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Numerical Simulation of the Motion of a Pile Thrust Into the Soil.

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In this paper we present a numerical study of the variational inequality proposed in ref. 1 as a model for a pile driven into the ground under the action of a pile-hammer. To solve numerically that inequality, we approximate the non-differentiable functional involved by differentiable ones, thus obtaining the so-called regularized equations. Then we use the Galerkin technique in the space variable and a predictor-corrector type scheme for integration in time, getting an unconditionally convergent algorithm. Results of some numerical experiments performed by implementing this algorithm with a basis of quadratic finite elements are exibitted.

Apresentamos, neste artigo, um estudo numérico da inequação variacional proposta na ref. 1 como um modelo para a penetração de uma estaca no solo, sob a ação de um bate-estaca. Para resolver numericamente aquela inequação, aproximamos por funcionais diferenciáveis o funcional não diferenciável envolvido, de modo a obter as chamadas equações regularizadas. Usamos então o método de Galerkin na variável espacial e um esquema tipo "predictor-corrector" na variável temporal, obtendo assim um algoritmo incondicionalmente convergente. Resultados de alguns experimentos numéricos realizados implementando este algoritmo com uma base de elementos finitos quadráticos são apresentados.

INTRODUCTION

To simulate the motion of a one-dimensional pile which is driven into the ground under the action of a pile hammer, a variational inequality was proposed in ref.1. The aim of this article is to present theoretical and computational results about an algorithm designed to calculate approximate solutions of that inequality. Some of these results were announced in a slightly weaker form in ref.2 and ref.3.

In Section 1 the physical model we adopted is explained. The functional framework employed and a theorem asserting that we have a mathematically well-posed problem are described in Section 2. Section 3 contains a description of the numerical algorithm and the convergence results deduced, all proofs being postponned to Section 5. The numerical experiments performed are exhibitted in Section 4.

1. THE PHYSICAL MODEL

The physical model developed in ref.1 can be understood through the diagram in Fig.1 and the postulates adopted for the forces acting upon the system.

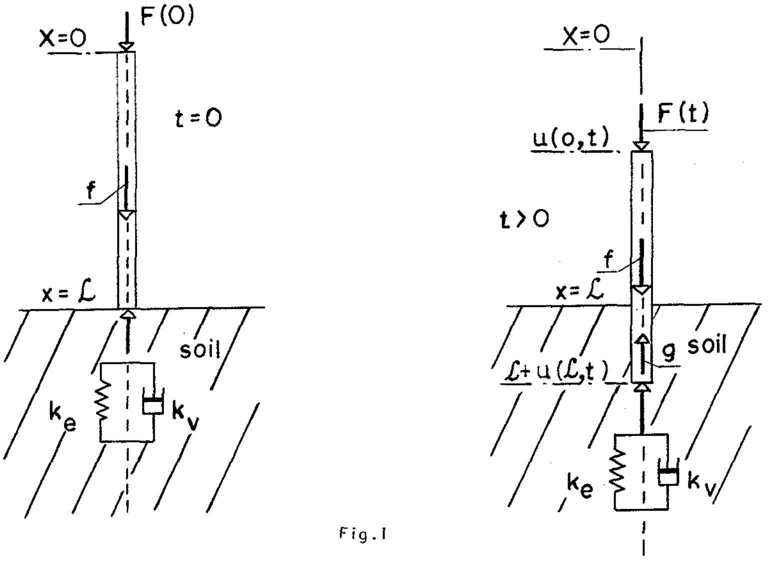

Fig.1

In Fig. 1 two states of the system are shown: the initial ($t=0$) and the state at some subsequent instant ($t>0$).

We assume the pile to be a cylinder of length L and uniform density ρ, a uni-dimensional representation of it being thus adopted. A particle which is at position x in the initial configuration, occupies the position $x + u$ at instant t, $u = u(x,t)$ being the displacement field.

Let us describe now the postulates we have adopted.

(i) Constitutive Law: The pile is described as made of a visco-elastic material with short memory. - That is, we assume that, under small deformations (although large displacements are expected), the stress tensor has the form

$$\sigma = a_e \,\partial u/\partial x + a_v \,\partial \dot{u}/\partial x \;. \tag{1.1}$$

Here the dot denotes $\partial/\partial t$ and the constants a_e and a_v are the modulus of elasticity and the viscosity factor of the material, respectively.

ii) A Visco-Elastic Character for the Soil Reaction to Penetration: By this we mean that the force exerted at particle $x = L$ is given by

$$r = - k_e \, u(L,t) - k_v \, \dot{u}(L,t) \;, \tag{1.2}$$

k_e and k_v being the elastic and viscous coefficients of the soil.

(iii) Rankine's Law for the Normal Force F_N Exerted by the Soil onto the Particle x: This law describes that force as having a hydrostatic behavior so that

$$|F_N| = R \,\omega\, \ell \;\cdot\; (x + u - L)\; H(x + u - L) \;, \tag{1.3}$$

where R is the Rankine coefficient, ω is the specific weight of the soil, and ℓ is the perimeter of the pile cross-section, (cf. ref.4, pp.240).

Of course, this force only acts after the particle x has penetrated the ground, and this is the reason for the appearance of Heaviside step-function $H(s)$ in (1.3).

(iv) Coulomb's Law for the Behavior of the Friction Force g: This law asserts (cf. ref.5, pp.135):

"at a time t and at any point of the contact region,

$$\begin{array}{lll} \text{i) if } |g| < F\,|F_N| & \text{then } \dot{u} = 0, & \\ \text{ii) if } |g| = F\,|F_N| & \text{then there exists } \lambda \geqslant 0 & \quad (1.4) \\ & \text{such that } \dot{u} = -\lambda g\ , & \end{array}$$

where F is the friction coefficient".

v) A Force Balance Law, the Principle of Virtual Powers: This principle states (cf. Ref.6, pp.25):

"In a Galilean frame and for an absolute chronology, the virtual power associated to the inertial forces of a system S equals the power generated by all forces applied to the system, internal as well as external, for any virtual motion of the system S we consider".

We emphasize that, as long as we are considering the friction force, the use of the principle of virtual powers can not be dispensed, of since we must deal with global equilibrium conditions.

Let us denote by δu an arbitrary virtual displacement and by $F(t)$ the external force that acts on the particle $x = 0$. Further let $f(t)$ be a body force and let S be the area of the cross-section of the pile. Then the principle of virtual powers (v) implies that

$$\int_0^L S\,\rho\,\ddot{u}\,(\delta\dot{u})\,dx = -\int_0^L S\sigma(\delta\dot{u}_x)\,dx + [\underline{r}(\delta\dot{u})]_{x=L} + \int_0^L f(\delta\dot{u})\,dx + \int_0^L g(\delta\dot{u})\,dx + F(t)\,[\underline{\delta}\dot{u}]_{x=0} \qquad (1.5)$$

holds for any admissible virtual displacement δu.

As shown in Ref.1, consideration of Eq. (1.1) -(1.4) implies the equivalence between (1.5) and

$$\int_0^L S\,\rho\,\ddot{u}\,(\delta\dot{u})\,dx + \int_0^L S\,a_e\,u_x(\delta\dot{u}_x)dx + k_e\,[u(\delta\dot{u})]_{x=L} +$$

$$+ \int_0^L S\,a_v\,\dot{u}_x(\delta\dot{u}_x)\,dx + k_v\,[\dot{u}(\delta\dot{u})]_{x=L}$$

$$+ \int_0^L R\,\omega\,\ell\,\mathrm{F}\cdot H(x+u-L)(x+u-L)\{|\dot{u}+\delta\dot{u}| - |\dot{u}|\}\,dx \geq$$

$$\int_0^L f(\delta\dot{u})\,dx + F(t)\,[\delta\dot{u}]_{x=0} \qquad (1.6)$$

for any virtual displacement δu.

Relation (1.6) is called a *variational inequality*. It should be mentioned that it does encompass all properties described in postulates (i) - (v). Its mathematical aspects will be analyzed in the following section.

2. THE MATHEMATICAL SET-UP

We introduce now the terminology and functional framework shall make use of.

Let Ω be an open set of the real line $\mathbb{R}$ or the plane $\mathbb{R}^2$. By $H^0(\Omega) \equiv L^2(\Omega)$ we mean the set of (classes of) square-integrable real functions on Ω. For $\Omega \equiv (0,1)$, we shall consider also $H^1(\Omega)$, the subset of $L^2(\Omega)$ consisting of functions whose distributional derivatives of first order are also in $L^2(\Omega)$. For u, v in $L^2(\Omega)$ and $\tilde{u}$, $\tilde{v}$ in $H^1(\Omega)$, we put

$$<u|v> \equiv \int_\Omega u(x)\ v(x)\ dx\ ,$$

$$<\tilde{u}|\tilde{v}>_1 \equiv <\tilde{u}|\tilde{v}> + <\tilde{u}_x|\tilde{v}_x>\ ,$$

$$|u|_0 \equiv <u|u>^{1/2}$$

and

$$|u|_1 \equiv <\tilde{u}|\tilde{u}>_1^{1/2}$$

Finally, for $m = 0$ or 1 and T in $(0,\infty]$ fixed, we denote:

$$L^2(0,T;H^m(\Omega)) \equiv \{w : (0,T) \to H^m(\Omega) \;;$$

$$|w|_{L^2(0,T;H^m(\Omega))} \equiv \left[\int_0^T |w(t)|_m^2 \, dt \right]^{1/2} < \infty \} ,$$

$$L^\infty(0,T;H^m(\Omega)) \equiv \{w : (0,T) \to H^m(\Omega) \;;$$

$$|w|_{L^\infty(0,T;H^m(\Omega))} \equiv \operatorname*{ess\,sup}_{0<t<T} |w(t)|_m < \infty \} .$$

For the sake of simplicity we shall take from now on $S\equiv 1$, $L\equiv 1$ and assume that the system it at rest for $t=0$. We are thus led to the following initial-value problem:

For any fixed $T \in (0,\infty]$ and $\Omega \equiv (0,1)$, find the function

$$u : [0,T] \to H^1(\Omega)$$

with $\dot{u} \equiv du/dt$ in $H^1(\Omega)$ almost everywhere (a.e.) and $\ddot{u} \equiv d^2u/dt^2$ in $L^2(\Omega)$ a.e. such that

$$\rho\langle\ddot{u}|v-\dot{u}\rangle + A_e(u;v-\dot{u}) + A_v(\dot{u};v-\dot{u}) + J(u;v) - J(u;\dot{u}) \geq \langle L|v-\dot{u}\rangle_1 , \tag{2.1}$$

for any v in $H^1(\Omega)$, and

$$u(0) = 0, \tag{2.2}$$

$$\dot{u}(0) = 0 . \tag{2.3}$$

The notation employed in (2.1) is:

i) $A_e(u;v) \equiv a_e \langle u_x \, v_x \rangle + k_e \, u(1) \, v(1)$

and

$A_v(u;v) \equiv a_v \langle u_x | v_x \rangle + k_v \, u(1) \, v(1)$

are bilinear, coercive and bounded forms on $H^1(\Omega)$.

(This implies the equivalence between the $|\cdot|_1$ - norm and the norms associated to both quadratic forms

$$A_e(w) \equiv A_e(w;w)$$

and

$$A_v(w) \equiv A_v(w;w)$$

In other words, there exist some positive constants α_e, β_e, α_v, β_v, for which the relations

$$\begin{cases} \alpha_e |w|_1^2 \leqslant A_e(w) \leqslant \beta_e |w|_1^2 \\ \alpha_v |w|_1^2 \leqslant A_v(w) \leqslant \beta_v |w|_1^2 \end{cases} \tag{2.4}$$

hold, for any w in $H^1(\Omega)$.)

ii) $J(u;v) \equiv c \int_0^1 H(x+u(x)-1)\ [x+u(x)-1]\ |v(x)|dx$

is, for each fixed u in $H^1(\Omega)$, a continuous and convex functional on $v \in H^1(\Omega)$, c standing for $R\omega\ell F$.

iii) $\langle L|u\rangle_1 \equiv \langle f|u\rangle + F(t)\ u(0,t)$

is a bounded linear functional on $H^1(\Omega)$.

A theoretical justification for adopting our model is given by the following result quoted from ref.1, which essentially guarantees that we are dealing with a well-posed mathematical problem:

Theorem 1

Given f in $L^2(0,\infty;\ L^2(\Omega))$ and F in $L^2(0,\infty)$ such that $df/dt \in L^2(0,\infty;\ L^2(\Omega))$, $dF/dt \in L^2(0,\infty)$ and $F(0) = 0$, there exists, for any given T in $(0,\infty]$, a unique u in $L^\infty(0,T;\ H^1(\Omega))$ with

$$\begin{cases} du/dt & \text{in } L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^1(\Omega)) , \\ d^2u/dt^2 & \text{in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) , \end{cases} \tag{2.5}$$

satisfying (2.1) - (2.3). Furthermore,

$$\lim_{t\to\infty} \mid du/dt \mid_1 = 0 , \tag{2.6}$$

which means that the motion tends asymptotically to rest.

On the other hand, the numerical experiments described in Section 4 indicate that our mathematical model does provide a good simulation of the physical system.

3. THE APPROXIMATION ALGORITHM

To compute approximate solutions of (2.1) - (2.3) we first regularize the inequality. By this we mean: take in (2.1) instead of $J(u;v)$ a functional $J_\varepsilon(u;v)$ obtained through replacement of the functions $\tilde{\psi}(s) \equiv H(s)s$ and $\phi(s) \equiv |s|$, appearing in $J(u;v)$, by convex, differentiable approximations $\tilde{\psi}_\varepsilon(s)$ and $\phi_\varepsilon(s)$. As shown in ref.1, this procedure leds us to the variational inequality for approximate solutions u_ε:

$$\rho\langle\ddot{u}_\varepsilon|v-\dot{u}_\varepsilon\rangle + A_e(u_\varepsilon;v-\dot{u}_\varepsilon) + A_v(\dot{u}_\varepsilon;v-\dot{u}_\varepsilon)$$

$$+ J_\varepsilon(u_\varepsilon;v) - J_\varepsilon(u_\varepsilon;\dot{u}_\varepsilon) \gtrsim \langle L|v-\dot{u}_\varepsilon\rangle_1 \tag{3.1}$$

for any v in $H^1(\Omega)$.

On the other hand, (3.1) is seen to be equivalent to the variational equation

$$\rho\langle\ddot{u}_\varepsilon|w\rangle + A_e(u_\varepsilon;w) + A_v(\dot{u}_\varepsilon;w) + c\langle\psi_\varepsilon(u_\varepsilon)\phi'_\varepsilon(\dot{u}_\varepsilon)|w\rangle = \langle L|w\rangle_1 \tag{3.2}$$

for any w in $H^1(\Omega)$.

In (3.2) and in the sequel $\psi_\varepsilon(u_\varepsilon)$ stands for $\tilde{\psi}_\varepsilon(x+u_\varepsilon(x,t)-1)$.

From now on we'll suppose a certain regularization was chosen for a fixed value of $\varepsilon>0$. We shall take $\tilde{\psi}_\varepsilon$ and ϕ_ε satisfying, for real s,

$$\tilde{\psi}_\varepsilon(s) = 0, \text{ if } s\leq 0; \quad |\tilde{\psi}_\varepsilon(s)| \leq s , \quad |\tilde{\psi}_\varepsilon(s) - \tilde{\psi}(s)| \leq \varepsilon, \tag{3.3}$$

and

$$\phi_\varepsilon(0) = 0, \quad |\phi'_\varepsilon(s)| \leq 1, \quad |\phi_\varepsilon(s) - \phi(s)| \leq \varepsilon , \tag{3.4}$$

and shall drop the index ε in the notation for the exact solution of (3.2), (2.2) and (2.3).

We first consider semi-discrete Galerkin approximation to u: take the variational equation (3.2) restricted to a finite-dimensional space $W_h \subset H^1(\Omega)$ and not in the whole space, getting

$$\rho\langle\ddot{u}_h|w\rangle + A_e(u_h;w) + A_v(\dot{u}_h;w) + c\,\langle\psi_\varepsilon(u_h)\phi'_\varepsilon(\dot{u}_h)\,|w\rangle = \langle L|w\rangle_1 \tag{3.5}$$

for any w in W_h.

The parameter $h > 0$ is associated to the discretization in the x-variable and we assume that as $h \to 0$ the subspaces W_h yield strong approximations of $H^1(\Omega)$, in the sence that for any $w \in H^1(\Omega)$ there exist functions $w_h \in W_h$ such that

$$\lim_{h\to 0} |w - w_h|_1 = 0.$$

Taking a fixed basis $\{e^1,\ldots,e^{M_h}\}$ of W_h, equation (3.5) when coupled with initial conditions (2.2) - (2.3) is equivalent to a Cauchy problem for a system of M_h second-order non-linear ordinary differential equations: the unknowns are the coefficients of u_h with respect to $\{e^j\}$ as shown in Section 4. Before showing how to deal with this system, let us introduce once more a piece of terminology.

Let $T > 0$ be finite and fixed once for all, let N be an arbitrary positive integer and let us denote

$$k \equiv T/(N+1), \quad t_n \equiv nk, \quad n = 0,1,\ldots,N+1 .$$

We intend to define approximations $U^n \equiv U^n(x)$ for $u_n \equiv u(\cdot,t_n)$.

For any function $G^n \equiv G^n(x) \in W_h$ associated to the time level t_n, we'll consider, for $1 \leq n \leq N$:

i) The quotient differences

$$\partial_t G^n \equiv (G^{n+1} - G^n)/k \quad , \qquad (n=0 \text{ also}) \tag{3.6a}$$

$$\delta_t G^n \equiv (\partial_t G^n + \partial_t G^{n-1})/2 = (G^{n+1} - G^{n-1})/2k \ , \tag{3.6b}$$

$$\partial_t^2 G^n \equiv (\partial_t G^n - \partial_t G^{n-1})/k = (G^{n+1} - 2G^n + G^{n-1})/k^2 \ ; \tag{3.6c}$$

ii) The weighted averages

$$G^{n+1/2} \equiv (G^{n+1} + G^n)/2 \quad , \quad (n=0 \text{ also}) \tag{3.6d}$$

$$W_\Theta \, G^n \equiv \Theta G^{n+1} + (1-2\Theta) G^n + \Theta G^{n-1} \tag{3.6e}$$

with $0 \leqslant \Theta \leqslant 1/2$;

iii) The step functions

$$G_{h,k}(x,t) \equiv \sum_{n=0}^{N} \chi_n^k(t) \; G^n(x) \quad , \tag{3.6f}$$

$$\partial_t G_{h,k}(x,t) \equiv \sum_{n=0}^{N} \chi_n^k(t) \, \partial_t G^n(x) \quad , \tag{3.6g}$$

$$\delta_t G_{h,k}(x,t) \equiv \sum_{n=1}^{N} \chi_n^k(t) \; \delta_t G^n(x) \, , \tag{3.6h}$$

$$\partial_t^2 G_{h,k}(x,t) \equiv \sum_{n=1}^{N} \chi_n^k(t) \; \partial_t^2 G^n(x) \, , \tag{3.6i}$$

and

$$\tilde{\partial}_t^2 G_{h,k}(x,t) \equiv \sum_{n=1}^{N-1} \chi_n^k(t) \; \partial_t^2 G^{n+1/2}(x) \quad , \tag{3.6j}$$

where $\chi_n^k(t)$ is the characteristic function of the interval $[t_n, t_{n+1})$.

To discretize (3.5) with respect to the time variable we use a predictor-corrector type scheme: at each time level t_n, we simulate (3.5) replacing the time derivatives by quotient differences. This leds us to obtaining an initial approximation $\hat{U}^{n+1}$, which is used to define the final approximation U^{n+1}. For the ease of notation, the following convention will be employed in the sequel: whenever we replace U^{n+1} by $\hat{U}^{n+1}$ for any of the operators ∂_t, δ_t ∂_t^2 or W_Θ in (3.6), that operator will be denoted by $\hat{\partial}_t$, $\hat{\delta}_t$, $\hat{\partial}_t^2$ or $\hat{W}_\Theta$, respectively . In this fashion, we put for example

$$\hat{\partial}_t U^n \equiv (\hat{U}^{n+1} - U^n)/k \ , \tag{3.6k}$$

and so on.

For any Θ in $[0,1/2]$, the algorithm we propose to study is the following: define

$$U^0 = U^1 \equiv 0 \tag{3.7}$$

and for $n = 1,2,\ldots,N$, characterize $\hat{U}^{n+1}$ and U^{n+1} recursively by

$$\rho\langle\partial_t^2 U^n|w\rangle + A_e(\hat{W}_\Theta U^n;w) + A_v(\delta_t U^n;w) + c\,\langle\psi_\varepsilon(U^n)\ \phi_\varepsilon'(\partial_t U^{n-1})\,|w\rangle = \langle L^n|w\rangle_1 \tag{3.8a}$$

for any w in W_h ,

$$\rho\langle\partial^2 U^n|w\rangle + A_e(W_\Theta U^n;w) + A_v(\delta_t U^n;w) + c\,\langle\psi_\varepsilon(U^n)\ \phi_\varepsilon'(\hat{\delta}_t U^n)\,|w\rangle = \langle L^n|w\rangle_1 \tag{3.8b}$$

for any w in W_h, where

$$\langle L^n|w\rangle_1 \equiv \langle f(\cdot,t_n)\,|w\rangle + F(t_n)w(0) \ .$$

The positivity of the forms A_e and A_v implies that $\hat{U}^{n+1}$ and U^{n+1} are uniquely determined through (3.8a) and (3.8b). Further, an important feature of the scheme should be mentioned: as shown in Section 4, the coefficient matrix associate to equation (3.8) is independent of n, so that triangularization is done once for all.

Notice that the equation for the predictor has a (local) accuracy of order $O(k)$, while the local error for the corrected equation is of order $O(k^2)$. We shall not obtain a global estimate for the error because our interest lies on inequality (2.1) whose solution may lack regularity properties. Any error estimates gotten for the regularized equations could not be carried over for the original inequality.

The algorithm (3.7) - (3.8) is unconditionally stable for $0<\Theta\leq 1/2$, while for $\Theta=0$ stability is proven only if $h,k \to 0$ under the condition

$$k\ S(h) < C \ , \tag{3.9}$$

for some constant C, Here $S(h)$ satisfies

$$|w|_1 \lesssim S(h) \ |w|_0 \ , \tag{3.10}$$

any w in W_h, and is called the stability function of the pair (L^2,H^1),(cf. ref.7, pp. V-16).

In more precise terms, we can state the following result, the proof of which will be postponned to Section 5:

Lemma 1

Assume that the hypothesis of Theorem 1 in Section 2 hold, take $\Theta \ \varepsilon(0,1/2]$ and construct with the solutions U^n of (3.7) - (3.8) the functions described in (3.6f) - (3.6j). Then

i) $\{U_{h,k}\}$ and $\{\partial_t \ U_{h,k}\}$ are both bounded families in $L^\infty(0,T;H^1(\Omega))$;

ii) $\{\tilde{\partial}_t^2 \ U_{h,k}\}$ is bounded family in $L^2(0,T;\ H^1(\Omega))$;

iii) $\{\partial_t^2 \ U_{h,k}\}$ is also bounded in $L^\infty(0,T;L^2(\Omega))$.

The same conclusions hold for $\Theta=0$ provided h and k always satisfy (3.9).

The estimates in this lemma are basic to prove the following result, which corresponds to Theorem 4.1 in ref.8:

Theorem 2

Let $\Theta \ \varepsilon(0,1/2]$ and $\varepsilon>0$ be fixed, and assume the same hypothesis as in theorem 1. Choose regularizing functions $\tilde{\psi}_\varepsilon$, ϕ_ε satisfying (3.3) - (3.4) and let $U_{h,k} \equiv U_{h,k}^\varepsilon$ be the approximations to the solution u_ε of (3.2), (2.2) and (2.3), computed through (3.7) - (3.8). Then, letting $Q_T \equiv (0,1) \ x \ (0,T)$, as $h,k \to 0$ we have

$$\left.\begin{aligned} |U_{h,k}^\varepsilon - u_\varepsilon|_{L^2(Q_T)} &\to 0 \qquad (3.11)\\ |\partial_t U_{h,k}^\varepsilon - \dot{u}_\varepsilon|_{L^2(Q_T)} &\to 0 \qquad (3.12)\end{aligned}\right\} \quad \text{(strong convergence)}$$

$$\int_0^T \langle \partial_t^2 U_{h,k}^{\varepsilon} - \ddot{u}_{\varepsilon} | w \rangle_1 \; dt \to 0 \;\forall\; w \in L^2(0,T;H^1(\Omega)) \text{ (weak convergence)} \tag{3.13}$$

The same conclusions hold for $\Theta=0$ provided $h,k \to 0$ subjected to condition

$$k\, S(h) \leqslant \sqrt{\rho\, \alpha_e}/\beta_e, \tag{3.14}$$

with the constant α_e, β_e as appearing in (2.4).

For the proof of this theorem, we again refer the reader to Section 5. When this result is put together with the convergence properties of the solutions of the regularized equations, as obtained in the proof of theorem 1, we are able to formulate the following.

Corollary. Under the hypothesis of theorems 1 and 2, the following equalities hold for the iterated limits:

$$\lim_{\varepsilon\to 0} \lim_{h,k\to 0} |u - U_{h,k}^{\varepsilon}|_{L^2(Q_T)} = 0 ,$$

$$\lim_{\varepsilon\to 0} \lim_{h,k\to 0} |\dot{u} - \partial_t U_{h,k}^{\varepsilon}|_{L^2(Q_T)} = 0,$$

u being the solution of (2.1) - (2.3) and $U_{h,k}^{\varepsilon}$ being defined by (3.7)-(3.8).

4. THE NUMERICAL EXPERIMENTS

In this section we present some numerical results obtained by implementing the proposed algorithm with the finite element method for constructing approximating spaces W_h.

The pile length is set equal to L and the interval $[0,L]$ is partitioned into E sub-intervals (elements). In each of them we take

$$u_h^{(e)} \equiv \sum_i u_h^{(e_i)}(t)\; \Phi_i(x) \tag{4.1}$$

as a local representation for the solution u_h of (3.5), where:

(i) $u_h^{(e_i)}(t)$ is the displacement of the i - th node of element e at instant t;

(ii) $\phi_i(x)$ is the interpolating function associated to the i - th node.

Using this representation, equation (3.5) implies the following relations for the nodal displacement vector. $\underset{\sim}{u}_h^{(e)}$ associated to element e:

$$M^{(e)}\ddot{\underset{\sim}{u}}_h^{(e)} + A_v^{(e)}\dot{\underset{\sim}{u}}_h^{(e)} + A_e^{(e)}\underset{\sim}{u}_h^{(e)} = \underset{\sim}{L}^{(e)}(t) - \underset{\sim}{J}_\varepsilon^{(e)}(u_h^{(e)};\dot{u}_h^{(e)}), \qquad (4.2)$$

for $e = 1,\ldots,E$ and $t \in [0,T]$.

The notation employed above is:

$$(M^{(e)})_{ij} \equiv \rho<\Phi_i|\ \Phi_j> \quad ,$$

$$(A_e^{(e)})_{ij} \equiv A_e\ (\Phi_i;\Phi_j) \quad ,$$

$$(A_v^{(e)})_{ij} \equiv A_v\ (\Phi_i;\Phi_j) \quad ,$$

$$(\underset{\sim}{L}^{(e)}(t))_i \equiv <f(t)|\Phi_i> \qquad i \neq 0 \ ,$$

$$(\underset{\sim}{L}^{(e)}(t))_0 \equiv <f(t)|\Phi_0> + F(t) ,$$

$$[\underset{\sim}{J}_\varepsilon^{(e)}(u_h^{(e)};\dot{u}_h^{(e)})]_i \equiv c<\psi_\varepsilon(u_h^{(e)})\phi'_\varepsilon(\dot{u}_h^{(e)})|\Phi_i> .$$

Assemblage of equations (4.2) leds us to the following second order non-linear ordinary differential system for the global displacement vector of all nodes, which is represented by $\underset{\sim}{U}_h$:

$$M\ddot{\underset{\sim}{U}}_h + A_v\dot{\underset{\sim}{U}}_h + A_e\underset{\sim}{U}_h = \underset{\sim}{L}(t) - \underset{\sim}{J}_\varepsilon(\underset{\sim}{U}_h;\dot{\underset{\sim}{U}}_h) \qquad (4.3)$$

where the notation for the matrices appearing is self-explanatory. The initial conditions are

$$\underset{\sim}{U}_h(0) = 0 \ , \ \dot{\underset{\sim}{U}}_h(0) = 0 \qquad (4.4)$$

Equations (3.8) assume thus the form

$$M\, \hat{\partial}_t^2\, \underset{\sim}{U}_h^n + A_e\, \hat{W}_\Theta\, \underset{\sim}{U}_h^n + A_v\, \hat{\delta}_t\, \underset{\sim}{U}_h^n = \underset{\sim}{L}(t_n) - \underset{\sim}{J}_\varepsilon(\underset{\sim}{U}_h^n; \partial_t\, \underset{\sim}{U}_h^{n-1}) \qquad (4.5a)$$

$$M\, \partial_t^2\, \underset{\sim}{U}_h^n + A_e\, W_\Theta \underset{\sim}{U}_h^n + A_v\, \delta_t\, \underset{\sim}{U}_h^n = \underset{\sim}{L}(t_n) - \underset{\sim}{J}_\varepsilon(\underset{\sim}{U}_h^n; \hat{\delta}_t\, \underset{\sim}{U}_h^n)\ , \qquad (4.5b)$$

with $1 \leq n \leq N$, $T = (N+1)k$, $\Theta \in [0,1/2]$ and a natural vector extension for the operators introduced in (3.6).

These equations together with

$$\underset{\sim}{U}_h^0 = \underset{\sim}{U}_h^1 \equiv 0 \qquad (4.5c)$$

uniquely define the approximations $\underset{\sim}{U}_h^n$ for $\underset{\sim}{U}_h(nk)$.

We implemented algorithm (4.5) with three-node quadratic elements. The interpolating functions are then

$$\Phi_i(x(\xi)) \equiv \delta_{i1}\, \xi(\xi-1)/2 + \delta_{i2}(1-\xi^2) + \delta_{i3}\, \xi(\xi+1)/2\ ,$$

with

$$x(\xi) \equiv \{\xi(x_b - x_a) + x_b + x_a\}/2\ , \qquad -1 \leq \xi \leq 1\ ,$$

x_a and x_b being the coordinates of the extreme points of the element under consideration and δ_{ij} as Kronecker's delta.

The choice for the regularizing functions was $\tilde{\psi}_\varepsilon(s) \equiv \tilde{\psi}(s)$ and

$$\phi_\varepsilon(s) \equiv \begin{cases} |s| - \varepsilon/3 & |s| \geq \varepsilon \\ \varepsilon[\, s/\varepsilon^{\,2} - |s/\varepsilon|^3/3] & |s| \leq \varepsilon \end{cases}$$

so that

$$\phi'_\varepsilon(s) \equiv \begin{cases} \operatorname{sgn}(s) & |s| \geq \varepsilon \\ 2\, s/\varepsilon - (s/\varepsilon)^2 \operatorname{sgn}(s) & |s| \leq \varepsilon\ . \end{cases}$$

Using a non-differentiable Ψ_ε did not affect the quality of the results obtained. In all examples a five-element regular partition was used, and the calculations were performed on the IBM 370/145 at CBPF with double precision. The parameter Θ was chose as equal to 1/4.

Example 1

We study a bar with one of its extremities fixed. This must be understood as a test case for validating the codem as no friction is involved.

The value adopted for $\Delta t \equiv k$ was .1, the other data being shown in Fig.2. For the elastic case, the solution is compared with the one proposed in ref.9. The stress was also computed, its values being plotted in Fig.3.

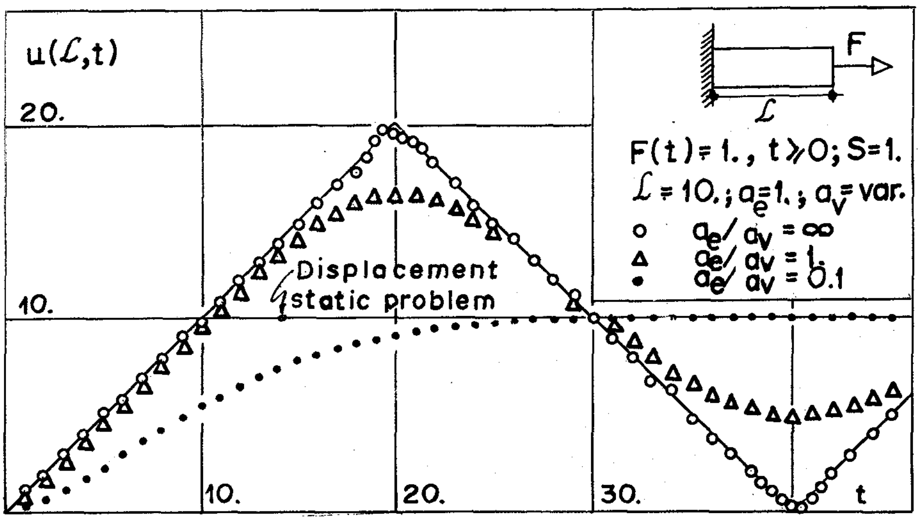

Fig.2

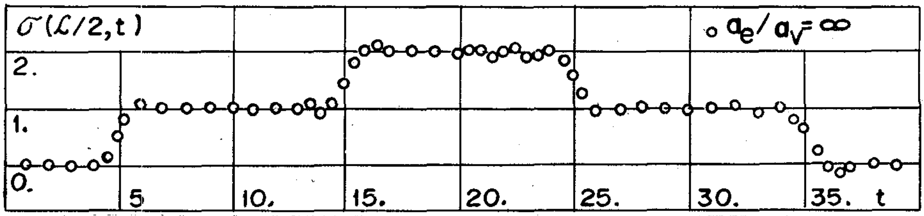

Fig.3

Example 2

We study the motion of a pile for a system with the following characteristics:

$$a_e = 100;\ a_v = .1\ ;\ k_e = .1\ ;\ k_v = 1\ ;$$

$$\rho = 10^{-3}\ ;\ c = 0.5;\quad L = 10\ ;$$

$$F(t) \equiv \begin{cases} t/5 & 0 \leqslant t \leqslant 10 \\ 2 & 10 \leqslant t \end{cases}$$

In Fig.4 the displacements of the extremity $x = L$ are plotted and compared with the solution obtained in the quasi-static case. We took $\Delta t = .05$ and $\varepsilon = 10^{-7}$.

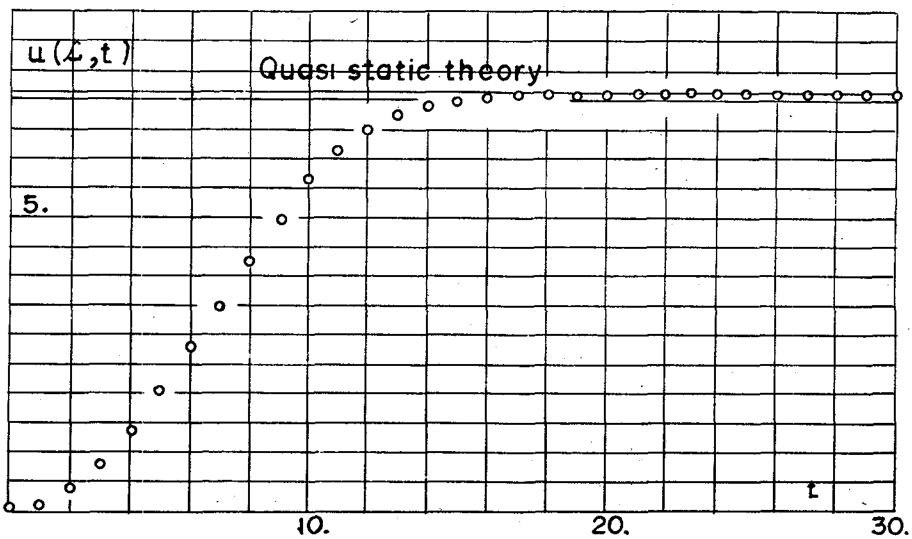

Fig.4

Example 3

A better simulation of the motion of a pile hammer was considered by taking the external force $F(t)$ as described in Fig. 5. In that figure, the displacements of both extremities of the pile are plotted. The physical constants are the same as in Example 2, except for $a_v = 1$ and $\rho = .01$.

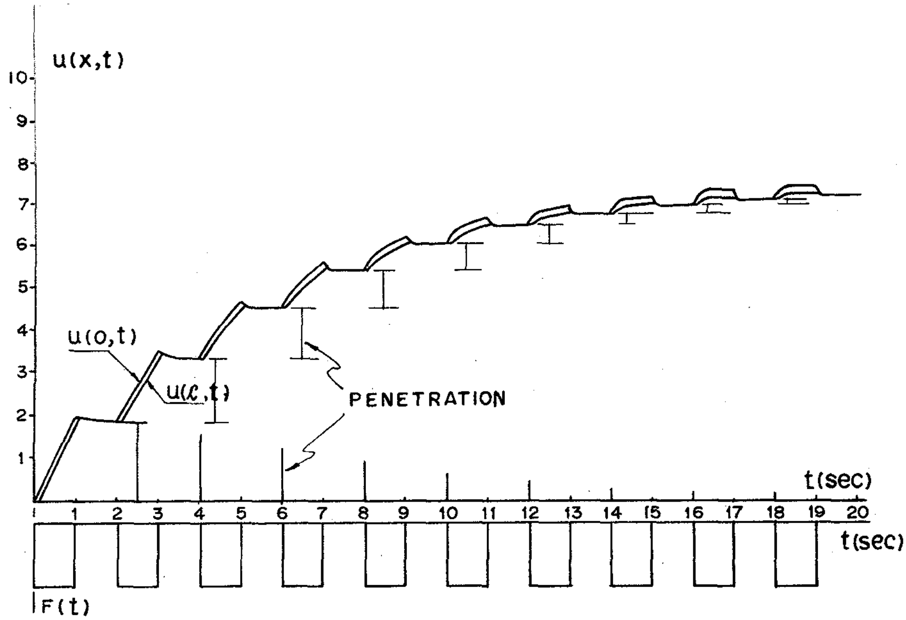

Fig.5

5. THE MATHEMATICAL PROOFS

In this section the technical details of the proofs of Lemma 1 and Theorem 2 from Section 3 are presented.

•Proof of Lemma 1.

The first aim is to reach an estimate of the form

$$|U_{h,k}|_{L^\infty(0,T;H^1(\Omega))} + |\delta_t U_{h,k}|_{L^2(0,T;H^1(\Omega))}$$

$$+ |\delta_t U_{h,k}|_{L^\infty(0,T;L^2(\Omega))} \leq C \tag{5.1}$$

for the step-functions defined in (3.6f) - (3.6h), with $C = C(T)$ independent of ε, h and k. For this, as well as for all other estimates, the technique is to choose convenient test-functions and to operate upon the relations thus obtained.

Take in (3.8b), $w \equiv \delta_t U^n$ to get

$$\frac{\rho}{2k}\{|\partial_t U^n|_0^2 - |\partial_t U^{n-1}|_0^2 + A_e(W_\Theta U^n;\ \delta_t U^n)$$

$$+ A_v(\delta_t U^n) = - c\langle\psi_\varepsilon(U^n)\phi'_\varepsilon(\hat{\delta}_t U^n)|\delta_t U^n\rangle + \langle L^n|\delta_t U^n\rangle_1 \tag{5.2}$$

Thanks to the hypothesis about f and F, the estimate

$$|L^n|_1 \leq K_1$$

holds for some constant K_1 independent of T, k and n and thus

$$|\langle L^n|\delta_t U^n\rangle|_1 \leq K_1|\delta_t U^n|_1 . \tag{5.3}$$

Using the assumptions (3.3) and (3.4) on the functions $\tilde{\psi}_\varepsilon$ and ϕ'_ε, we get

$$c|\langle\psi_\varepsilon(U^n)\phi'_\varepsilon(\hat{\delta}_t U^n)|\delta_t U^n\rangle| \leq c\ |\psi_\varepsilon(U^n)|_0\ |\delta_t U^n|_0 \leq$$

$$2c(1 + |U^n|_1\ |\delta_t U^n|_0 . \tag{5.4}$$

To handle the term $A_e(W_\Theta U^n;\delta_t U^n)$ when $\Theta\ \varepsilon(0,1/2]$ we make use of the identity

$$A_e(\Theta\ U^{n+1} + (1-2\Theta)U^n + \Theta\ U^{n-1}\ ;\ U^{n+1} - U^{n-1}) =$$

$$\Theta[A_e(U^{n+1}) - A_e(U^{n-1})] + (1-2\Theta)A_e(U^n; \frac{U^{n+1}-U^{n-1}}{2k})2k$$

and of the inequality

$$|A_e(U^n;\delta_t U^n| \leq A_e(U^n)^{1/2} A_e(\delta_t U^n)^{1/2} \leq \beta_e[\gamma|\delta_t U^n|_1^2 + |U^n|_1^2 /4\gamma] ,$$

getting from (5.2):

$$\begin{aligned}
\rho\{|\partial_t U^n|_0^2 - |\partial_t U^{n-1}|_0^2\} &+ 2\alpha_v k|\delta_t U^n|_1^2 \\
&+ \Theta[A_e(U^{n+1}) - A_e(U^{n-1})] \leq \\
(2-4\Theta)\ k\ \beta_e&\{\gamma|\delta_t U^n|_1^2 + |U^n|_1^2/4\gamma\} \\
&+ k\{\gamma|\delta_t U^n|_1^2 + K_1^2/\gamma\} \\
&+ 2k\ \{c^2(1+|U^n|_1)^2/\gamma + \gamma|\delta_t U^n|_1^2\} \\
&\leq C_1^\Theta\ \gamma k|\delta_t U^n|_1^2 + C_2^\Theta k|U^n|_1^2/\gamma \\
&+ k(K_1^2 + 4c^2)/\gamma \\
&\leq \alpha_v\ k|\delta_t U^n|_1^2 + k\ K(1 + |U^n|_1^2)\ .
\end{aligned}$$

Here K is a fixed constant and we assigned the value $\gamma \equiv \alpha_v/C_1^\Theta$. When we sum these relations for n varying from 1 to L, $1 \leq L \leq N$ taking (3.7) into account,

$$\rho|\partial_t U^L|_0^2 + \alpha_v \sum_{n=1}^{L} k|\delta_t U^n|_1^2 + \Theta[A_e(U^{L+1}) + A_e(U^L)] \leq K(T + \sum_{n=1}^{L} k|U^n|_1^2)$$

follows, and thus:

$$\rho|\partial_t U^L|_0^2 + \alpha_v \sum_{n=1}^{L} k|\delta_t U^n|_1^2 + \Theta\ \alpha_e|U^{L+1}|_1^2 \leq K\ (T + \sum_{n=1}^{L+1} k|U^n|_1^2\ ).$$

In the case of the implicit schemes that is, for $0 < \Theta \leq 1/2$, we can recall the discrete form of Gronwall's lemma to get

$$|\partial_t U^{L-1}|_0^2 + \sum_{n=1}^{L-1} k|\delta_t U^n|_1^2 + |U^L|_1^2 \leq \text{constant},$$

independently of h, k and $L \leq T/k$. This inequality implies (5.1).

For the explicit scheme, i.e., $\Theta=0$, we must follow another approach to take care of the term $A_e(U^n;\delta_t U^n)$.

Using inequalities (5.3.), (5.4) in (5.2) we get

$$\rho\{|\partial_t U^n|_0^2 - |\partial_t U^{n-1}|_0^2\} + A_e(U^n;U^{n+1} - U^{n-1}) + 2\ k\ A_v(\delta_t U^n) \leq$$

$$k\{\gamma|\delta_t U^n|_1^2 + K(1 + |U^n|_1^2)\ /\ \gamma\ \}$$

and then sum from $n=1$ to $L \leq N$:

$$\rho|\partial_t U^L|_0^2 + \sum_{n=1}^{L} k\ A_v(\delta_t U^n) + A_e(U^L;U^{L+1})$$

$$- A_e(U^1;U^0) \leq \gamma \sum_{n=1}^{L} k\{\delta_t U^n|_1^2$$

$$+ K(T + \sum_{n=1}^{L} k\ |U^n|_1^2\ )/\gamma\ .$$

Now we recall (3.7), the coercivity of A_e and A_v, take $\gamma \equiv \alpha_v/2$ and rewrite

$$A_e\ (U^L;U^{L+1}) = A_e(U^L) + k\ A_e\ (U^L;\partial_t U^L)$$

to obtain

$$\rho|\partial_t U^L|_0^2 + (\alpha_v/2) \sum_{n=1}^{L} k|\delta_t U^n|_1^2 + \alpha_e\ |U^L|_1^2$$

$$\leq 2\ K(T + \sum_{n=1}^{L} k|U^n|_1^2\ )/\alpha_v\ +$$

$$k\ A_e(U^L)^{1/2}\ \ A_e(\partial_t U^L)^{1/2}$$

The last term is estimated as follows:

$$k\, A_e(U^L)^{1/2}\, A_e(\partial_t U^L)^{1/2} \leq k\, \beta_e |U^L|_1 |\partial_t U^L|_1$$

$$\leq \alpha_e |U^L|_1^2/\, 2 + (\beta_e |\partial_t U^L|_1\, k)^2 \,/\, 2\,\alpha_e$$

$$\leq \alpha_e |U^L|_1^2/2 + \beta_e^2\, k^2\, S(h)^2\, |\partial_t U^L|_0^2 \,/\, 2\,\alpha_e ,$$

where $S(h)$ is the stability function mentioned in (3.10). If we let $h,k\to 0$ in such a way that (3.14) always holds, we obtain

$$\rho |\partial_t U^L|_0^2 + \alpha_v \sum_{n=1}^{L} k |\delta_t U^n|_1^2 + \alpha_e |U^L|_1^2 \leq 4\, K(T + \sum_{n=1}^{L} k |U^n|_1^2) ,$$

which again implies (5.1).

For the sake of brevity, from now on we shall deal only with the implicit schemes. The computations shown above should point up clearly the changes one has to carry in the proofs to make them valid for the explicit case.

To get another fundamental estimate, this time for the predicted approximation, take as a test-function in (3.8a), $w \equiv \hat{\delta}_t U^n$ with $1 \leq n \leq N$, and use the relations

$$\hat{\delta}_t U^n = (\hat{\partial}_t U^n + \partial_t U^{n-1})/2$$

and

$$\hat{\partial}_t^2 U^n = (\hat{\partial}_t U^n - \partial_t U^{n-1})/k$$

that follows from the conventions in (3.6k). We are left with

$$\frac{\rho}{2k} |\hat{\partial}_t U^n|_0^2 + \frac{\Theta}{2k} A_e(\hat{U}^{n+1}) + A_v(\hat{\delta}_t U^n) =$$

$$(1-2\Theta)\; A_e(U^n;\hat{\delta}_t U^n) - c \langle \psi_\varepsilon(U^n)\phi_\varepsilon'(\partial_t U^{n-1}) | \hat{\delta}_t U^n\rangle$$

$$+ \langle L^n | \hat{\delta}_t U^n\rangle_1 + \frac{\rho}{k} |\partial_t U^{n-1}|_0^2 + \frac{\Theta}{2k} A_e(U^{n-1}) ,$$

and operating in the same fashion as before we reach

$$|\hat{\partial}_t U^n|_0^2 + |\bar{U}^{n+1}|_1^2 + k|\hat{\delta}_t U^n|_1^2 \leq C\,\{|\partial_t U^{n-1}|_0^2 +$$

$$+ |U^{n-1}|_1^2 + |U^n|_1^2 + k|L^n|_1^2\,\} \qquad (5.5)$$

A bound for the quotient differences of second order constructed with the predicted approximations is gotten as follows.

Thanks to (3.6c) and the identity

$$\hat{\partial}_t^2 U^n - \hat{\partial}_t^2 U^{n-1} = 2\{(\hat{\delta}_t U^n - \hat{\delta}_t U^{n-1})/k + \partial_t^2 U^{n-1}\}\ ,$$

when we subtract equation (3.8a) considered for n-1 from the same equation taken for n, $2 \leq n \leq N$, with $w \equiv (\hat{\delta}_t U^n - \hat{\delta}_t U^{n-1})/k$, we obtain

$$2\rho\ \left\{\left|\frac{\hat{\delta}_t U^n - \hat{\delta}_t U^{n-1}}{k}\right|_0^2 - \langle \partial_t^2 U^{n-1} | w\rangle\right\} + k\ A_v\left(\frac{\hat{\delta}_t U^n - \hat{\delta}_t U^{n-1}}{k}\right)$$

$$+ k^2 \Theta A_e\left(\frac{\hat{\delta}_t U^n - \hat{\delta}_t U^{n-1}}{k}\right) \leq k|A_e((1-2\Theta)\ \partial_t U^{n-1} + 2\Theta \partial_t U^{n-2}; w)|$$

$$+ k\left|\left\langle \frac{L^n - L^{n-1}}{k} \middle| w\right\rangle_1\right| + ck\left|\left\langle\left\{\frac{\psi_\varepsilon(U^n) - \psi_\varepsilon(U^{n-1})}{k}\right\}\phi'_\varepsilon(\partial_t U^{n-1})\,|w\right\rangle\right|$$

$$ck\left|\ \left\langle \psi_\varepsilon(U^{n-1})\ \frac{\phi'_\varepsilon(\partial_t U^{n-1}) - \phi'_\varepsilon(\partial_t U^{n-2})}{k}\ |w\right\rangle\right|\ .$$

Denoting by α the righ-hand side of this inequality, it is seen that

$$\alpha \leq k\{(1-2\Theta)\ A_e(\partial_t U^{n-1})^{1/2} + 2\Theta\ A_e(\partial_t U^{n-2})^{1/2}\}A_e(w)^{1/2}$$

$$+ k\ |\partial_t L^{n-1}|_1\ |w|_1 + ck\ C_1|\partial_t U^{n-1}|_0\ \ |w|_0$$

$$+ ck\ C_1\ C_2|\partial_t^2 U^{n-1}|_0\ \ |w|_0,$$

where $C_1 = C_1(\phi_\varepsilon')$ and $C_2 = C_2(\phi_\varepsilon'')$. This dependence of C_2 on ϕ_ε'' implies our estimates will vary with ε, as we must have $|\phi_\varepsilon''|_\infty \uparrow \infty$ with $\varepsilon \downarrow 0$. It is possible to handle the above inequality in the standart way to get:

$$\left|\frac{\hat\delta_t U^n - \hat\delta_t U^{n-1}}{k}\right|_0^2 + k\left|\frac{\hat\delta_t U^n - \hat\delta_t U^{n-1}}{k}\right|_1^2 + k^2\left|\frac{\hat\delta_t U^n - \hat\delta_t U^{n-1}}{k}\right|_1^2$$

$$\leqslant C\,(|\partial_t^2 U^{n-1}|_0^2 + k\,\{|\partial_t U^{n-1}|_1^2 + |\partial_t U^{n-2}|_1^2$$

$$+ |\partial_t L^{n-1}|_1^2 + |\partial_t^2 U^{n-1}|_0^2\})\ . \tag{5.6}$$

The aim now is to bound $\tilde\delta_t^2 U_{h,k}$ and $\partial_t^2 U_{h,k}$. Consider equation (3.8b) for n and $n+1$, and take for both

$$w \equiv \delta_t U^{n+1} - \delta_t U^n = k\,\delta_t\,\partial_t U^n = k\,\partial_t^2 U^{n+1/2}\ .$$

After subtraction, the following relation is reached

$$\frac{k}{2}\,\rho\{|\partial_t^2 U^{n+1}|_0^2 - |\partial_t^2 U^n|_0^2\} + \frac{k^2}{2}\,A_v(\partial^2 U^{n+1/2})$$

$$+ \frac{k}{2}\,A_e(W_\Theta \partial_t U^n; \partial_t U^{n+1} - \partial_t U^{n-1})$$

$$= k^2 < \partial_t L^n \mid \partial_t^2 U^{n+1/2} >_1$$

$$- ck<[\psi_\varepsilon(U^{n+1}) - \psi_\varepsilon(U^n)]\ \phi_\varepsilon'(\hat\delta_t U^{n+1})\,|\partial_t^2 U^{n+1/2}>$$

$$- ck<\psi_\varepsilon(U^n)\ [\phi_\varepsilon'(\hat\delta_t U^{n+1}) - \phi_\varepsilon'(\hat\delta_t U^n)]\,|\partial_t^2 U^{n+1/2}>$$

The left-hand side of this relation has the same formas that of (5.2). To deal with the last term in the right-hand side, (5.6) must be used, while estimates for the other terms are straight-forward:

i) $\quad k^2\ |<\partial_t L^n|\partial_t^2 U^{n+1/2}>_1| \leqslant k^2|\partial_t L^n|_1^2/4\gamma + \gamma|\partial_t^2 U^{n+1/2}|_1^2$;

ii) $ck[<\psi_\varepsilon(U^{n+1}) - \psi_\varepsilon(U^n)]\ \phi'_\varepsilon(\hat{\delta}_t U^{n+1})\ |\partial_t^2 U^{n+1/2}>|$

$$\leq k^2\ C_1\ \{|\partial_t U^n|_0^2/4\gamma + \gamma|\partial_t^2 U^{n+1/2}|_0^2 \quad ;$$

iii) $ck|<\psi_\varepsilon(U^n)\,[\phi'_\varepsilon(\hat{\delta}_t U^{n+1}) - \phi'_\varepsilon(\hat{\delta}_t U^n)]\,|\partial_t^2 U^{n+1/2}>|$

$$\leq\ k^2\ c\ C\ C_2\ \left|\frac{\hat{\delta}_t U^{n+1} - \hat{\delta}_t U^n}{k}\right|_0\ |\partial_t^2 U^{n+1/2}|_0$$

$$\leq k^2\ C_\varepsilon\ \{\ \left|\frac{\hat{\delta}_t U^{n+1} - \hat{\delta}_t U^n}{k}\right|_0^2 / 4\gamma + \gamma|\partial_t^2 U^{n+1/2}|_0^2\ \}$$

$$\leq k^2\ C_\varepsilon\ \{\gamma|\partial_t^2 U^{n+1/2}|_0^2 + (|\partial_t^2 U^n|_0^2 + k\ [|\partial_t U^n|_1^2 + |\partial_t U^{n-1}|_1^2$$

$$|\partial_t L^n|_1^2 + |\partial_t^2 U^n|_0^2])/4\gamma\ \}\ .$$

Putting these relations together we reach

$$\rho\{|\partial_t^2 U^{n+1}|_0^2 - |\partial_t^2 U^n|_0^2 + 2\ k\ A_v(\partial_t^2 U^{n+1/2})$$

$$+ \Theta[A_e(\partial_t U^{n+1}) - A_e(\partial_t U^{n-1})] \leq (1-2\Theta)2k\ A_e(\partial_t U^n)^{1/2}\ A_e(\delta_t \partial_t U^n)^{1/2}$$

$$+ 2k\{|\partial_t L^n|_1^2/4\gamma + C_\varepsilon|\partial_t U^n|_0^2/4\gamma + C_\varepsilon|\partial_t^2 U^n|_0^2/4\gamma$$

$$+ k(|\partial_t U^n|_1^2 + |\partial_t U^{n-1}|_1^2 + |\partial_t L^n|_1^2 + |\partial_t^2 U^n|_0^2)/4\gamma\ \}$$

$$+ 12\ \gamma\ k|\partial_t^2 U^{n+1/2}|_0^2 \quad ,$$

from which

$$\{|\partial_t^2 U^{n+1}|_0^2 - |\partial_t^2 U^n|_0^2\} + 4k|\partial_t^2 U^{n+1/2}|_1^2$$

$$+ [A_e(\partial_t U^{n+1}) - A_e(\partial_t U^{n-1})] \lesssim C\, k\, \{|\partial_t U^n|_1^2 +$$

$$+ |\partial_t L^n|_1^2 + |\partial_t U^n|_0^2 + |\partial_t^2 U^n|_0^2 \}$$

$$+ Ck^2(|\partial_t U^n|_1^2 + |\partial_t U^{n-1}|_1^2 + |\partial_t L^n|_1^2 + |\partial_t^2 U^n|_0^2)$$

$$\lesssim C\, k\, \{|\partial_t L^n|_1^2 + |\partial_t U^n|_1^2 + |\partial_t U^{n-1}|_1^2 + |\partial_t^2 U^n|_0^2 \}$$

follows, by assuming $k < 1$. Summing up from $n = 1$ to $n = L \lesssim N - 1$ we obtain

$$|\partial_t^2 U^{L+1}|_0^2 + k \sum_{n=1}^{L} |\partial_t^2 U^{n+1/2}|_1^2 + A_e(\partial_t U^{L+1}) \lesssim |\partial_t^2 U^1|_0^2$$

$$+ A_e(\partial_t U^1) + A_e(\partial_t U^0) +$$

$$+ Ck \sum_{n=1}^{L} \{|\partial_t L^n|_1^2 + |\partial_t U^n|_1^2 + |\partial_t^2 U^n|_0^2\} ,$$

and therefore, by recalling that $\partial_t U^0 = 0$:

$$|\partial_t^2 U^{L+1}|_0^2 + k \sum_{n=1}^{L} |\partial_t^2 U^{n+1/2}|_1^2 + |\partial_t U^{L+1}|_1^2 \lesssim$$

$$\lesssim C \{|\partial_t U^1|_1^2 + |\partial_t^2 U^1|_0^2 + k \sum_{n=1}^{L} [|\partial_t^2 U^n|_0^2 + |\partial_t L^n|_1^2 + |\partial_t U^n|_1^2]\} .$$

The bound we are looking for will be obtained as sonn as we can estimate $|\partial_t U^1|_1$ and $|\partial_t^2 U^1|_0$. With $w \equiv \partial_t U^1 = 2\delta_t U^1 = k\partial_t^2 U^1 = U^2/k$ in (3.8b), we obtain

$$\rho k|\partial_t^2 U^1|_0^2 + \Theta\, A_e(U^2)/k + A_v(\partial_t U^1)/2 = \langle L^1|\partial_t U^1\rangle_1 ,$$

where use was made of (3.3) and (3.7). This relation implies that

$$|\partial_t U^1|_1 \lesssim \text{constant} .$$

In the same fashion, by taking $w \equiv \partial_t^2 U^1 = U^2/k^2 = 2\delta_t U^1/k = \partial_t U^1/k$ in (3.8b) we get

$$\rho|\partial_t^2 U^1|_0^2 + \Theta\, A_e(U^2/k) + \frac{k}{2} A_v(\partial_t^2 U^1) = <\partial_t L^1|\partial_t U^1>_1 + <f(0)|\partial_t^2 U^1>$$

$$\leqslant \{|\partial_t L^1|_1^2 + |\partial_t U^1|_1^2 + |f(0)|_0^2/\rho + \rho|\partial_t^2 U^1|_0^2\}/2$$

and consequently

$$|\partial_t^2 U^1|_0^2 \leqslant \text{constant}.$$

Thus we can conclude:

$$|\partial_t U_{h,k}|_{L^\infty(0,T;H^1(\Omega))} + |\tilde{\partial}_t^2 U_{h,k}|_{L^2(0,T;H^1(\Omega))}$$

$$|\partial_t^2 U_{h,k}|_{L^\infty(0,T;L^2(\Omega))} \leqslant C \qquad (5.7)$$

where $C = C(\varepsilon,T)$ does not depend on h and k. This finishes the proof.

We remark that (5.1) and (5.7) are the discrete analogous to estimates (3.28) and (3.32) in ref.1.

Proof of Theorem 2

The conclusion of Lemma 1 implies the existence of a sub--sequence of $\{U_{h,k}\}$, for which we shall use the same notation, as well as the existence of functions $U = U^\varepsilon$, U_1, $U_2 \in L^2(0,T;H^1(\Omega))$ such that, as $h,k \to 0$ we have the weak convergence

$$U_{h,k} \to U,\ \partial_t U_{h,k} \to U_1,\ \partial_t^2 U_{h,k} \to U_2\ . \qquad (5.8)$$

(Notice that the weak-convergence of $\partial_t^2 U_{h,k}$ is a consequence of having $\tilde{\partial}_t^2 U_{h,k} \to U_2$ weakly).

We claim that $U_1 = \dot{U}$ and $U_2 = \ddot{U}$. Indeed, denote by $D(0,T)$ the space of infinitely differentiable real functions ψ on $|0, T|$ that vanish outside a sub-interval $|\delta, T-\delta|$ with $\delta = \delta(\psi) > 0$. Now take the vector-valued distribution $D'(0,T;H^1(\Omega))$, that is, the set of continuous linear mappings from $D(0,T)$ to $H^1(\Omega)$. It is seen that the functions $f \in L^2(0,T;H^1(\Omega))$ are naturally associated to the distributions $T(f)$ in $D'(0,T;H^1(\Omega))$ through

$$\psi \in D(0,T) \to \langle T(f),\psi\rangle \equiv \{ \int_0^T \psi(t) f(t)\,dt \} \in H^1(\Omega),$$

in such a way that, for f, g, $f_n \in L^2(0,T;H\ (\Omega))$, $n = 0,1\ \ldots\ ,$:

i) if $\langle T(f_1),\psi\rangle = \langle T(f_2),\psi\rangle \quad , \quad \forall\ \psi \in D(0,T)$ then $f_1 = f_2$;

ii) whenever $f_n \to f_0$ weakly, then

$$\langle T(f_n),\psi\rangle \to \langle T(f_0),\psi\rangle , \quad \forall \quad \psi \in D(0,T) ,$$

that is $T(f_n) \to T(f_0)$ in the sense of distributions.

For fixed $\psi \in D(0,T)$, if $k > 0$ is less than $\delta(\psi)$, useof the summation-by-parts formula

$$\sum_{j=0}^{L-1} a_j(b_{j+1}-b_j) = - \sum_{j=0}^{L-1} (a_{j+1}-a_j)b_{j+1} + a_L b_L - a_0 b_0$$

gives

$$\langle T(\partial_t U_{h,k}),\psi\rangle = \sum_{n=0}^{N} \frac{U^{n+1}-U^n}{k} \int_{t_n}^{t_{n+1}} \psi(t)\,dt =$$

$$= - \sum_{n=1}^{N} U^n \{ \int_{t_n}^{t_{n+1}} \psi(t)\,dt - \int_{t_{n-1}}^{t_n} \psi(t)\,dt \} / k$$

$$= - \langle T(U_{h,k}) , \partial_t \psi \rangle,$$

with $\partial_t \psi(s) \equiv \{\psi(s+k) - \psi(s)\}/k$.

Now recall that if $\psi \in D(0,T)$, $|\partial_t\psi - \dot{\psi}|_{L^2(0,T)} \to 0$, so that

$$\langle T(\partial_t U_{h,k}),\psi\rangle \to - \langle T(U),\dot{\psi}\rangle = \langle T(\dot{U}),\psi\rangle \quad ,$$

from which we conclude that $U_1 = \dot{U}$ because from (5.8) we can infer that

$$<T(\partial_t U_{h,k}),\psi> \to <T(U_1),\psi>$$

The inequality $U_2 = \ddot{U}$ is deduced in a similar way, and therefore we can rewrite (5.8) as

$$U_{h,k} \to U, \quad \partial_t U_{h,k} \to \dot{U}, \quad \partial_t^2 U_{h,k} \to \ddot{U} , \qquad (5.9)$$

the weak convergence in $L^2(0,T;H^1(\Omega))$ always being meant.

The aim now is to show that U equals the solution u_ε of (2.2) - (2.3) and (3.2) and that the modes of convergence described in (3.11) - (3.13) do hold.

Observe that, as long as we show the equality $U=u_\varepsilon$, by using the uniqueness property of (2.2) - (2.3) and (3.2), we can deduce that the choice of a sub-sequence is unnecessary: convergence of the whole family $U_{h,k}$ occurs.

To prove that $U = u_\varepsilon$ we show that U is a weak solution of (3.2), in the sense that

$$\int_0^T \{\rho<\ddot{U}|v> + A_e(U;v) + A_v(\dot{U};v)\}dt$$

$$-\int_0^T <L|v>_1 \, dt = - \int_0^T c<\psi_\varepsilon(U) \;\; \phi'_\varepsilon(\dot{U})\,|v> dt \qquad (5.10)$$

holds for any $v \in L^2(0,T;H^1(\Omega))$. Then we make use of the same reasoning as in Theorem 3.1 in ref.1 to deduce that any solution of (5.10) satisfies (3.2), so that by the uniqueness property, the conclusion $U = u_\varepsilon$ is reached.

In order to get (5.10), denote

$$L_k(t) \equiv \sum_{n=0}^{N} L^n \chi_n^k(t) \in L^2(0,T;H^1(\Omega)) , \qquad (5.11)$$

so that

$$\lim_{k\to 0} |L_k - L|_{L^2(0,T;H^1(\Omega))} = 0 . \qquad (5.12a)$$

Take also an arbitrary $v \in L^2(0,T;H^1(\Omega))$ ans strong approximations $V^n \in W_h$ in such a way that

$$\lim_{k,h\to 0} |V_{h,k} - v|_{L^2(0,T;H^1(\Omega))} = 0 . \qquad (5.12b)$$

For $n = 1,\dots, N$, take $w \equiv V^n$ in (3.8b); multiplication by k and addition yields

$$\sum_{n=1}^{N} k \{\rho<\partial_t^2 U^n|V^n> + A_e(W_\Theta U^n;V^n) + A_v(\delta_t U^n;V^n)\} + c \sum_{n=1}^{N} <\psi_\varepsilon(U^n)\phi'_\varepsilon(\hat{\delta} U^n)|V^n> = \sum_{n=1}^{N} k < L^n | V^n >_1 .$$

Employing definitions (3.6f) - (3.6i) and (5.11), we rewrite this relation in the equivalent form

$$\int_0^T \{\rho<\partial_t^2 U_{h,k}|V_{h,k}> + A_e(W_\Theta U_{h,k};V_{h,k}) + A_v(\delta_t U_{h,k};V_{h,k})\}dt$$

$$- \int_0^T <L_k|V_{h,k}>_1 \, dt = - c \int_0^T <\psi_\varepsilon(U_{h,k})\phi'_\varepsilon(\hat{\delta}_t U_{h,k})|V_{h,k}> dt , \qquad (5.13)$$

where we are denoting

$$\psi_\varepsilon(U_{h,k}) \equiv \sum_{n=1}^{N} \chi_n^k(t) \; \psi_\varepsilon(U^n) ,$$

$$\phi'_\varepsilon(\hat{\delta}_t U_{h,k}) \equiv \sum_{n=1}^{N} \chi_n^k(t) \; \phi'_\varepsilon(\hat{\delta}_t U^n) .$$

Relations (5.9) and (5.12) imply that the two integrals in the left - hand side of (5.13) converge to the corresponding ones in (5.10). All left to show is the convergence of the non-linear term. For this result, stronger modes of convergence than the ones described in (5.9) must be obtained.

Let $x \in [0,1]$ be fixed. The family $U_{h,k}(x,\cdot)$ is uniformly bounded in $L^2(0,T)$ and satisfies

$$\lim_{\sigma\to 0} \int_0^{T-\sigma} |U_{h,k}(x,t+\sigma) - U_{h,k}(x,t)|^2 \, dt = 0$$

uniformly with respect to h and k. Thus by Frechet-Kolmogorov Theorem, (see ref.10, pp.275), $U_{h,k}$ remains within a compact set in $L^2(0,T)$. By using Tychonov's Theorem on product of compact spaces (see ref.10,pp.6), we conclude that $U_{h,k}$ is contained in a compact set of $L^2(Q_T)$, so that the existence of a sub-sequence of $U_{h,k}$ -still denoted by $U_{h,k}$-for which

$$\lim_{h,k\to 0} |U_{h,k} - U|_{L^2(Q_T)} = 0 \qquad (5.14a)$$

is guaranteed. By a similar argument we arrive at a sub-sequence for which

$$\lim_{h,k\to 0} |\partial_t U_{h,k} - \dot{U}|_{L^2(Q_T)} = 0 \qquad (5.14b)$$

holds.

The last step in the proof is to obtain the pointwise limit

$$\lim_{h,k\to 0} \phi'_\varepsilon(\hat{\delta}_t U_{h,k}) = \phi'_\varepsilon(\dot{U}) \; . \qquad (5.15)$$

This relation together with (5.12b) and (5.14a) plus an application of Lebesgue's Dominated Convergence Theorem shall yield the sought conclusion:

$$\lim_{h,k\to 0} \int_0^T \langle\psi_\varepsilon(U_{h,k})\phi'_\varepsilon(\hat{\delta}_t U_{h,k})\,|V_{h,k}\rangle dt = \int_0^T \langle\psi_\varepsilon(U)\phi'_\varepsilon(\dot{U})\,|v\rangle dt. \qquad (5.16)$$

Notice that, since

$$\hat{\delta}_t U^n = \delta_t U^n + (\bar{U}^{n+1} - U^{n+1})/2k \; ,$$

(5.14b) would imply $\hat{\delta}_t U_{h,k} \to \dot{U}$ in $L^2(Q_T)$ provided we have

$$\sup_{0\leq n\leq N-1} \left| \frac{\tilde{U}^{n+1} - U^{n+1}}{2\,k} \right|_0^2 = 0(k) \tag{5.17}$$

uniformly with respect to h.

Picking as a test-function in both (3.8a) and (3.8b) $w \equiv (\tilde{U}^{n+1}-U^{n+1})/2$, after subtracting these equations we are left with

$$\frac{2\rho}{k} \left|\frac{\tilde{U}^{n+1}-U^{n+1}}{2k}\right|_0^2 + \frac{\Theta\alpha_e}{2k} \left|\tilde{U}^{n+1} - U^{n+1}\right|_1^2 + \alpha_v \left|\frac{\tilde{U}^{n+1}-U^{n+1}}{2k}\right|_1^2$$

$$\leq \frac{2C_2^2}{\alpha_v} \left|\hat{\delta}_t U^n - \partial_t U^{n-1}\right|_0^2 + \frac{\alpha_v}{2} \left|\frac{\tilde{U}^{n+1}-U^{n+1}}{2k}\right|_0^2$$

which implies

$$\left| \frac{\tilde{U}^{n+1}-U^{n+1}}{k} \right|_0^2 \leq \frac{4C_2^2 k}{\alpha_v\, \rho} \left|\hat{\delta}_t U^n - \partial_t U^{n-1}\right|_0^2$$

Consequently (5.17) follows, if we use the identity $2\hat{\delta}_t U^n = \hat{\partial}_t U^n + \partial_t U^{n-1}$ and the estimates (5.1) and (5.5). The proof is thus concluded.

CONCLUSIONS

For a numerical simulation of the motion of a pile propelled into the soil by a pile-hammer, a variational inequality is proposed in Eqs. (1.6) and (2.1)-(2.3). Theorem 1 in Section 2 asserts that they give rise to a mathematically well-posed problem, while a physical justification of the modelling is shown by the computer experiments exhibited in Section 4. These calculations also confirm the theoretical studies of the algorithm used, as described in Section 3.

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