements in a Grassmann algebra) and z^{a} are Fermi-type (odd elements in a Grassmann algebra). For definiteness, we take $\underline{\alpha}$ as a Lorentz index and the Fermi index $\underline{a}$ as a spinor index, ranging from one to four. By appending an extra internal index to z^{a}, we can accommodate internal symmetries as well as conventional supersymmetry.

Let S be an even action functional.

$$S = \int d^8z \; L(\phi, \phi_{,A}) \tag{1}$$

where ϕ compactly denotes a set of superfields, assumed to be of even Grassmann parity. Comma denotes right graded derivation and increments are accordingly written to the left. By requiring S to be stationary, under superfield variations that vanish on the "boundary" Σ of the integration region Ω, [8]

$$\delta S \quad \int_{\Omega} d^8z \; \delta L = 0$$

with

$$\delta L = \delta\phi S_{,\phi} - (-1)^{\sigma_A}(\delta\phi \; L_{,\phi_{,A}})_{,A} \equiv -(-1)^{\sigma_A}(\delta\phi\pi^A)_{,A} + \delta\phi S_{,\phi} \tag{2}$$

where the conjugate momentum π^A is defined as usual. We get

$$S_{,\phi} \equiv L_{,\phi} - (-1)^{\sigma_A}\pi^A{}_{,A} = 0 \tag{3}$$

as the Euler-Lagrange superfield equations. (σ^A is the Grassmann parity of z^A, $\sigma^A = 0$ if $A = \alpha$, $\sigma^A = 1$ if $A = a$).

Let $T(\varepsilon, \omega)$ be a parameter-dependent nonsingular transformation of the SS coordinates

$$z^A \to z^{A'} = z^{A'}(\varepsilon,\omega) \tag{4}$$

where ε denotes a set of anticommuting parameters and ε a set of commuting ones. Like the Fermionic coordinates z^{α}, the parameters ε are odd elements of a Grassmann algebra. The total number of such parameters is left unspecified. The transformations (4) can include the usual super-gauge transformations where the ε's are taken to be anticommuting Majorana spinors,[1] as well as the Poincaré group. As a result of T, S changes by

$$\delta_T S = \int d^8z \, [L' \det J - L] \tag{5}$$

where $L' \equiv L(\phi'(z'),$ etc.) and $J = J(\varepsilon,\omega)$ is the super-Jacobian matrix of the mapping T, whose Bose-Fermi structure is the usual one, namely

$$\sigma(J_A{}^B) = \sigma_A + \sigma_B \tag{6}$$

Infinitesimally, the transformation T reads

$$z^A \to z^{A'} = z^A + \xi^A(\varepsilon,\omega) \tag{7}$$

where ξ^A has the same Grassmann parity as z^A.

The assumed non-singularity of T certainly holds if the set of such transformations form a supergroup,[7] where $T^{-1}(\varepsilon,\omega) = T(-\varepsilon,-\omega)$. Also, it can be characterized as a condition on the existence and local non-vanishingness of det J, which is defined, following reference 7, as

$$\det J = \det \left[\begin{array}{c|c} z^{\beta}{}_{,\alpha'} & z^{\beta}{}_{,a'} \\ \hline z^{b}{}_{,\alpha'} & z^{b}{}_{,a'} \end{array}\right] = \frac{\det(z^{\beta}{}_{,\alpha'})}{\det(z^{b}{}_{,a'} - z^{\beta}{}_{,a'} z^{\alpha'}{}_{,\beta} z^{b}{}_{,\alpha'})} \tag{8}$$

where $z^A{}_{,B'}$ denotes right graded derivative of z^A with respect to $z^{B'}$,

$$z^A{}_{,B'} \equiv \frac{\partial_R z^A}{\partial z^{B'}} \; .$$

Such definition assures the usual properties, for example, $\det(J_1 J_2) = \det J_1 \det J_2$ and $\det J = \exp[\mathrm{tr}(\ln J)]$, where the trace of SS matrices has also been given in reference 7 as

$$\mathrm{tr}\ M^A{}_B \equiv \sum_A (-1)^{\sigma_A} M^A{}_A .$$

If the action is invariant under T, then δL is a total superdivergence,

$$\delta_T S = \int d^8z\ \delta L = \int d^8z (-1)^{\sigma_A}\ \delta C^A{}_{,A} \tag{9}$$

where δC^A is some infinitesimal super vector field on SS vanishing on Σ. The signs involved in the above equation are needed for supercovariance (a tensor calculus in superspaces has been given in references 5,6.

Comparing (9) and (2), we get

$$\delta\phi S_{,\phi} = -(-1)^{\sigma_A} (\delta\phi\pi^A)_{,A} = (-1)^{\sigma_A}\ \delta C^A{}_{,A} .$$

$$\delta J^A = \delta\phi\pi^A + \delta C^A$$

satisfies, along the superfield equations (3), the generalized continuity equation

$$(-1)^{\sigma_A} \delta J^A{}_{,A} = 0 \tag{10}$$

As an example we see, from (5), that if L transforms as a superscalar density (like the supergravity Lagrangian density does) then

$$L = \det JL'$$

implies

$$\delta L = (-1)^{\sigma_A} (\xi^A L)_{,A} \tag{11}$$

so that S is automatically invariant, since δL is a total superdivergence and thus can only contribute through surface terms, leaving unaffected the Euler-Lagrange equations. The form of δJ^A is then

$$\delta\mathcal{J}^A = \delta\phi\pi^A + \xi^A L \qquad (12)$$

The canonical expression of the supercurrent j^A may not be the most useful. Indeed, it is arbitrary up to the addition of a generalized superpotential V^A defined as

$$V^A = (-1)^{\sigma_B} \Lambda^{AB}{}_{,B}$$

where Λ^{AB} has the symmetry

$$\Lambda^{AB} = (-1)^{\sigma_A \sigma_B + 1} \Lambda^{BA}$$

It is immediately seen that the modified supercurrent $\mathcal{J}^A$ given by

$$\mathcal{J}^A = \delta j^A + V^A$$

also obeys the generalized continuity equation.

3. CONCLUSION

If the *SS* coordinate transformations form a finite dimensional supergroup (that is, one possessing a finite number of generators in the graded Lie algebra[10], then one gets one conserved current, which of course may be trivial, associated with each parameter (commuting or anticommuting). The supergroup T is defined as the invariance group of some *SS* geometrical object and this provides a systematic way of constructing invariant actions. Without dwelling explicitly upon the *SS* geometry, it is clear that in general "classical" currents will be super vectors obeying generalized continuity equations.

If the *SS* coordinate transformation group is infinite (as for example, if it comprises "general graded diffeomorphisms" of *SS* the then it may happen that some conserved currents are identities involving the superfield equations themselves (generalized Bianchi identities). This happens in local gauge theories, and by the same token, will also happen in local supergauge theories, like supergravity.

We have left aside the question of exploring the physical consequences, to the level of immediate interpretation and measurement, arising from the mixture of Grassmann variables and usual space-time coordinates. Likewise, deeper mathematical questions concerning the nature of the *SS* "manifold" have not been considered. It is in this sense that our derivation is a formal one.

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8. This seems to be a meaningless statement if applied to the Fermi coordinates, for which no sensible notion of definite integral over a domain of integration" seems possible. However, for such variables integration is purely algebraically defined (cf. ref. 5).
9. It should be realized that a Grassmann algebra is not (as is the case for quaternions) a division algebra.
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