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Functional Theory of Stochastic Neutron Transport

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A functional theory of the stochastic transport of neutrons is presented. The new formulation generalizes the methods which have been applied so far in order to obtain the singlet and doublet densities. The functional approach leads to a single master equation which carries all information on the stochastic properties of the system. Therefore, kinetic equations for moments and correlations of arbitrary order as well as for probability generating functionals can be obtained in a straightforward way. All calculations are performed in continuous phase space through the appropriate definitions of a singular transition probability.

É apresentada uma teoria funcional estocástica do transporte de neutrons. A nova formulação generaliza os métodos que têm sido aplicados para a obtenção das densidades de primeira e segunda ordem. O tratamento funcional conduz a uma Única equação-mestra a qual contém toda a informação sobre as propriedades estocásticas do sistema. Pode-se obter de uma maneira direta, portanto, equações cinéticas de momentos e correlações de ordem arbitrária bem como de funcionais geratrizes de probabilidade. Todos os cálculos são efetuados no espaço de fases continuo definindo - se para isso probabilidades de transição singulares adequadas.

1. INTRODUCTION

We propose to formulate in this paper a compact theory of stochastic neutron transport, bearing in mind possible applications to some less explored areas of research in the field of reactor noise analysis. Although the theory of zero power reactor noise may be considered established on a sound theoretical basis, we feel that there is still room

for fundamental research in at least two particular areas, namely (a) analysis of higher-than-the-second order effects: high order moments and correlations are increasingly difficult to compute but, nevertheless, they are required in order to investigate the possibility of obtaining new information from high order experiments as well as to obtain error estimates in conventional second order ones; (b) evaluation of detector effects: a more unifying approach in the treatment of detection processes is still lacking even in the framework of the point reactor one velocity model.

The subject of stochastic transport of neutrons has been treated using severa1 approaches and degrees of sophistication. Pal[1,2], appears to be the first to derive a backward transport equation forthe singlet probability density of neutrons. Osborn and Yip[3,4] formulated a transport theory for the singlet and doublet densities in coarse grained phase space through a quantum statistical mechanics formulation. Otsuka and Saito[5] developed a simpler classical formulation for the singlet density also in coarse grained phase space. Akcasu and Osborn[6] used Langevin's technique to compute correlation functions of the neutron density and detection rate. Williams[7], and Cassel and Williams[8] applied diffusion and transport models to the evaluation of space-dependent effects on some noise experiments.

In an attempt to develop a stochastic transport formulation simple and compact enough to allow easy computation of high ordermoments and correlations as well as detector effects, we have avoided using Langevin's approach since it gives a correct description of uptosecond order effects only. We have chosen to use instead the forward Kolmogorov formulation because of its inherent advantages over the backward one: (a) the forward equation is linear; (b) an equation for the probability generating function (pgf) of the state variable can be derived for a system with an external source (e.g., a stationary subcritical reactor); (c) the forward formulation is suitable for the analysis of detector effects by the method of the non-homogeneous Poisson distribution developed by Watson[9,10], which requires only the computation of moments and correlations of theneutronicstatevariable (neutronandprecursordensities). We formulate the stochastic transport of neutrons in continuous phase space, thus abstaining from dealing with the cumbersome coarse grained model. We

also avoid the unnecessary complications of the quantum statistical mechanics model, since quantum effects (except for those already lumped in the cross sections) are not expected to influence the results. Furthermore, our functional formulation yields a single forward equation which carries all information at all phase space points, in the sence that moments and correlations of any order at different phase space points can be obtained from the same master equation.

To keep only its essential features, the theory presented is based on the prompt neutron, zero power reactor model. Extensions that take into account delayed neutrons and counts are sketched in the Appendix.

As has been shown by Watson[11], high order moments of state variables including delayed neutron precursors are easier to obtain in matrix form directly from the master equation instead of through successive derivations of the pgf. However, an equation for the pgf is obtained here as a by-product. This may serve as a guide-line for the derivation of pgf equations that include counts in the state variable as well (analysis of a few noise experiments still requires the approximate solutions of these equations).

The development of the theory is carried out as faras possible independent of the particular form of transition probabilities involved. These are introduced at a later stage so that explicit equations for moments and correlations up to an arbitrary order can be derived.

2. THE PROBABILITY DENSITY FUNCTIONAL

The stochastic transport of neutrons will be formulated interms of a probability density functional of the neutron density, position, and velocity functions

$$\begin{aligned} iv &: U \to \mathbb{R}^1 & N &\in H \\ r &: U \to R & \underset{\sim}{r} &\in H_r \\ \underset{\sim}{v} &: U \to V & \underset{\sim}{v} &\in H_v \end{aligned}$$

defined in a 6-dimensional phase space $U = R \times V \subseteq \mathbb{R}^6$, where $R \subseteq \mathbb{R}^3$ and $V \subseteq \mathbb{R}^3$ are the 3-dimensional positionandvelocity spaces. Points of these spaces will be denoted by $\underset{\sim}{u}$, $\underset{\sim}{r}$ and $\underset{\sim}{v}$, respectively, with $\{\underset{\sim}{u}\}=\{\underset{\sim}{r},\underset{\sim}{v}\}$.

We tacitly assume that the above H_r and H_v functional spaces include the following functions*

$$\begin{aligned} \underset{\sim}{r}(\underset{\sim}{u}) &= \underset{\sim}{r} \\ \underset{\sim}{v}(\underset{\sim}{u}) &= \underset{\sim}{v} \end{aligned} \tag{1}$$

Let $t \in \mathbb{R}^1$ be the time variable. We consider now the functional

$$P : H \times H_r \times H_v \times \mathbb{R}^1 \to \mathbb{R}^{1+} ,$$

such that $P[N(\underset{\sim}{u}),\underset{\sim}{r}(\underset{\sim}{u}),\underset{\sim}{v}(\underset{\sim}{u}),t]$ will measure the joint probability density of the state variable $N(\underset{\sim}{u})$ at every position and velocity points $\underset{\sim}{r}(\underset{\sim}{u})$ and $\underset{\sim}{v}(\underset{\sim}{u})$ at time t, and subjected to the normalization

$$\int_H dN(\underset{\sim}{u})\, P[N(\underset{\sim}{u}),\underset{\sim}{r}(\underset{\sim}{u}),\underset{\sim}{v}(\underset{\sim}{u}),t] = 1 . \tag{2}$$

The integral appearing above is to be performed in the functional space H and it may be heuristicaly regarded, as has been pointed outbe Beran[12], as a limit when $n \to \infty$ of an integral in $\mathbb{R}^n$.

In the sequel, functional derivatives with respect to $\underset{\sim}{r}(\underset{\sim}{u})$ will be taken at the point (function) $\underset{\sim}{r}(\underset{\sim}{u})$, as displayed in the abreviated notation below

$$\frac{\delta P}{\delta \underset{\sim}{r}(\underset{\sim}{u})} = \left[\frac{\delta P}{\delta \underset{\sim}{r}(\underset{\sim}{u})}\right]_{\underset{\sim}{r}(\underset{\sim}{u}) = \underset{\sim}{r}(\underset{\sim}{u})} . \tag{3}$$

Joint probability functionals at different times may also be defined. Due to the Markovian property of the stochastic process, the doublet probability will carry all information on the process. The following relations will hold

* In order to avoid unnecessary diversity of notation, the same simbols for the functions and their values have been adopted in Eq. (1).

$$P[N,\underset{\sim}{r},\underset{\sim}{v},t] = \int_H dN' P[N,\underset{\sim}{r},\underset{\sim}{v},t;N',\underset{\sim}{r}',\underset{\sim}{v}',t'] \quad , \tag{4}$$

$$P[N,\underset{\sim}{r},\underset{\sim}{v},t;N',\underset{\sim}{r}',\underset{\sim}{v}',t'] = P[N,\underset{\sim}{r},\underset{\sim}{v},t\,|N',\underset{\sim}{r}',\underset{\sim}{v}',t'] \; P[N',\underset{\sim}{r}',\underset{\sim}{v}',t'] \quad . \tag{5}$$

In the above equations, all u-dependences have been omitted for convenience.

3. THE FORWARD CHAPMAN-KOLMOGOROV EQUATION

By factorizing the joint probability $P[N,\underset{\sim}{r}+\underset{\sim}{v}\Delta t,\underset{\sim}{v},t+\Delta t;N',\underset{\sim}{r},\underset{\sim}{v},t]$ into the conditional and singlet probability, according to Eq.(5), and integrating N', using Eq.(4), the Chapman-Kolmogorov equation is obtained

$$P[N,\underset{\sim}{r}+\underset{\sim}{v}\Delta t,\underset{\sim}{v},t+\Delta t] = \int_H dN' P[N,\underset{\sim}{r}+\underset{\sim}{v}\Delta t,\underset{\sim}{v},t+\Delta t\,|N',\underset{\sim}{r},\underset{\sim}{v},t] P[N',\underset{\sim}{r},\underset{\sim}{v},t] \tag{6}$$

Expansion of the conditional probability in terms of Δt gives

$$P[N,\underset{\sim}{r}+\underset{\sim}{v}\Delta t,\underset{\sim}{v},t+\Delta t\,|N',\underset{\sim}{r},\underset{\sim}{v},t] = (1 - \Gamma_N \Delta t)\,\delta[N-N'] + Q_{NN'}\Delta t + 0(\Delta t^2), \tag{7}$$

where the δ-functional is defined, for an arbitrary functional F, by

$$\int_H dN'(\underset{\sim}{u})\, F[N'(\underset{\sim}{u})]\,\delta[N(\underset{\sim}{u}) - N'(\underset{\sim}{u})] = F[N(\underset{\sim}{u})] \quad , \tag{8}$$

and $Q_{NN'}$ and Γ_N are shorthand notations for

$$\begin{aligned} Q_{NN'} &= Q_{N(\underset{\sim}{u}),N'(\underset{\sim}{u})}[\underset{\sim}{r}(\underset{\sim}{u}),\ \underset{\sim}{v}(\underset{\sim}{u}),t] \\ \Gamma_N &= \Gamma_{N(\underset{\sim}{u})}[\underset{\sim}{r}(\underset{\sim}{u}),\underset{\sim}{v}(\underset{\sim}{u}),t] \quad , \end{aligned} \tag{9}$$

with

$$\Gamma_N = \int_H dN' Q_{N'N} \quad . \tag{10}$$

The transition probability per unit time, $Q_{N'N}$, will be derived on physical grounds later (c.f Part 6).

From Eq. (6) and (7), we obtain the master equation

$$\frac{\partial P[\underline{N},\underline{r},\underline{v},t]}{\partial t} + \int_U d\underline{u}'\underline{v}(\underline{u}') \cdot \frac{\delta P[\underline{N},\underline{r},\underline{v},t]}{\delta \underline{r}(\underline{u}')} = \int_H dN' \Big[Q_{\underline{NN}}, P[\underline{N}',\underline{r},\underline{v},t] - Q_{N',N} P[\underline{N},\underline{r},\underline{v},t] \Big] \qquad (11)$$

Two kinds of solutions of Eq.(11) will be considered:

1) the steady-state (time-independent) solution $P_s[N,\underline{r},\underline{v}]$;

2) the conditional solution $P[\underline{N},\underline{r},\underline{v},t|N_0,\underline{r},\underline{v},t]$.

A steady-state solution may exist when $Q_{NN'}$ is time-independent. A conditional solution may be obtained by solving Eq.(11) subjected to the initial condition $\delta[N-N_0]$ at $t=t_0$

We term **RHS** and **LHS** the right and left hand sides of Eq.(11) as well as of its transformations that will appear in the sequel.

4. MOMENTS AND PROBABILITY GENERATING FUNCTIONALS

The average of a generally non-linear functional F is defined in the usual way

$$\left\langle F[\underline{N},\underline{r},\underline{v},t] \right\rangle = \int_H dN\, P[\underline{N},\underline{r},\underline{v},t] F[\underline{N},\underline{r},\underline{v},t] , \qquad (12)$$

a similar definition holding for the steady-state average

$$\left\langle F[\underline{N},\underline{r},\underline{v}] \right\rangle_s .$$

A special notation is reserved for the steady-state average of the state variable at the phase space point $\underline{u}=\underline{u}_1$

$$\bar{N}(\underline{r}_1,\underline{v}_1) = \bar{N}(\underline{r}(\underline{u}_1),\underline{v}(\underline{u}_1)) = \left\langle N(\underline{u}_1) \right\rangle_s . \qquad (13)$$

Averages of several kinds of F-functional will now be defined.

4.1 — Non-central averages

Three types of non-central averages will be considered:

a) the local moment

$$M_k^{loc}(\underset{\sim}{r}_1,\underset{\sim}{v}_1,t) = \left\langle N(\underset{\sim}{u}_1)^k \right\rangle \; ; \qquad (14)$$

b) the non-local moment

$$M_k(\underset{\sim}{r}_1,\ldots,\underset{\sim}{r}_k,\underset{\sim}{v}_1,\ldots,\underset{\sim}{v}_k,t) = \left\langle N(\underset{\sim}{u}_1)\ldots N(\underset{\sim}{u}_k) \right\rangle \; ; \qquad (15)$$

c) the exponential pgf

$$G[\underset{\sim}{x}(\underset{\sim}{u}),\underset{\sim}{r}(\underset{\sim}{u}),\underset{\sim}{v}(\underset{\sim}{u}),t] = \left\langle \exp\left[\int_U d\underset{\sim}{u}\; N(\underset{\sim}{u})\; x(\underset{\sim}{u})\right] \right\rangle \; . \qquad (16)$$

It is worth mentioning that the corresponding F-functionals associated to Eq.(14) through (16) do not depend explicitly on $\underset{\sim}{r}$, $\underset{\sim}{v}$ and t.

4.2 — Central Averages

Similar definitions can be given for functionais acting on the central variable

$$n(\underset{\sim}{u}) = N(\underset{\sim}{u}) - \bar{N}(\underset{\sim}{r},\underset{\sim}{v}) \; . \qquad (17)$$

The definitlons are:

a) the central local moment

$$m_K^{loc}(\underset{\sim}{r}_1,\underset{\sim}{v}_1,t) = \left\langle \left[N(\underset{\sim}{u}_1) - \bar{N}(\underset{\sim}{r}_1,\underset{\sim}{v}_1)\right]^k \right\rangle ; \qquad (18)$$

b) the central non-local moment

$$m_k(\underset{\sim}{r}_1,\ldots,\underset{\sim}{r}_k,\underset{\sim}{v}_1,\ldots,\underset{\sim}{v}_k,t) = \left\langle \left[N(\underset{\sim}{u}_1)-\bar{N}(\underset{\sim}{r}_1,\underset{\sim}{v}_1)\right]\ldots\left[N(\underset{\sim}{u}_k)-\bar{N}(\underset{\sim}{r}_k,\underset{\sim}{v}_k)\right] \right\rangle \qquad (19)$$

c) the central exponential pgf

$$g[\underset{\sim}{x}(\underset{\sim}{u}),\underset{\sim}{r}(\underset{\sim}{u}),\underset{\sim}{v}(\underset{\sim}{u}),t] = \left\langle \exp\left[\int_U [d\underset{\sim}{u}\ N(\underset{\sim}{u}) - \bar{N}(\underset{\sim}{r}(\underset{\sim}{u}),\underset{\sim}{v}(\underset{\sim}{u}))]\right]\right\rangle \tag{20}$$

Now, due to the presence of the steady-state moment $\bar{N}$ in all expressions, the corresponding F are also functional of $\underset{\sim}{r}(\underset{\sim}{u})$ and $\underset{\sim}{v}(\underset{\sim}{u})$.

Functional derivatives of the exponential pgf's lead tothe non-local moments

$$M_k(\underset{\sim}{r}_1,\ldots,\underset{\sim}{r}_k,\underset{\sim}{v}_1,\ldots,\underset{\sim}{v}_k,t) = \left[\frac{\delta^k G[\underset{\sim}{x}(\underset{\sim}{u}),\underset{\sim}{r}(\underset{\sim}{u}),\underset{\sim}{v}(\underset{\sim}{u}),t]}{\delta x(\underset{\sim}{u}_1)\ldots\delta x(\underset{\sim}{u}_k)}\right]_{x(u)=0} \tag{21}$$

$$m_k(\underset{\sim}{r}_1,\ldots,\underset{\sim}{r}_k,\underset{\sim}{v}_1,\ldots,\underset{\sim}{v}_k,t) = \left[\frac{\delta^k g[\underset{\sim}{x}(\underset{\sim}{u}),\underset{\sim}{r}(\underset{\sim}{u}),\underset{\sim}{v}(\underset{\sim}{u}),t]}{\delta x(\underset{\sim}{u}_1)\ldots\delta x(\underset{\sim}{u}_k)}\right]_{x(u)=0} \tag{22}$$

5. FUNCTIONAL KINETIC EQUATIONS

Let us find a kinetic equations for the average ofafunctional F. Multiplying Eq.(11) by F and integrating the variable N, we obtain

$$\text{LHS} = \frac{\partial \langle F\rangle}{\partial t} - \left\langle \frac{\partial F}{\partial t}\right\rangle + \int_U d\underset{\sim}{u}'\underset{\sim}{v}(\underset{\sim}{u}')\cdot\left[\frac{\delta\langle F\rangle}{\delta\underset{\sim}{r}(\underset{\sim}{u}')} - \left\langle\frac{\delta F}{\delta\underset{\sim}{r}(\underset{\sim}{u}')}\right\rangle\right] \tag{23}$$

and

$$\text{RHS} = \int_H dN'\ Q_{N'N}\left[F[\bar{N}'] - F[\bar{N}]\right] \quad . \tag{24}$$

Using a new notation for the transition probability

$$W_{\Delta N,N} = Q_{N+\Delta N,N} \quad , \tag{25}$$

where

$$\Delta N(u) = N'(u) - N(u) \tag{26}$$

is the net transition of the neutron density, we obtain

$$\text{RHS} = \int_H d\Delta N \; W_{\Delta N,N} \left[F[N+\Delta N] - F[N] \right] . \qquad (27)$$

Kinetic equation for the averages of the functionals definedin Part 4 will now be derived.

5.1 — Kinetic Equations for Non-Central Averages

Cmbining Eq. (14) through (16) with Eq. (23) and (27), we obtain the

a) kinetic equation for the local mment

$$\frac{\partial M_k^{loc}}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla}_1 \; M_k^{loc} = \sum_{j=1}^{k} \binom{k}{j} \left\langle \mathcal{D}_j[N(\underset{\sim}{u}),\underset{\sim}{u}_1] N(\underset{\sim}{u}_1)^{k-j} \right\rangle , \qquad (28)$$

where

$$\mathcal{D}_j[N(\underset{\sim}{u}),\underset{\sim}{u}_1] = \int_H d\Delta N(\underset{\sim}{u}) \; W_{\Delta N(\underset{\sim}{u}),N(\underset{\sim}{u})} \left[\Delta N(\underset{\sim}{u}_1)\right]^j ; \qquad (29)$$

b) kinetic equation for the non-local moment

$$\frac{\partial M_k}{\partial t} + \sum_{j=1}^{k} \underset{\sim}{v}_j \cdot \underset{\sim}{\nabla}_j M_k =$$

$$= \sum_{j=1}^{k} \sum_{\substack{i_1<\ldots<i_j \\ =1}}^{k} \left\langle \mathcal{D}_j\left[N(\underset{\sim}{u}),\underset{\sim}{u}_{i_1},\ldots,\underset{\sim}{u}_{i_j}\right] \prod_{\substack{i\neq i_1,\ldots,i_j \\ =1}}^{k} N(\underset{\sim}{u}_i) \right\rangle , \qquad (30)$$

where

$$\mathcal{D}_j[N(\underset{\sim}{u}),\underset{\sim}{u}_1,\ldots,\underset{\sim}{u}_j] = \int_H d\Delta N(\underset{\sim}{u}) \; W_{\Delta N(\underset{\sim}{u}),N(\underset{\sim}{u})} \; \Delta N(\underset{\sim}{u}_1)\ldots\Delta N(\underset{\sim}{u}_j) ; \qquad (31)$$

c) kinetic equation for the exponential pgf

$$\frac{\partial G}{\partial t} + \int_U d\underset{\sim}{u}' \underset{\sim}{v}(\underset{\sim}{u}') \cdot \frac{\delta G}{\delta \underset{\sim}{x}(\underset{\sim}{u}')} = \left\langle R[N(\underset{\sim}{u}), x(\underset{\sim}{u})] F[N(\underset{\sim}{u}), x(\underset{\sim}{u})] \right\rangle \quad , \tag{32}$$

where

$$F[N(\underset{\sim}{u}), x(\underset{\sim}{u})] = \exp\left[\int_U d\underset{\sim}{u}\, N(\underset{\sim}{u})\, x(\underset{\sim}{u}) \right] \quad , \tag{33}$$

$$R[N(\underset{\sim}{u}), x(\underset{\sim}{u})] = G^{\omega}[N(\underset{\sim}{u}), x(\underset{\sim}{u})] - G^{\omega}[N(\underset{\sim}{u}), 0] \tag{34}$$

and G^{ω} is the exponential pgf of the trançition probability, namely

$$G^{\omega}[N(\underset{\sim}{u}), x(\underset{\sim}{u})] = \int_H d\Delta N\, W_{\Delta N, N}\, F[\Delta N, x] \quad , \tag{35}$$

with, according to Eq.(10) and (25),

$$G^{\omega}[N(\underset{\sim}{u}), 0] = \Gamma_{N(\underset{\sim}{u})} \quad . \tag{36}$$

5.2 — Kinetic Equations for Central Averages

Combining Eq. (18) through (20) with Eq. (23) and (27), we obtain the

a) kinetic equation for the central local moment

$$\frac{\partial m_k^{loc}}{\partial t} + \underset{\sim}{v}_1 \cdot \underset{\sim}{\nabla}_1 m_k^{loc} + k\, m_{k-1}^{loc}\, \underset{\sim}{v}_1 \cdot \underset{\sim}{\nabla}_1 \bar{N}_1 =$$

$$= \sum_{j=1}^{k} \binom{k}{j} \left\langle D_j[N(\underset{\sim}{u}), \underset{\sim}{u}_1]\; n(\underset{\sim}{u}_1)^{k-j} \right\rangle \quad ; \tag{37}$$

b) kinetic equation for the central non-local moment

$$\frac{\partial m_k}{\partial t} + \sum_{j=1}^{k} \underset{\sim}{v}_j \cdot \underset{\sim}{\nabla}_j m_k + \sum_{j=1}^{k} m_{k-1}^{(j)}\, \underset{\sim}{v}_j \cdot \underset{\sim}{\nabla}_j \bar{N}_j =$$

$$= \sum_{j=1}^{k} \sum_{i_1<\ldots<i_j=1}^{k} \left\langle D_j\left[N(\underset{\sim}{u}), \underset{\sim}{u}_{i_1}, \ldots, \underset{\sim}{u}_{i_j}\right] \prod_{i\neq i_1,\ldots,i_j=1}^{k} n(\underset{\sim}{u}_i) \right\rangle , \qquad (38)$$

where

$$m_{k-1}^{(j)} = m_{k-1}(\underset{\sim}{r}_1, \ldots, \underset{\sim}{r}_{j-1}, \underset{\sim}{r}_{j+1}, \ldots, \underset{\sim}{r}_k, \underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_{j-1}, \underset{\sim}{v}_{j+1}, \ldots, \underset{\sim}{v}_k, t) \; ; \qquad (39)$$

c) kinetic equation for the central exponential pgf

$$\frac{\partial g}{\partial t} + \int_U d\underset{\sim}{u}' \underset{\sim}{v}(\underset{\sim}{u}') \cdot \frac{\delta g}{\delta \underset{\sim}{r}(\underset{\sim}{u}')} + g \int_U d\underset{\sim}{u}' x(\underset{\sim}{u}') \underset{\sim}{v}(\underset{\sim}{u}') \cdot \underset{\sim}{\nabla} N(\underset{\sim}{r}', \underset{\sim}{v}')$$

$$= \left\langle R[N(\underset{\sim}{u}), x(\underset{\sim}{u})] F[n(\underset{\sim}{u}), x(\underset{\sim}{u})] \right\rangle , \qquad (40)$$

where F and R are given by Eq.(33) and (34).

We mention that the general central moments kinetic equations cannot be trivially obtained from the corresponding non-central equations and the transformation given by Eq. (17).

In order to simplify the kinetic equations further, we anticipate the result (c.f. Part 6)

$$R[N(\underset{\sim}{u}), x(\underset{\sim}{u})] = R[\bar{N}, x(\underset{\sim}{u})] + \Delta R[\bar{n}(\underset{\sim}{u}), x(\underset{\sim}{u})] , \qquad (41)$$

claiming tliat ΔR is a linear functional of its argument $n(\underset{\sim}{u})$.

From the general relations between moments and pgf derivatives, given by Eq.(21), and Eq.(41) and (31), we also obtain

$$\mathcal{D}_j[N(\underset{\sim}{u}), \underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_j] = \mathcal{D}_j[\bar{N}, \underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_j] + \Delta\mathcal{D}_j[\bar{n}(\underset{\sim}{u}), \underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_j] , \qquad (42)$$

with $\Delta\mathcal{D}_j$ linear with respect to $n(\underset{\sim}{u})$.

We shall apply the above properties to obtain simplified versions for the kinetic equations associated to the central averages given by Eq. (38) and (40). Similar equations can be obtained for non-central averages which, however, are not our main concern in this paper. We shall not be concerned also with Eq. (37) since in its RHS appear non-local moments.

Performing the averaging operation indicated in Eq.(38), taking into account Eq. (42), we obtain

$$\frac{\partial m_k}{\partial t} + \sum_{j=1}^{k} \underset{\sim}{v}_j \cdot \underset{\sim}{\nabla}_j m_k + \sum_{j=1}^{k} m_{k-1}^{(j)} \underset{\sim}{v}_j \cdot \underset{\sim}{\nabla}_j \bar{N}_j$$

$$= \sum_{j=1}^{k} \binom{k}{j} \mathcal{D}_j \left[\bar{N}, \underset{\sim}{u}_{i_1}, \ldots, \underset{\sim}{u}_{i_j} \right] m_{k-j}^{(i_1, \ldots, i_j)} \qquad (43)$$

$$+ \sum_{j=1}^{k} \sum_{i_1 < \ldots < i_j = 1}^{k} \Delta \mathcal{D}_j \left[m_{k-j+1}^{(i_1, \ldots, i_j)}, \underset{\sim}{u}_{i_1}, \ldots, \underset{\sim}{u}_{i_j} \right]$$

where, in a notation consistent with Eq. (39), we term

$$m_{k-j}^{(i_1, \ldots, i_j)} = m_{k-j}(\underset{\sim}{r}_1, \ldots, \underset{\sim}{r}_\ell, \ldots, \underset{\sim}{r}_k, \underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_\ell, \ldots, \underset{\sim}{v}_k, t)$$

$$m_{k-j+1}^{(i_1, \ldots i_j)} = m_{k-j+1}(\underset{\sim}{r}, \underset{\sim}{r}_1, \ldots, \underset{\sim}{r}_\ell, \ldots, \underset{\sim}{r}_k, \underset{\sim}{v}, \underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_\ell, \ldots, \underset{\sim}{v}_k, t)$$

$$\ell \neq i_1, \ldots, i_j \quad . \qquad (44)$$

It will be instructive to specialize Eq.(43) for the cases k=1 and k=2. The former case leads to the first order central moment equation

$$\frac{\partial m_1}{\partial t} + \underset{\sim}{v}_1 \cdot \underset{\sim}{\nabla}_1 m_1 + \underset{\sim}{v}_1 \cdot \underset{\sim}{\nabla}_1 \bar{N}_1 = \mathcal{D}_1 [\bar{N}, \underset{\sim}{u}_1] + \Delta \mathcal{D}_1 [m_1, \underset{\sim}{u}_1] \quad . \qquad (45)$$

If the reactor is stationary (i.e. if the transition probabilities $Q_{NN'}$, and therefore, the $\mathcal{D}_j$ do not depend on time), the first order steady-state central moment is zero and the following steady-state equation is obtained

$$\underset{\sim}{v}_1 \cdot \nabla \bar{N}_1 = \mathcal{D}_1 [\bar{N}, \underset{\sim}{u}_1] \quad . \tag{46}$$

For this case, Eq. (45) reduces to the form

$$\frac{\partial m_1}{\partial t} + \underset{\sim}{v}_1 \cdot \nabla_1 m_1 = \Delta \mathcal{D}_1 [m_1, \underset{\sim}{u}_1] \tag{47}$$

or, in terms of the Boltzmann operator B,

$$\frac{\partial m_1}{\partial t} + B_1 m_1 = 0 \tag{48}$$

with

$$B = \underset{\sim}{v} \cdot \nabla - \Delta \mathcal{D}_1 [\cdot, \underset{\sim}{u}] \quad . \tag{49}$$

The second order moment equation is obtained from Eq.(43) with k=2. For a stationary reactor, we have the result

$$\frac{\partial m_2}{\partial t} + \underset{\sim}{v}_1 \cdot \nabla_1 m_2 + \underset{\sim}{v}_2 \cdot \nabla_2 m_2 = \Delta \mathcal{D}_1 [m_2^{(1)}, \underset{\sim}{u}_1] + \Delta \mathcal{D}_1 [m_2^{(2)}, \underset{\sim}{u}_2] +$$

$$+ \mathcal{D}_2 [\bar{N}, \underset{\sim}{u}_1, \underset{\sim}{u}_2] + \Delta \mathcal{D}_2 [m_1, \underset{\sim}{u}_1, \underset{\sim}{u}_2] \tag{50}$$

where, according to Eq. (44),

$$m_2^{(1)} = m_2(\underset{\sim}{r}, \underset{\sim}{r}_2, \underset{\sim}{v}, \underset{\sim}{v}_2)$$

$$m_2^{(2)} = m_2(\underset{\sim}{r}_1, \underset{\sim}{r}, \underset{\sim}{v}_1, \underset{\sim}{v}) \quad . \tag{51}$$

In terms of the Boltzmann operator, Eq.(49), and keeping in mind Eq.(51), we cast Eq. (50) into the form

$$\frac{\partial m_2}{\partial t} + B_1 m_2 + B_2 m_2 = \mathcal{D}_2 [\bar{N}, \underset{\sim}{u}_1, \underset{\sim}{u}_2] + \Delta \mathcal{D}_2 [m_1, \underset{\sim}{u}_1, \underset{\sim}{u}_2] \tag{52}$$

Next, using Eq. (41) in Eq. (40), we obtain the RHS of the kinetic equation for the central exponential pgf

$$\text{RHS} = R[\bar{N},\ x(\underset{\sim}{u})]g + \Delta R\left[\frac{\delta g}{\delta x(\underset{\sim}{u})},\ x(\underset{\sim}{u})\right] . \tag{53}$$

Let $\Delta\tilde{R}$ be the Kernel of the linear functional ΔR. Then, we can write Eq. (40) in the form

$$\frac{\partial g}{\partial t} + \int_U d\underset{\sim}{u}'\, \underset{\sim}{v}(\underset{\sim}{u}') \cdot \frac{\delta g}{\delta \underset{\sim}{x}(\underset{\sim}{u}')} + g \int_U d\underset{\sim}{u}'\, x(\underset{\sim}{u}')\, \underset{\sim}{v}(\underset{\sim}{v}') \cdot \nabla \underset{\sim}{N}(\underset{\sim}{x}', \underset{\sim}{v}')$$

$$= R[\bar{N}, x(\underset{\sim}{u})]g + \int_U d\underset{\sim}{u}'\, \Delta\tilde{R}[x(\underset{\sim}{u}), \underset{\sim}{u}']\, \frac{\delta g}{\delta x(\underset{\sim}{u}')} . \tag{54}$$

5.3 — Kinetic Equation for the Second Order Covariance

Covariances of up to an arbitrary order can be derived taking into account the Markovian property of the stochastic process*. The stationary second order non-local covariance, in particular, is defined by the doublet steady-state average

$$m_{21}(\underset{\sim}{u}_2, t_2; \underset{\sim}{u}_1, t_1) = \left\langle n(\underset{\sim}{u}_2, t_2) n(\underset{\sim}{u}_1, t_1) \right\rangle_s \tag{55}$$

Applying the conditional relation, given by Eq. (5), we obtain

$$m_{21}(\underset{\sim}{u}_2, t_2; \underset{\sim}{u}_1, t_1) = \left\langle m_1(\underset{\sim}{u}_2, t_2 | t_1) n(\underset{\sim}{u}_1, t_1) \right\rangle_s \tag{56}$$

where m_1 is the solution of the first order moment equation

$$\begin{aligned} &\frac{\partial m_1}{\partial t} + B_2 m_1 = 0 \\ &m_1(\underset{\sim}{u}_2, t_1 | t_1) = n(\underset{\sim}{u}_2, t_1) . \end{aligned} \tag{57}$$

* In fact, the derivation of only third and higher order covariances requires the Markovian property to be invoked.

Therefore, m_{21} is the solution of the equation

$$\frac{\partial m_{21}}{\partial t} + B_2 m_{21} = 0$$
$$m_{21}(\underset{\sim}{u}_2, t_1; \underset{\sim}{u}_1, t_1) = m_2^s(\underset{\sim}{u}_1, \underset{\sim}{u}_2) \quad , \tag{58}$$

where

$$m_2^s(\underset{\sim}{u}_1, \underset{\sim}{u}_2) = \left\langle n(\underset{\sim}{u}_2, t_1) n(\underset{\sim}{u}_1, t_1) \right\rangle_s \quad . \tag{59}$$

6. TRANSITION PROBABILITY FOR THE NEUTRON TRANSPORT PROCESS

We shall make the usual assumption that the only possible transition events are source emission, capture and production (scattering and fission). Moreover, in a vanishingly small time interval, the probability of occurence of more than one type of event at more than one phase space point is also vanishingly small. Therefore, all events at all phase space points are independent of each other and the total transition probability is made up of summed contributions of every individual event and integrated contributions of every phase space point.

The net variation of the number of incoming $(\underset{\sim}{u}')$ and outcoming $(\underset{\sim}{u}'')$ neutrons for each event is given in the table below

event	Δ incoming neutron $(\underset{\sim}{u}')$	Δ outcoming neutron $(\underset{\sim}{u}'')$
source emission	0	1
capture	-1	0
production	-1	ν

Table 1 - possible transition events for the prompt neutron model

The relative probabilities of the events will be measured by the following material parameters:

external source: $S(\underset{\sim}{r}'',\underset{\sim}{v}'')$

capture cross section: $\Sigma_c(\underset{\sim}{r}',\underset{\sim}{v}')$

production cross section: $\Sigma_p(\underset{\sim}{r}',\underset{\sim}{v}' \to \underset{\sim}{r}'',\underset{\sim}{v}'';\nu)$.

Due to the local nature of the production event, we have

$$\Sigma_p(\underset{\sim}{r}',\underset{\sim}{v}' \to \underset{\sim}{r}'',\underset{\sim}{v}'';\nu) = \delta(\underset{\sim}{r}'-\underset{\sim}{r}'')\Sigma_p(\underset{\sim}{r}';\underset{\sim}{v}' \to \underset{\sim}{v}'';\nu) \quad . \tag{60}$$

With the above assumptions, the transition probability is given by

$$\begin{aligned} W_{\Delta N(\underset{\sim}{u}),N(\underset{\sim}{u})} = & \int_U d\underset{\sim}{u}''\ S(\underset{\sim}{r}'',\underset{\sim}{v}'')\,\delta\left[\Delta N(\underset{\sim}{u}) - \delta(\underset{\sim}{u}-\underset{\sim}{u}'')\right] \\ & + \int_U d\underset{\sim}{u}'\ \nu' N(\underset{\sim}{u}')\,\Sigma_c(\underset{\sim}{r}',\underset{\sim}{v}')\,\delta\left[\Delta N(\underset{\sim}{u}) + \delta(\underset{\sim}{u}-\underset{\sim}{u}')\right] \\ & + \sum_\nu \int_U d\underset{\sim}{u}' \int_U d\underset{\sim}{u}''\ \nu' N(\underset{\sim}{u}')\,\Sigma_p(\underset{\sim}{r}';\underset{\sim}{v}' \to \underset{\sim}{v}'';\nu)\,\delta(\underset{\sim}{r}'-\underset{\sim}{r}'')\,\delta\left[\Delta N(\underset{\sim}{u}) + \right. \\ & \left. + \delta(\underset{\sim}{u}-\underset{\sim}{u}') - \nu\delta(\underset{\sim}{u}-\underset{\sim}{u}'')\right] \quad . \end{aligned} \tag{61}$$

We proceed by computing the exponential pgf of $W_{\Delta N,N}$. From Eq. (35) we obtain

$$\begin{aligned} G^\omega\left[N(\underset{\sim}{u}),x(\underset{\sim}{u})\right] = & \int_U d\underset{\sim}{u}''\ e^{x(\underset{\sim}{u}'')}\,S(\underset{\sim}{r}'',\underset{\sim}{v}'') + \int_U d\underset{\sim}{u}'\ e^{-x(\underset{\sim}{u}')}\,\nu' N(\underset{\sim}{u}')\,\Sigma_c(\underset{\sim}{r}',\underset{\sim}{v}') \\ & + \sum_\nu \int_U d\underset{\sim}{u}' \int_U d\underset{\sim}{u}''\ e^{\nu x(\underset{\sim}{u}'')-x(\underset{\sim}{u}')}\,\nu' N(\underset{\sim}{u}')\,\delta(\underset{\sim}{r}'-\underset{\sim}{r}'')\,\Sigma_p(\underset{\sim}{r}';\underset{\sim}{v}' \to \underset{\sim}{v}'';\nu) \quad . \end{aligned} \tag{62}$$

From this exponential pgf we can compute moments of up to any order. Explicit expressions for the first two moments (the mobility and diffusion functionals) are given below

$$\mathcal{D}_1\left[N(\underset{\sim}{u}),\underset{\sim}{u}_1\right] = \left[\frac{\delta G^\omega\left[N(\underset{\sim}{u}),x(\underset{\sim}{u})\right]}{\delta x(\underset{\sim}{u}_1)}\right]_{x(\underset{\sim}{u})=0}$$

$$= S(\underset{\sim}{r}_1,\underset{\sim}{v}_1) - v_1 N(\underset{\sim}{r}_1,\underset{\sim}{v}_1)\,\Sigma_t(\underset{\sim}{r}_1,\underset{\sim}{v}_1) +$$

$$+ \int_V d\underset{\sim}{v}'\; v' N(\underset{\sim}{r}_1,\underset{\sim}{v}')\,\overline{v\Sigma_p}(\underset{\sim}{r}_1;\underset{\sim}{v}'\to\underset{\sim}{v}_1) \tag{63}$$

$$\mathcal{D}_2[N(\underset{\sim}{u}),\underset{\sim}{u}_1,\underline{u}_2] = \left[\frac{\delta G^{(2)}[N(\underset{\sim}{u}),x(\underset{\sim}{u})]}{\delta x(\underset{\sim}{u}_1)\;\delta x(\underset{\sim}{u}_2)}\right]_{x(\underset{\sim}{u})=0}$$

$$= \delta(\underset{\sim}{r}_1-\underset{\sim}{r}_2)\left[\delta(\underset{\sim}{v}_1-\underset{\sim}{v}_2)\left[S(\underset{\sim}{r}_1,\underset{\sim}{v}_1) + v_1 N(\underset{\sim}{r}_1,\underset{\sim}{v}_1)\,\Sigma_t(\underset{\sim}{r}_1,\underset{\sim}{v}_1)\right.\right.$$

$$\left. + \int_V d\underset{\sim}{v}'\; v' N(\underset{\sim}{r}_1,\underset{\sim}{v}')\;\overline{v^2\Sigma_p}(\underset{\sim}{r}_1;\underset{\sim}{v}'\to\underset{\sim}{v}_1)\right]$$

$$\left. - \left[v_2 N(\underset{\sim}{r}_1,\underset{\sim}{v}_2)\,\overline{v\Sigma_p}(\underset{\sim}{r}_1;\underset{\sim}{v}_2\to\underset{\sim}{v}_1) + v_1 N(\underset{\sim}{r}_1,\underset{\sim}{v}_1)\,\overline{v\Sigma_p}(\underset{\sim}{r}_1;\underset{\sim}{v}_1\to\underset{\sim}{v}_2)\right]\right] \tag{64}$$

where

$$\overline{v^i\Sigma_p}(\underset{\sim}{r};\underset{\sim}{v}'\to\underset{\sim}{v}) = \sum_\nu \nu^i\Sigma_p(\underset{\sim}{r};\underset{\sim}{v}'\to\underset{\sim}{v};\nu) \tag{65}$$

$$\Sigma_t(\underset{\sim}{r},\underset{\sim}{v}) = \Sigma_c(\underset{\sim}{r},\underset{\sim}{v}) + \Sigma_p(\underset{\sim}{r},\underset{\sim}{v}) \tag{66}$$

$$\Sigma_p(\underset{\sim}{r},\underset{\sim}{v}) = \int_V d\underset{\sim}{v}'\;\Sigma_p(\underset{\sim}{r},\underset{\sim}{v}'\to\underset{\sim}{v}) \;. \tag{67}$$

We are now in a position to calculate explicit expressions for the RHS terms of Eq.(47), (52) and (54). The RHS of Eq. (47) was included in Eq. (48) in terms of the Boltzmann operator, Eq. (49), which can be written in the familiar form

$$\mathcal{B}m = \underset{\sim}{u}\cdot\nabla m + vm\Sigma_t - \int_V d\underset{\sim}{v}'\;\; v' m(\underset{\sim}{r},\underset{\sim}{v}')\;\overline{v\Sigma}_p(\underset{\sim}{r};\underset{\sim}{v}'\to\underset{\sim}{v}) \;. \tag{68}$$

The RHS of Eq. (52) springs out directly from Eq. (64) through straightforward substitutions. The RHS of Eq. (54) is obtained from thefollowing two terms

$$R[N,x(\underset{\sim}{u})] = \int_U d\underset{\sim}{u}\;\left[e^{x(\underset{\sim}{u})}-1\right]\,S(\underset{\sim}{r},\underset{\sim}{v}) + \int_U d\underset{\sim}{u}\;\left[e^{-x(\underset{\sim}{u})}-1\right]vN(\underset{\sim}{u})\,\Sigma_c(\underset{\sim}{r},\underset{\sim}{v})$$

$$+ \sum_{\nu} \int_U d\underset{\sim}{u} \int_U d\underset{\sim}{u}' \left[e^{\nu x(\underset{\sim}{u}) - x(\underset{\sim}{u}')} - 1\right] \nu' N(\underset{\sim}{u}') \delta(\underset{\sim}{r} - \underset{\sim}{r}') \Sigma_p(\underset{\sim}{r}; \underset{\sim}{v}' \to \underset{\sim}{v}; \nu) \tag{69}$$

$$\Delta R[x(\underset{\sim}{u}), \underset{\sim}{u}'] = \left[e^{-x(\underset{\sim}{u}')} - 1\right] \nu' \Sigma_c(\underset{\sim}{r}', \underset{\sim}{v}')$$

$$+ \sum_{\nu} \int_U d\underset{\sim}{u} \left[e^{\nu x(\underset{\sim}{u}) - x(\underset{\sim}{u}')} - 1\right] \nu' \delta(\underset{\sim}{r} - \underset{\sim}{r}') \Sigma_p(\underset{\sim}{r}; \underset{\sim}{v}' \to \underset{\sim}{v}; \nu) \tag{70}$$

APPENDIX A — Stochastic Transport with Delayed Neutrons

The functional approach presented sofar can be easily extended in order to include delayed neutron precursors in the model. We do not intend to duplicate here all results that have been obtained in the framework of the prompt neutron model but rather indicate the main points which have to be modified.

The state variable containing the neutron density, $Y_0(\underset{\sim}{u}) = N(\underset{\sim}{u})$, and the precursor densities, $Y_j(\underset{\sim}{u})$, $j=1,\ldots,d$, in phase space is denoted by

$$\underset{\sim}{Y}(\underset{\sim}{u}) = \text{col.}\left[Y_0(\underset{\sim}{u}) \quad Y_1(\underset{\sim}{u}) \quad \ldots \quad Y_d(\underset{\sim}{u})\right] . \tag{A.1}$$

With the assumption that the precursors are fixed in the position space, we have

$$Y_j(\underset{\sim}{u}) = C_j(\underset{\sim}{r}) \; \delta(\underset{\sim}{v}) \qquad j=1,\ldots,d \tag{A.2}$$

The probability density functional is defined in terms of the state variable, the position and velocity functions, and time,

$$P\left[\underset{\sim}{Y}(\underset{\sim}{u}), \underset{\sim}{r}_0(\underset{\sim}{u}), \ldots, \underset{\sim}{r}_d(\underset{\sim}{u}), \underset{\sim}{v}_0(\underset{\sim}{u}), \ldots, \underset{\sim}{v}_d(\underset{\sim}{u}), t\right] ,$$

at the point

$$
\begin{aligned}
\underset{\sim}{r}_0(\underset{\sim}{u}) &= \underset{\sim}{r}(\underset{\sim}{u}) = \underset{\sim}{r} & \underset{\sim}{v}_0(\underset{\sim}{u}) &= \underset{\sim}{v}(\underset{\sim}{u}) = \underset{\sim}{v} \\
\underset{\sim}{r}_1(\underset{\sim}{u}) &= \underset{\sim}{r}(\underset{\sim}{u}) & \underset{\sim}{v}_1(\underset{\sim}{u}) &= \underset{\sim}{0} \\
&\vdots & &\vdots \\
\underset{\sim}{r}_d(\underset{\sim}{u}) &= \underset{\sim}{r}(\underset{\sim}{u}) & \underset{\sim}{v}_d(\underset{\sim}{u}) &= \underset{\sim}{0}
\end{aligned}
\qquad (A.3)
$$

The transition events are given in the table below.

event:	A incoming		A outcoming	
	neutron($\underset{\sim}{u}_0'$)	precursor($\underset{\sim}{u}_j'$)	neutron($\underset{\sim}{u}_0''$)	precursor($\underset{\sim}{u}_j''$)
source emission	0	0	1	0
capture	-1	0	0	0
precursor emission	0	-1	1	0
production	-1	0	ν_0	ν_j

Table A.I - possible transition events for the delayed neutron model.

The relevant material parameters are:

external source	$S(\underset{\sim}{r}_0'', \underset{\sim}{v}_0'')$
capture cross section	$\Sigma_c(\underset{\sim}{r}_0', \underset{\sim}{v}_0')$
production cross section	$\Sigma_p(\underset{\sim}{r}_0', \underset{\sim}{v}_0' \to \underset{\sim}{r}_0'', \ldots, \underset{\sim}{r}_d'', \underset{\sim}{v}_0'', \ldots, \underset{\sim}{v}_d''; \nu)$
probability density of neutrons produced from a precursor decay	$p_j(\underset{\sim}{r}_j \to \underset{\sim}{r}_0'', \underset{\sim}{v}_0'')$.

Due to the local nature of the production events and to the fact that precursors are fixed in space, we have

$$
\begin{aligned}
&\Sigma_p(\underset{\sim}{r}_0', \underset{\sim}{v}_0' \to \underset{\sim}{r}_0', \ldots, \underset{\sim}{r}_d'', \underset{\sim}{v}_0'', \ldots, \underset{\sim}{v}_d''; \underset{\sim}{v}) \\
&= \delta(\underset{\sim}{r}_0' - \underset{\sim}{r}_0'') \prod_{j=1}^{d} \delta(\underset{\sim}{r}_0' - \underset{\sim}{r}_j'') \; \delta(\underset{\sim}{v}_j'') \; \Sigma_p(\underset{\sim}{r}_0'; \underset{\sim}{v}_0' \to \underset{\sim}{v}_0''; \underset{\sim}{v})
\end{aligned}
\qquad (A.4)
$$

$$p_j(\underset{\sim}{r}'_j \to \underset{\sim}{r}''_0, \underset{\sim}{v}''_0) = \delta(\underset{\sim}{r}''_0 - \underset{\sim}{r}'_j)\ \chi_j(\underset{\sim}{r}''_0, \underset{\sim}{v}''_0)\ , \tag{A.5}$$

where the delayed neutron spectrum is normalized

$$\int_V d\underset{\sim}{v}''_0\ \chi_j(\underset{\sim}{r}''_0, \underset{\sim}{v}''_0) = 1 \tag{A.6}$$

The transition probability is given by

$$\begin{aligned} W_{\Delta Y, Y} = & \int_U d\underset{\sim}{u}''_0\ S(\underset{\sim}{r}''_0, \underset{\sim}{v}''_0)\ \delta\left[\Delta Y_0(\underset{\sim}{u}) - \delta(\underset{\sim}{u} - \underset{\sim}{u}''_0)\right] \prod_{j=1}^{d} \delta\left[\Delta Y_j(\underset{\sim}{u})\right] \\ & + \int_U d\underset{\sim}{u}'_0 v'_0\ Y_0(\underset{\sim}{u}'_0)\, \Sigma_c(\underset{\sim}{r}'_0, \underset{\sim}{v}'_0)\, \delta\left|\Delta Y_0(\underset{\sim}{u}) + \delta(\underset{\sim}{u} - \underset{\sim}{u}'_0)\right] \prod_{j=1}^{d} \delta\left|\Delta Y_j(\underset{\sim}{u})\right] \\ & + \sum_{j=1}^{d} \int_U d\underset{\sim}{u}'_j \int_U d\underset{\sim}{u}''_0\ \lambda_j Y_j(\underset{\sim}{u}'_j)\, \delta(\underset{\sim}{r}''_0 - \underset{\sim}{r}'_j)\, \chi_j(\underset{\sim}{r}''_0, \underset{\sim}{v}''_0)\, \delta\left[\Delta Y_0(\underset{\sim}{u}) - \delta(\underset{\sim}{u} - \underset{\sim}{u}''_0)\right] \\ & \qquad \delta\left[\Delta Y_j(\underset{\sim}{u}) + \delta(\underset{\sim}{u} - \underset{\sim}{u}'_j)\right] \\ & + \sum_{\nu} \prod_{j=1}^{d} \int_U d\underset{\sim}{u}'_0 \int_U d\underset{\sim}{u}''_0 \int_U d\underset{\sim}{u}''_j\ \nu'_0\ Y_0(\underset{\sim}{u}'_0)\, \delta(\underset{\sim}{r}'_0 - \underset{\sim}{r}''_0)\, \delta(\underset{\sim}{r}'_0 - \underset{\sim}{r}''_j)\, \delta(\underset{\sim}{v}''_j) \\ & \Sigma_p(\underset{\sim}{r}'_0; \underset{\sim}{v}'_0 \to \underset{\sim}{v}; \underset{\sim}{v})\ \delta\left[\Delta Y_0(\underset{\sim}{u}) + \delta(\underset{\sim}{u} - \underset{\sim}{u}'_0) - \nu_0\, \delta(\underset{\sim}{u} - \underset{\sim}{u}''_0)\right] \delta\left[\Delta Y_j(\underset{\sim}{u}) - \nu_j\, \delta(\underset{\sim}{u} - \underset{\sim}{u}_j)\right] \ . \end{aligned} \tag{A.7}$$

The exponential pgf of the transition probability is defined by

$$G^{\omega}\left[\underset{\sim}{x}(\underset{\sim}{u}), \underset{\sim}{Y}(\underset{\sim}{u})\right] = \int_H d\underset{\sim}{\Delta Y}(\underset{\sim}{u})\ W_{\underset{\sim}{\Delta Y}(\underset{\sim}{u}), \underset{\sim}{Y}(\underset{\sim}{u})}\ \exp\left[\int_U d\underset{\sim}{u}\ \underset{\sim}{x}^T(\underset{\sim}{u})\, \underset{\sim}{\Delta Y}(\underset{\sim}{u})\right] \ , \tag{A.8}$$

and given by

$$\begin{aligned} G^{\omega}\left[\underset{\sim}{x}, \underset{\sim}{Y}\right] = & \int_U d\underset{\sim}{u}''_0\ \ S(\underset{\sim}{r}''_0, \underset{\sim}{v}''_0)\ e^{x_0(\underset{\sim}{u}''_0)} + \int_U d\underset{\sim}{u}'_0\ \underset{\sim}{v}'_0\ Y_0(\underset{\sim}{u}'_0) \Sigma_c(\underset{\sim}{r}'_0, \underset{\sim}{v}'_0) e^{-x_0(\underset{\sim}{u}'_0)} \\ & + \sum_{j=1}^{d} \int_U d\underset{\sim}{u}'_j \int_U d\underset{\sim}{u}''_0\ \lambda_j Y_j(\underset{\sim}{u}'_j)\, \delta(\underset{\sim}{r}''_0 - \underset{\sim}{r}'_j)\, \chi_j(\underset{\sim}{r}''_0, \underset{\sim}{v}''_0) e^{x_0(\underset{\sim}{u}''_0) - x_j(\underset{\sim}{u}'_j)} \end{aligned}$$

$$+ \sum_{\nu} \prod_{j=1}^{d} \int_U d\underset{\sim}{u}_0' \int_U d\underset{\sim}{u}_0'' \int_U d\underset{\sim}{u}_j'' \; \nu_0' \; Y_0(\underset{\sim}{u}_0') \, \delta(\underset{\sim}{r}_0' - \underset{\sim}{r}_0'') \, \delta(\underset{\sim}{r}_0' - \underset{\sim}{r}_j'') \, \delta(\underset{\sim}{v}_j'')$$

$$\Sigma_p(\underset{\sim}{r}_0'; \underset{\sim}{v}_0' \to \underset{\sim}{v}_0''; \underset{\sim}{v}) \, e^{\nu_0 x_0(\underset{\sim}{u}_0'') - x_0(\underset{\sim}{u}_0') + \nu_j x(\underset{\sim}{u}_j'')} \tag{A.9}$$

From Eq.(A.9), all previously obtained results can be reformulated. The first two moments of W in particular, can be calculated through the functional derivatives

$$\underset{\sim}{\mathcal{D}}_1[\underset{\sim}{Y}(\underset{\sim}{u}), \underset{\sim}{u}_1] = \left[\frac{\delta G^w}{\delta \underset{\sim}{x}(\underset{\sim}{u})} \right]_{\underset{\sim}{x}(\underset{\sim}{u})=\underset{\sim}{0}} \tag{A.10}$$

$$\underset{\sim}{\mathcal{D}}_2[\underset{\sim}{Y}(\underset{\sim}{u}), \underset{\sim}{u}_1, \underset{\sim}{u}_2] = \left[\frac{\delta G^w}{\delta \underset{\sim}{x}(\underset{\sim}{u}_1) \, \delta \underset{\sim}{x}^T(\underset{\sim}{u}_2)} \right]_{\underset{\sim}{x}(\underset{\sim}{u})=\underset{\sim}{0}} \tag{A.11}$$

where $\underset{\sim}{\mathcal{D}}_1$ is a column matrix operator and $\underset{\sim}{\mathcal{D}}_2$ a symmetric square matrix operator

$$\underset{\sim}{\mathcal{D}}_2[\underset{\sim}{Y}(\underset{\sim}{u}), \underset{\sim}{u}_1, \underset{\sim}{u}_2] = \underset{\sim}{\mathcal{D}}_2^+[\underset{\sim}{Y}(\underset{\sim}{u}), \underset{\sim}{u}_1, \underset{\sim}{u}_2] = \underset{\sim}{\mathcal{D}}_2^T[\underset{\sim}{Y}(\underset{\sim}{u}), \underset{\sim}{u}_2, \underset{\sim}{u}_1]. \tag{A.12}$$

Explicit calculations for the first moment of W lead tothere-sult

$$\underset{\sim}{\mathcal{D}}_1[\underset{\sim}{Y}(\underset{\sim}{u}), \underset{\sim}{\bar{u}}]_0 = S(\underset{\sim}{r}, \underset{\sim}{v}) - \nu Y_0(\underset{\sim}{r}, \underset{\sim}{v}) \; \Sigma_t(\underset{\sim}{r}, \underset{\sim}{v})$$

$$+ \sum_{j=1}^{d} \lambda_j \chi_j(\underset{\sim}{r}, \underset{\sim}{v}) \int_V d\underset{\sim}{v}' \; Y_j(\underset{\sim}{r}, \underset{\sim}{v}')$$

$$+ \int_V d\underset{\sim}{v}' \; \nu' \; Y_0(\underset{\sim}{r}, \underset{\sim}{v}') \; \overline{\nu_0 \Sigma_p}(\underset{\sim}{r}; \underset{\sim}{v}' \to \underset{\sim}{v}) \tag{A.13a}$$

$$\underset{\sim}{\mathcal{D}}_1[\underset{\sim}{Y}(\underset{\sim}{u}), \underset{\sim}{\bar{u}}]_j = \delta(\underset{\sim}{v}) \int_V d\underset{\sim}{v}' \; \nu' \; Y_0(\underset{\sim}{r}, \underset{\sim}{v}') \; \overline{\nu_j \Sigma_p}(\underset{\sim}{r}, \underset{\sim}{v}') - \lambda_j \; Y_j(\underset{\sim}{r}, \underset{\sim}{v}) \tag{A.13b}$$

From Eq. (A.13), (A. 2) and an equivalent version of Eq.(47),

we obtain, in obvious notations, the central first moment kinetic equation

$$\frac{\partial \langle n \rangle}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} \langle n \rangle + v \langle n \rangle \Sigma_t = \sum_{j=1}^{d} \lambda_j \chi_j \langle c_j \rangle + \int_V d\underset{\sim}{v}' \; v' \langle n' \rangle \overline{\nu_0 \Sigma_p'} \qquad \text{(A.14a)}$$

$$\frac{\partial \langle c_j \rangle}{\partial t} = \int_V d\underset{\sim}{v}' \; v' \langle n' \rangle \overline{\nu_j \Sigma_p'} - \lambda_j \langle c_j \rangle \qquad \text{(A.14b)}$$

Higher order moment equations can be obtained in a similar fashion.

APPENDIX B — The Non-Hornogeneous Poisson Distríbution

Let $\lambda(\underset{\sim}{r},\underset{\sim}{v})$ be the probability of detection per unit phase space per neutron absorbed. In terms of the stochastic variable

$$C_t(\tau) = \int_t^{t+\tau} dt' \int_R d\underset{\sim}{r} \int_V d\underset{\sim}{v} \; \lambda(\underset{\sim}{r},\underset{\sim}{v}) \; N(\underset{\sim}{r},\underset{\sim}{v},t') \qquad \text{(A.15)}$$

the probability of obtaining K counts in the interval $(t,t+\tau)$ is given by

$$P_\tau(K) = \left\langle \frac{C_t(\tau)^K \, e^{-C_t(\tau)}}{K!} \right\rangle \qquad \text{(A.16)}$$

where the average is taken over the neutronic process. In case the reactor is stationary, $P_\tau(K)$ is independent of t.

Eq.(A.16) can be generalized in order to include counts in more than one time interval. For a review on the application of the non-homogeneous Poisson distribution, the reader i's refered to Watson[10,11].

7. CONCLUSIONS

A functional approach has been applied to formulate the stochastic transportofneutrons in a simple and compact form. A singlefunctional master equation has been used in the derivation of kinetic equations for severa1 quantities of interest such as centraland non-central moments, exponential pgf's, etc. The particular form of the transition probability does not play any role in the theory until all relevant kinetic equations are obtained in compact form. Inorder to mantain consistency with the continuous phase space formulation, a singular transition probability is introduced and explicit formsof kineticequations can then be obtained.

The first order and the stationary second order non-local moment kinetic equations agree with results previously obtained through other methods, e.g., the Langevin approach used by Akcasu and Osborn[6]. However, as has been pointed out by Watson[11] in the framework of the point reactor model, the Langevin rnethod does not predict a correct kinetic equation for the non-stationary second order moment: an additional term appears in the equation (c.f. second RHS term in Eq.(52)), which is due to the stochastic nature of the diffusion functional $\mathcal{D}_2$. It is worth mentioning that a (correct) *non-stationary* second order moment is required in order to obtain a *stationary* third order covariance kinetic equation.

Much like in the case of conventional (deterministic) transport theory, we have not attempted to "solve" the kinetic equations obtained. Our formulation should serve rather as a starting point in the derivation of more tractable approximate equations in a consistent way, likethe P_1 and multigroup-diffusion equations,a task which is beyond the scope of this paper.

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