Revista Brasileira de Física, Vol. 9, Nº 2, 1979

Theory of Superfluidity*

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Recebido em 1º de Dezembro de 1978

We propose a new description of superfluidity based upon the fact that Bogoliubov's theory of superfluidity exhibits some so far unsuspected macroscopic quantum waves *(MQws)*, which have a topological nature and trave1 within the fluid at subsonic velocities. In the presence of a MQW, besides the usual phonons, there exists also a new type of quasi-particles, which are bound in the wave and have no counterpart in Bogoliubov's original theory. The spectrum of these new excitations looks much like the liquid 4 ~ spectrum observed in neutron scattering experiments. To quantize the bound quasi-particles we employ the field theoretic version of the Bohr-Somnerfeld quantization rule, andalso resort to a variational computation. At any instantaneous configuration the MQWs cut the condensate into blocks of phase, providing, by analogy with ferromagnetism, a nice explanation of what could be the X-transition. A crude estimate of the critica1 temperature givesTc = 2-4 K. An attempt is made to understand Tiszals two-fluid model in terms of the MQWS, and we conjecture that they play an important role in the motion of second sound. We present also a qualitative prediction concerning of the behavior of the "phonon-roton" peak below 1.OK, and propose two experiments to look for MQWs.

A teoria da superfluidez de Bogoliubov descreve certas ondas quanticas macroscópicas (U&Ms) de natureza topológica, que se inovem no fluido a velocidades subsônicas e eram,até o presente, desconhecidas. Com base nesse fato, propomos uma nova descrição da superfluidez.Na presença de uma Q@ além dos fonons usuais, existe também um novo tipo de

* *Dedicated* to the *memory of Nassim Bechara.*

quasi-partículas, ligadas á onda, que não têm análogos na teoria original de Bogoliubov. O espectro dessas novas excitações é bastante parecido com o espectro do 4 ~ êíquido, observado através do espalhamento de neutrons. Na quantização das quasi-partículas ligadas, empregamos a generalização da regra de quantização de Bohr-Somnerfeld para teorias de campos, e além disso recorremos a uma aproximação variacional. Em qualquer configuração instantânea, as *O&Ms* cortam o condensado em blocos de fase, proporcionando assim, uma possível explicação para a transição h. Uma estimativa grosseira da temperatura critica indica que Tc ■ 2-4 K. Procuramos entender o modelo dos dois fluidos de Tisza em termos das *OQMs,* e conjecturamos que elas desempenham um papel importante no movimento do segundo som. Fazemos tarnb6m uma previsão a respeito do comportamento do pico de fonons-rotons abaixo de 1,0 K, e propomos duas experiências para procurar *OQNs.*

1. INTRBDUCTION

Helium II is one of the most fascinating and puzzling many-body systems[1]. Being made up of identical bosons, it should be amenable to a field theoretic treatment. Nevertheless, and paradoxically, the peculiar properties of liquid ^{4}He are yet needing a microscopic description.

We propose here a new strategy to solve that problem. We show that Bogoliubov's theory of superfluidity[2] displays some up to now unsuspected macroscopic quantum waves of topological nature, which are coherent states of the fluid, inequivalent to the ground state.

At any instantaneous configuration, these waves partition the condensate into domains of phase, and provide, by analogy with ferromagnetism an explanation about what could be the h-transition.

Besides that, bound within the macroscopic quantum waves, there exists a new type of excitations, different from the original Bogoliubov's phonons, whose spectrum is very similar to that of liquid Helium.

In 1941, to explain the Helium II specific heat data, Landau[3] proposed a parametrization for the fluid elementary excitations. His work gave rise to the phenomenological theory of superfluidity which accounts for many properties of liquid ^{4}He within the temperature range $0.8K \lesssim T \lesssim 1.8K$. Landau's theory was further developed by himself[4], Khalatnikov[5,6] and others[6,19-25].

The earliest neutron scattering experiments designed to look for Landau's quasi-particles[7] have found, in the energy-momentum plane, a cross section peak (the "phonon-roton" peak) near the curve of his parametrization. On the other hand, more recent measurements[8] are now indicating that the spectrum of Helium II is actually much more complex than that proposed by him.

Another important approach to superfluidity was formulated by Feynman[9,10] - see also Feynman and Cohen[11] and Feenberg[12] - in this scheme, the goal was to relate the quasi-particles dispersion relation to the liquid structure function.

In 1947, when attempting to deduce the quantum liquid spectrum from the interaction between Helium atoms, Bogoliubov[2] proposed a more ambitious project: the microscopic theory of superfluidity - theory which was later extended by Lee, Huang and Yang[13], Beliaev[14], Hugenholtz and Pines[15] and many others[16,19-22].

The formulation of Bogoliubov's theory is simple, accurate and beautiful, In spite of that, this theory is still resisting to exhibit the more remarkable properties of a superfluid.

Our strategy is just to say that the macroscopic quantum waves are the bridge needed to link the foundations of such a theory to the liquid Helium phenomenology.

These waves, which we show to exist in a large class of non relativistic bosonic models[26], will be studied here in the context of the $\lambda|\phi|^4$ theory.

Of course the $\lambda|\phi|^4$ theory, being no more than a simplified model of quantum fluids, cannot account for a perfect quantitative description of Helium II. We believe, however, that despite of its great simplicity, it displays all qualitative features of the real superfluid.

In this respect, Bogoliubov himself wrote that "all we can require from a molecular theory of superfluidity, at least at the first stage of investigation, is to be able to account for the qualitative picture of this phenomenon, being based on a certain simplified scheme" (Ref.2). These words are again opportune to define the spirit of this paper.

Concerning to the equation of motion of the $\lambda|\phi|^4$ theory, the macroscopic quantum waves (*MQWs*) are solutions of the type:

$$W_V = \sqrt{\rho}\{V - i\sqrt{1-V^2}\ \mathrm{tgh}\{\sqrt{1-V^2}\ mc(x-cVt)\}\}\exp(-imc^2t) \quad ,$$

In this expression, x is a particular coordinate, m, ρ and c are respectively the ^{4}He mass, the fluid density and the sound velocity, V is a real number whose modulus is never larger than unit.

W_V describes a subsonic coherent pulse moving in the condensate. Relatively to the fundamental state its energy is positive. The wave momentum and velocity are opposite, because it is a moving plane of low density. A *MQW* has also a charge of topological nature[27], since the phases of $W_V(-\infty)$ and $W_V(+\infty)$ are different. Thus, it cannot decay into elementary excitations of the fluid.

From the knowledge of this wave, it is natural to expect that more complex solutions, corresponding to multi-*MQWs* states, should also exist. At an instantaneous configuration of these states, the condensate is cut into blocks of phase (analogous to the ferromagnetic domains) which provide a simple description to the λ-transition.

By studing the infinitesimal fluctuations of W_V, we will show that, besides the usual phonons, there exist quasi-particles which are bound in the MQWs and have no countepart in Bogoliubov's condensate. The spectrum of these new excitations is quite similar to the spectrum observed in liquid Helium by means of neutron scattering experiments[8,28,29].

The plan of this paper is as follows:

Chapter II is a brief review of Bogoliubov's theory. There, instead of using the cannonical method, we quantize the phonons bymeans of the field theoretic version of the Bohr-Sommerfeld quantization rule[30-32]. The reason for this is that such a semiclassical method, which is more transparent than the cannonical one, will be employed later to treat the quasi-particles bound in the topological waves.

In section III we study in detail the *MQWs* solutions, computing the momentum and energy they carry per unit of area.

In the present stage of our research on superfluidity, themajor defficulty we face is the construction of the topological wavesstatistical mechanics. This deficiency inhibits a more precise exploration of the phenomenological implications of the macroscopic quantum waves phenomenon. We are forced to resort very frequently to analogies with magnetic systems.

By using these analogies we are able to understand what could be the nature of the X-transition, and to do a very crude estimate of its temperature (which gives $T_C \simeq 2\text{-}4K$). We also attempt to understand Tisza's two-fluid model[33] in terms of the *MQWs*, and conjecture that these waves may have much to do with the motion of second sound.

It is in chapter IV that we study the *MQWs* bound excitations. We first write their equation of motion and general expressions for their energy and momentum. The quasi-particle energy and the component of its momentum which is parallel to the *MQW* velocity are not good quantum numbers to characterize its state. However a linear combination of these quantities still is.

To quantize the bound excitations we employ again the Bohr--Sommerfeld rule. And to find their spectra we use a variational approach, because until now only a few exact solutions of the quasi-particles equation of motion are known. The spectrum so obtained is then faced with those of Woods and Cowley[8,28] and Dietrich et *al.*[29] We advance that our results are quite promising.

The sketch of an apparatus to look for MQWs in Helium II is shown in chapter V.

Section VI is left for conclusions whereas three appendices complement our calculations.

The main ideas of our theory of superfluidity are also discussed in reference 34.

2. SEMICLASSICAL QUANTIZATION OF BOGOLIUBOV'S THEORY

The matter of this chapter is not new. We present first a brief review of Bogoliubov's superfluidity theory[2], emphasizing, in the configuration space, that the $\lambda|\phi|^4$ theory is both a special case and an approximation for it.

The condensate excitations are quantized by means of the field theoretic version of the Bohr-Sommerfeld quantization rule. In section IV we shall employ such a semiclassical method to treat the quasi-particles of the *MQWs*, because it is more transparent than the cannonical procedure That is the reason For using this rule here too.

The cannonical formulation of the macroscopic quantum waves problem will be studied in a separate publication.

Let us consider the ^{4}He field $\phi(\vec{x})$ and its momentum $\Pi(\vec{x}) = i\,\phi^*(\vec{x})$. The microscopic superfluidity theory may be defined by the Hamiltonian[2,35]

$$H = H_{kin} + H_{pot} , \tag{1a}$$

being its kinetic and potential parts given by (m is the ^{4}He mass)

$$H_{kin} = -\frac{1}{2m} \int d\vec{x}\,\phi^*(\vec{x})\; \nabla^2\phi(\vec{x}) \quad , \tag{1b}$$

and

$$H_{pot} = \frac{1}{2} \int d\vec{x}\,d\vec{r}\,\phi^*(\vec{x})\,\phi(\vec{x})\,V(\vec{r})\,\phi^*(\vec{x}+\vec{r})\,\phi(\vec{x}+\vec{r}) \quad , \tag{1c}$$

where $V(\vec{r})$ is the interaction between any given pair of particleç.

The equation of motion of $\phi(\vec{x})$ will be

$$i\partial_t\phi(\vec{x}) = -\frac{1}{2m}\nabla^2\phi(\vec{x}) + \int d\vec{r}\phi^*(\vec{x}+\vec{r})\phi(\vec{x}+\vec{r})V(\vec{r})\phi(\vec{x}) \quad , \tag{2}$$

a) The role of the $\lambda|\phi|^4$ Theory

When $V(\vec{r}) = \lambda\delta(\vec{r})$ it is obvious that the model defined above turns to be the $\lambda|\phi|^4$ theory, whose Hamiltonian and equation of motion are (take λ to be positive):

$$H = \int d\vec{x}\, \{-\phi^* \frac{1}{2m}\nabla^2\phi + \frac{\lambda}{2}(\phi^*\phi)^2\} \quad , \tag{3}$$

and

$$i\partial_t\phi = -\frac{1}{2m}\nabla^2\phi + \lambda\phi^*\phi^2 \quad . \tag{4}$$

In many cases, however, even when $V(\vec{r})$ is not a delta function, we can use the $\lambda|\phi|^4$ theory to learn a lot of things about the original model. If the potential $V(\vec{r})$ displays a certain regularity[36], the product $\phi^*(\vec{x}+\vec{r})\phi(\vec{x}+\vec{r})$ appearing in Eq.(1c) can be expanded in $\vec{r}=(r_1,r_2,r_3)$ around the point $\vec{x} = (x_1,x_2,x_3)$.

$$\phi^*(\vec{x}+\vec{r})\ \phi(\vec{x}+\vec{r}) = \phi^*(\vec{x})\ \phi(\vec{x}) + \vec{r}\cdot\vec{\nabla}\{\phi^*(\vec{x})\phi(\vec{x})\} +$$

$$+ \frac{1}{2} r_i r_j\ \partial_{x_i}\ \partial_{x_j}\ \{\phi^*(\vec{x})\ \phi(\vec{x})\} + \ldots +$$

$$+ \frac{r_1^n\ r_2^m\ r_3^\ell}{n!\ m!\ \ell!}\ \partial^n_{x_1}\ \partial^m_{x_2}\partial^\ell_{x_3}\ \{\phi^*(\vec{x})\ \phi(\vec{x})\} + \ldots . \tag{5}$$

We point out that an expansion equivalent to this is usually done in momentum space[2,35].

Plugging this expanded product into the expression of H_{pot}, we obtain

$$H_{pot} = \frac{1}{2}\sum_{mn\ell} a_{nm\ell} \int d\vec{x}\ \phi^*(\vec{x})\phi(\vec{x})\ \partial^n_{x_1}\ \partial^m_{x_2}\ \partial^\ell_{x_3}\ \{\phi^*(\vec{x})\ \phi(\vec{x})\} \quad , \tag{6a}$$

where

$$a_{nm\ell} = \int d\vec{r} \frac{r_1^n r_2^m r_3^\ell}{n!\, m!\, \ell!} V(\vec{r}) . \quad (6b)$$

Whenever $V(\vec{r})$ has spherical symmetry - as happens for the interaction between ^{4}He atoms - it follows that:

1) The only coefficients $a_{nm\ell}$ different from zero are those having even indices n, m and ℓ.

2) Two coefficients having permuted indices are equal, i. e., $a_{mn\ell}=a_{nm\ell}=a_{\ell mn}$, etc. Thus, in this case H_{pot} can be written as

$$H_{pot} = \frac{\lambda}{2} \int d\vec{x}(\phi^*(\vec{x})\phi(\vec{x}))^2 + \frac{g}{2} \int d\vec{x}\phi^*(\vec{x})\phi(\vec{x})\nabla^2(\phi^*(\vec{x})\phi(\vec{x})) + \dots \quad (7)$$

Here, $\lambda = a_{000}$ and $g = a_{200}$.

Since the integral $(\frac{\lambda}{2}) \int d\vec{x}(\phi^*(\vec{x})\phi(\vec{x}))^2$ is the first term of an expansion of H_{pot}, it is natural to expect that the solution of the $\lambda|\phi|^4$ theory has, in many circunstances, much to do with the model defined by $V(r)$. A precise statement about what circunstances are those is hard to achieve, because they depend strongly on the shape of $V(r)$. There is a consensus that they correspond to low density and (or) to a weakly interacting bose gas. However, the results of this paper will suggest that the range of validity of the $\lambda|\phi|^4$ approximation may be much more wide .

The potential between He atoms has a negative tail. London[17] and Lee, Huang and Yang[13] have shown why an hypothetic model, which does nof have such a long range attractive force (the $\lambda|\phi|^4$ theory, for instance), may describe the qualitative features if the real fluid.

In threc dimensions, the $\lambda|\phi|^4$ theory is not renormalizable. This means that it only makes sense if we work with a cut-off, upon which many of the properties of the system will depend: However, in this paper we shall not consider such a cut-off dependence. For example, the change of the MQW's energy that is due to the quasi-particles fundamental state will not be computed here.

b) Classical Theory

Before going to the semiclassical quantization of the $\lambda|\phi|^4$ theory, it is important to discuss some properties and results of its classical version.

Consider a system of charge Q defined in a box of volume V.

The fundamental state of such a system has a degeneracy of infinite degree. A particular fundamental state is represented by the following solution of Eq. (4):

$$\Omega_\theta = \sqrt{\rho}\ \exp\ i(\theta-\lambda\rho t) \ , \tag{8}$$

where ρ is the charge density,

$$\rho = Q/V \ , \tag{9}$$

and θ is a real constant in the range $0 \leq \theta \leq 2\pi$. Associated with each value of θ in this range there exists a different fundamental state. Of course all them have the same energy

$$H\{\Omega_\theta\} = \frac{1}{2}\lambda\rho^2 V \tag{10}$$

The solutions of type (8) are usually called the fluid condensate.

To simplify the notation in many equations that will follow, let us define Ω as being the condensate of phase zero:

$$\Omega = \Omega_0 = \sqrt{\rho}\ \exp(-i\lambda\rho t) \tag{11}$$

Phonons are small excitations of the condensate. To study them we represent the field in the following way:

$$\phi = (\sqrt{\rho'} + \eta)\ \exp(-i\lambda\rho' t) \ , \tag{12}$$

where $\sqrt{\rho'}\ \exp(-i\lambda\rho' t)$ is the condensate modified by the existence of an excitation (ρ' is a constant slightly different from $\rho = Q/V$), and η, the excitation, is orthogonal to the condensate wave function. I.e.:

$$\int \sqrt{\rho'}\, \eta\, d\vec{x} = 0 \; . \qquad (13)$$

We can relate ρ' to the total charge of the system

$$Q = \int d\vec{x} \phi^* \phi = \rho' V + \int d\vec{x} \eta^* \eta \; , \qquad (14a)$$

or

$$\rho' = \rho - \frac{1}{V} \int d\vec{x} \eta^* \eta \; . \qquad (14b)$$

Plugging ϕ defined by Eq.(12) in the equation of motion (4), and retaining only terms up to first order in η, we get a linear equation of motion for this fluctuation

$$i\partial_t \eta = -\frac{1}{2m} \nabla^2 \eta + \lambda\rho(\eta^* + \eta) \; . \qquad (15)$$

Note that it is the same to write the last term of the above equation as $\lambda\rho'(\eta^*+\eta)$ or $\lambda\rho(\eta^*+\eta)$, since ρ and ρ' differ only by a term of second order in η.

To obtain the Hamiltonian associated with the field of Eq.(12), we must insert this field in Eq. (3). Taking into account only terms up to second order in η, and considering the orthogonality relation (13), it follows that

$$H\{\phi\} = \frac{\lambda}{2} (\rho')^2 V + \int d\vec{x} \left\{ \frac{1}{2m} \vec{\nabla}\eta^* \vec{\nabla}\eta + \frac{\lambda\rho'}{2} (4\eta^*\eta + \eta^{*2} + \eta^2) \right\} + O(\eta^3) \; . \qquad (16)$$

Using now Eq. (14b) to eliminate ρ' in terms of ρ, we obtain

$$H\{\phi\} = \frac{\lambda\rho^2}{2} V + H\{\eta\} + O(\eta^3) \; . \qquad (17a)$$

Here $(\lambda/2)\rho^2 V$ is the fundamental state energy (see Eq.(10)) and $H\{\eta\}$ is the excitation Hamiltonian:

$$H\{\eta\} = \int d\vec{x} \left\{ \frac{1}{2m} \vec{\nabla}\eta^* \vec{\nabla}\eta + \frac{\lambda\rho}{2} (2\eta^*\eta + \eta^{*2} + \eta^2) \right\} \; . \qquad (17b)$$

Combining Eqs. (15) and (17b), we get the expression for the quasi-particle energy

$$E = \frac{i}{2} \int d\vec{x} \{ \eta^* \partial_t - \partial_t \eta^*) \eta \} \; . \qquad (18a)$$

Since $\int d\vec{x}\eta^*\eta$ is time independent (see Eq. (14a)), we can still write E as

$$E = i \int d\vec{x}\ \eta^* \partial_t \eta \ . \tag{18b}$$

The momentum that corresponds to ϕ is given by

$$\vec{p} = -i \int d\vec{x}\ \phi^* \vec{\nabla} \phi \ , \tag{19a}$$

and the translational invariance of the condensate wave function implies that

$$\vec{p} = -i \int d\vec{x}\ \eta^* \vec{\nabla} \eta \ . \tag{19b}$$

If $\vec{k}$ is the wave number, and ω the following function of k ($k=|\vec{k}|$)

$$\omega = \sqrt{\frac{\lambda\rho}{m}}\, k \left\{ 1 + \frac{k^2}{4m\lambda\rho} \right\}^{1/2} \ , \tag{20}$$

we can verify by direct substitution that

$$\eta_{\vec{k}} = \frac{A}{\sqrt{V}} \left\{ \left(\frac{k^2}{2m} + \lambda\rho + \omega\right)^{1/2} \exp i(\vec{k}\vec{x}-\omega t) - \left(\frac{k^2}{2m} + \lambda\rho - \omega\right)^{1/2} \exp i(-\vec{k}\vec{x}+\omega t) \right\} \tag{21}$$

(where A is a real constant) is a solution of Eq. (15).

With the set $\{\eta_{\vec{k}}\}$ we can expand any other solution of the classical equation.

Observe that, if in Eq. (21) we change $\omega \to \omega$, η will correspond to an excitation of wave number $-\vec{k}$.

c) Quantization and the Quasi-Particles Dispersion Relation

In appendix A, by means of the Bohr-Sommerfeld quantization rule, we conclude that, if the amplitude A (see Eq. (21)) is given by

$$A = \sqrt{\frac{1}{2\omega}} \ , \tag{22}$$

the field $\eta_{\vec{k}}$ will describe the state of a single quasi-particle.

Then, using Eq.(22), and plugging $\eta_{\vec{k}}$ in Eqs.(18b) and (19b) we compute respectively the eneryy and the momentum of such an elementary excitation:

$$E(\vec{k}) = \omega(\vec{k}) \ , \tag{23}$$

and

$$\vec{p}(\vec{k}) = \vec{k} \ . \tag{24}$$

Finally, from Eqs. (20), (23) and (24) we can deduce the well known dispersion relation of phonons that move in the Bogoliubov's condensate

$$E(p) = \sqrt{\frac{\lambda\rho}{m}}\, p \ \{1 + \frac{p^2}{4m\lambda\rho}\}^{1/2} \tag{25}$$

The fact that this result agrees with the dispersion relation deduced through the cannonical quantization method[2,35] indicates that our semiclassical scheme works well.

The function $E(p)$ is shown in figure 1. For small values of p, it behaves like a straight line,

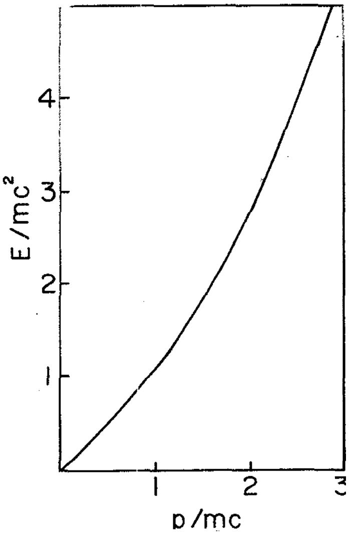

Fig.1 - Dispersion relation of Bogoliubov's phonons

$$E(p) \underset{p\to 0}{\approx} cp \quad , \tag{26}$$

where c, given by

$$c = \sqrt{\frac{\lambda\rho}{m}} \quad , \tag{27}$$

is the velocity of large wave length phonons on a medium of density ρ, which is at *absolute* zero. Absolute zero because the system considered here is solely the Bogoliubov's condensate plus a single quasi-particle.

In the rest of this paper we shall frequently use Eq.(27) to eliminate h in terms of c. Nevertheless, even when the fluid is considered at a finite temperature we must recall that c refers to the *absolute* zero *sound velocity*.

When p is very large, $E(p)$ has the shape of a parabola:

$$E(p) \underset{p\to\infty}{\approx} \frac{p^2}{2m} + mc^2 \quad . \tag{28}$$

3. MACROSCOPIC QUANTUM WAVES

In addition to Bogoliubov's condensate R, the equation of motion of the $\lambda|\phi|^4$ theory,

$$i\partial_t\phi = -\frac{1}{2m}\nabla^2\phi + \lambda\,\phi^*\,\phi^2 \quad , \tag{29}$$

describes also a set of macroscopic quantum waves (*MQWs*), which travel in the fluid at subsonic velocities, dividing the condensate into two sectors of different phases. These waves are the fundamental objects upon which we intend to base a new description of superfluidity.

If x is a particular coordinate, and V a real number such that $0 \leq |V| \leq 1$, then a macroscopic quantum wave is a solution of the type

$$W_{V,\theta} = (V - i\gamma \text{ tgh } \gamma\xi)\sqrt{\rho}\, \exp i(\theta - \lambda\rho t) \quad , \tag{30a}$$

where 8 is an arbitrary phase, and γ and ξ are given by

$$\gamma = \sqrt{1 - V^2} \quad , \tag{30b}$$

and

$$\xi = \sqrt{m\lambda\rho}\ (x - cVt) = mc(x - cVt) \quad . \tag{30c}$$

This new solution $W_{V,\theta}$ is a pulse that moves in the x-direction, with velocity cV, in a medium of density ρ. It is a coherent state of the bosonic system.

When $|V|>1$, $W_{V,\theta}$ is no longer a solution of Eq. (29). Then, since c is the lowest sound velocity, the *MQWs* will be always subsonic.

To simplify the notation we define W_V as being

$$W_V = W_{V,0} = (V - i\gamma\ \text{tgh}\ \gamma\xi)\sqrt{\rho}\ \exp(-i\ \lambda\rho t) \quad . \tag{31}$$

Sometimes we shall use only W to represent a MQW.

In reference (26) we show, in a semi-deductive way, how to reach the solution W_V, by starting from the equation of motion. There, it is also shown that the *MQWs* exist not only in the $\lambda|\phi|^4$ theory, but in a large class of nonrelativistic bosonic models.

Figure 2 exhibits the charge distribution

$$|W_V|^2 = \rho\ \{1 - \frac{\gamma^2}{(\cosh\ \gamma\xi)^2}\} \tag{32}$$

corresponding to certain *MQWs*. Each one of these waves is a moving rarefied plane, and the widths of the slower ones are nearly given by $\Delta \approx \approx 2/mc$ (~1.2 Å if c refers to the velocity of sound in Helium II at absolute zero and normal pressure, and m to ^{4}He mass).

Note that far from the *MQW* the fluid remains at rest.

A question of semantics: Even being solutions of the *classical* equation of motion of a field theoretic model, the functions W_V were called *quantum* waves because Eq.(29) is a nonlinear Schrödinger

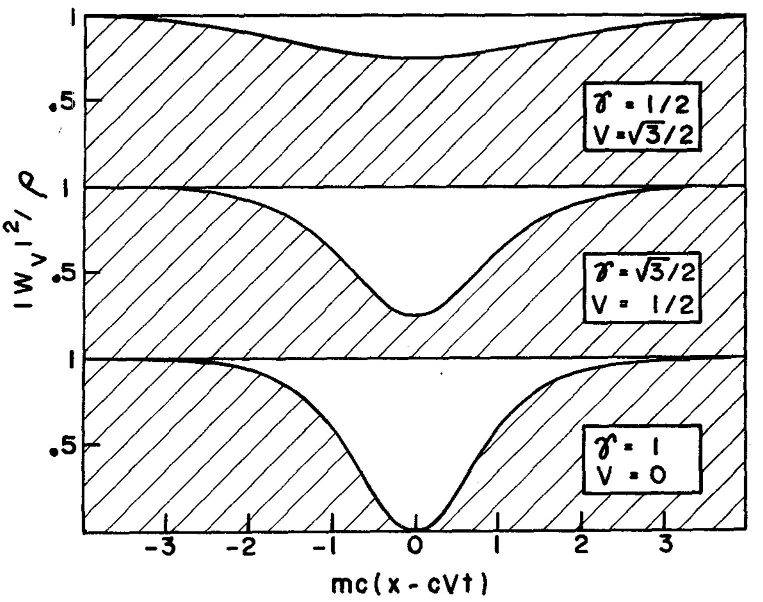

Fig.2 - Charge density for three different macroscopic quantum waves.

Equation. In other words, from the view point of first quantization these waves are quantum objects, whereas, in a second quantization scheme, they can be considered to be classical ones.

Before proceeding it is instructive to point out that two among the waves (31) were already known in the literature. One of them is W_1 - when V=1 (γ=0), we have

$$W_1 = \sqrt{\rho}\, \exp(-i\, \lambda\rho t) \ , \tag{33}$$

which means that W_1 is simply the fundamental state of the system.

The other known solution is W_0 (γ=1)

$$W_0 = -i\, \sqrt{\rho}\, \text{tgh}\, (mcx)\, \exp(-i\, \lambda\rho t) \ . \tag{34}$$

This static MQW shows how the condensate wave function changes near a wall at x=0 [37]. However it also can correspond to a system without the wall, where the condensate phase changes from $-\pi/2$ to $+\pi/2$ when we go from a half-space to the other.

a) Topology

If

$$\delta = \text{arc}\begin{cases}\sin\gamma \\ \cos V\end{cases}, \tag{35}$$

an inspection of the large $|\xi|$ behavior of W_V gives

$$\lim_{\xi\to-\infty} W_V = \sqrt{\rho}\,\exp i(\delta-\lambda\rho t) \tag{36a}$$

$$= \Omega_{+\delta}$$

and

$$\lim_{\xi\to+\infty} W_V = \sqrt{\rho}\,\exp i(-\delta-\lambda\rho t) \tag{36b}$$

$$= \Omega_{-\delta}$$

Therefore, a *MQW* cuts the condensate in two sectors of phase the difference in phase is given by 26. This difference in phase rernains time independent even when there exist quasi-particles in the system, being itself a charge of topological nature[27].

Figure 3 exhibits in the W-cornplex plane the " trajectories " of various *MQWs* when ξ varies from $-\infty$ to $+\infty$. In drawing this figure we choose the constant θ of Eq. (30a) to be $\theta=\delta$, and the time to be $t=0$.

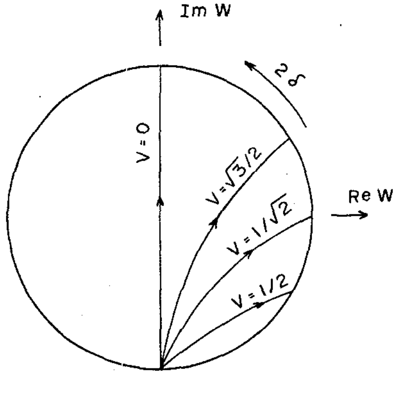

Fig.3 - "Trajectories" of various macroscopic quantum waves in the W-complex plane when $\theta=\delta$ and t=O (see Eqs.30).

In subsection C we shall compute the energy of a state where a MQW exists. It is larger then the energy of the condensate Ω. Nevertheless, the decay

$$W \to \Omega + \text{phonons}$$

is forbidden by topological charge conservation. A MQW is an stable object because its topology is inequivalent to that of the fundamental state.

b) Mornenturn

The momentum of a MQW is given by

$$\vec{p} = -i \int d\vec{x}\ W^* \vec{\nabla} W \ . \tag{37}$$

Since W_V is invariant under translations on the directions transverse to the x - axis, these waves have only longitudinal momentum.

From now on, we will put our system within a parallelepipedof length L (in the direction of the MQW's motion) and transverse section A. L and A are such that L, $\sqrt{A} >> 1/mc,\ \rho^{-1/3}$.

Putting W_V in Eq. (37) and integrating, we obtain the longitudinal momentum

$$\begin{aligned} P &= -2\ A\ \rho\ \gamma\sqrt{1-\gamma^2} \\ &= -2\ A\ \rho\ V\sqrt{1-V^2} \ . \end{aligned} \tag{38}$$

The momentum is opposite to the MQW's velocity because such a wave corresponds to the motion of a pulse which has density lower than ρ, and so, the motion of matter is opposite to the wave motion.

Now, dividing P by the parallelepiped transverse section, we get the momentum per unit of area carried by W_V

$$P = -2\ \rho\ \gamma\ \sqrt{1-\gamma^2} \ . \tag{39}$$

Since in our opinion the $\lambda|\phi|^4$ theory has much to do with the theory of ^{4}He superfluid, we may use Eq. (39) to estimate the values of

P carried by the MQWs that could exist in this system. At 1.0K and normal pressure, the ^4He mass density is 0.145 g/cm^3 [38], corresponding to a charge density of

$$\rho_4 = 0.0217/\text{Å}^3$$

Therefore, when the liquid satisfies the above condictions, the maximum P intensity would be 2.3×10^{-5} g/cm sec.

P is a very small quantity. That is why, we believe, in case of existence of *MQWs*, they could not be observed in mechanical experiments on liquid ^4He done until now. The shock of a topological wave against a recipient wall amounts to nearly the same momentumexchangeas that of a paper sheet which, having the same transverse dimensions, hits the wall with a velocity of 10^{-2} cm/sec (with the difference that the momenta have opposite signs).

c) Energy

To compute the energy carried by the MQWs is slightly more complicated than the calculation of its momentum, because it is necessary to be carefull to take away fhe ground state energy.

The charge of the *MQW* represented by W_V, and contained within the volume A L, is

$$\begin{aligned} Q &= \int_A d\vec{s} \int_L dx \, |W_V|^2 \\ &= A\, L\, \rho - \frac{A\rho}{\sqrt{m\lambda\rho}}\, 2\gamma \quad . \end{aligned} \tag{41}$$

Here $\vec{s}$ is a vector orthogonal to the direction of the wave motion.

When this system of charge Q is on its own fundamental state (or when it has not any *MQW*) it is described by the wave function (see Eqs. (8) or (11))

$$\Omega = \sqrt{Q/AL}\, \exp(-i\lambda\rho t) \quad , \tag{42}$$

and its energy will be (see Eq. (10))

$$H\{\Omega\} = \frac{1}{2} \lambda \frac{Q^2}{AL} \,. \qquad (43)$$

Cornbining Eqs. (41) and (43) it follows that

$$H\{\Omega\} = \frac{1}{2} AL\lambda\rho^2 - 2A \frac{\lambda\gamma\rho^2}{\sqrt{m\lambda\rho}} + 0(1/L) \,. \qquad (44)$$

The energy carried by the MQW is the difference between the energy related to the wave function W_V and that of the fundamental state

$$E(\gamma) = H\{W_V\} - H\{\Omega\} \quad , \qquad (45)$$

and $H\{W_V\}$ can be obtained by inserting W_V into Eq. (3). This procedure leads to

$$E(\gamma) = \frac{4}{3} \sqrt{\lambda\rho/m}\, \rho\, \blacksquare^3 A \,. \qquad (46)$$

So that, the energy per unit of area carried by the wave $(E(\gamma)/A)$ will be

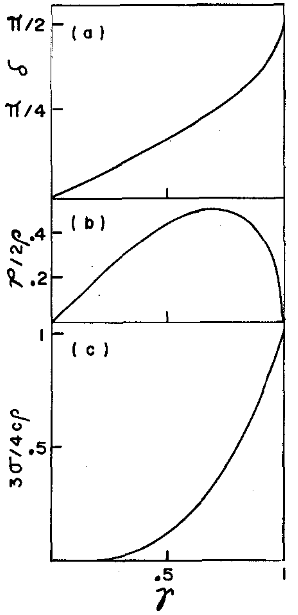

Fig.4 - Topological charge (δ), momentum and energy per unity of area (P and σ) carried by the macroscopic quantum waves.

$$\sigma(\gamma) = \frac{4}{3} c \rho \gamma^3 \tag{47}$$

Then, if the topological waves really exists in ^{4}He, at 1.0 K and normal pressure (recall that c is the sound velocity at absolute zero, $c \approx 247$ m/sec [39]) σ should lie within the range $0 < \sigma \lesssim 4.7 \times 10^{-5}$ eV/Å^2 = 0.54 K/Å^2.

d) The λ-transition

The nature of the λ-transition has not been understood up to now. It is a second order phase transition, similar to those of ferromagnetism, order-disorder, etc. Nevertheless, since the early work on liquid Helium theory, all attempts to treat the λ-point by means of lattice models- which are appropriate to study second order phase transitions - were never taken seriously due to the simple fact that the superfluid is a liquid, not a solid. Not having periodic structure, what kind of order could exist in this system? London[17] proposed that, near absolute zero, Helium II should gain *order in momentum space*, that, in a certain sense, is what happensin the fundamental state of Bogoliubov's theory, where we have a uniform condensate[13].

Now, the question that must be posed is: what kind of phenomenon can break the condensate uniformity to give rise to the λ-transition?

Here we propose that this phenomenon may be just the existence of *MQWs*. Our discussion will be carried out in analogy with ferromagnetism.

The solution W_V is a single MQW in the condensate. It is legitimate to expect that Eq. (29) should also exhibit more complex solutions, corresponding to multi - *MQWs* states. In this type of states the topological waves would then be *moving* boundaries between sectors of phase.

Certainly these boundaries must increase in quantity as the temperature grows, because they carry a *positive* energy per unit of area.

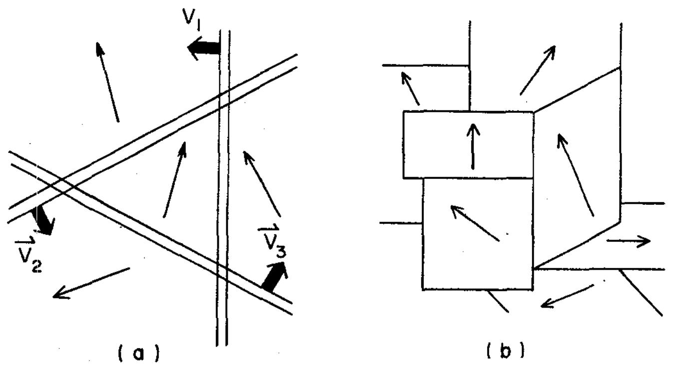

Fig.5 - (a) sketch of aninstantaneous configuration of macroscopic quantum waves (b) structure of domains of a ferromagnetic system.

Figure 5(a) is a "photograph" of a fluid segment at finite temperature. There we show a typical *MQWs* configuration, and we can seethe blocks of phase delimited by them at a fixed instant of time. The angle defined by each arrow and the horizontal lines gives the phase of its block.

To implement the statistical mechanics of the topological waves systems (for simplicity we are not considering the elementary excitations[40]), in the cannonical ensemble for instance, it is necessary to sum over an infinite set of configurations. In spite of the waves motion, *to sum over the instantaneous configurations is enough to build up the partition function,* since the energy of each *MQW* is a conserved quantity.

In Figure 5(b), that represents a cut of a ferromagnetic material, we show the well known structure of ferromagnetic domains. The magnetizatiori of a given block points in the direction of its arrow.

Just as in the case of topological waves, the energy of each magnetic blocks configuration is stored within the boundaries of the domains. The ferromagnetic system partition function can beobtained by means of a sum over all blocks configuration.

Figure 5 suggests that both sums - over *MQWs* and over magnetic 'configurations - are operations of the same nature. Therefore, quite probably, they will lead 'to similar results.

This fact explains why the fluid should undergo a second order phase transition, since this is what happens to the magnetic system.

The liquid order paranieter would be given by the statistical average of the field $\phi(\vec{x},t)$ that describes the *MQWs* motion

$$\psi(T) = \frac{\sum_{\{c\}} \frac{1}{V} \int d\vec{x}\phi(\vec{x},t) \exp(-E_c/k_BT)}{\sum_{\{c\}} \exp(-E_c/k_BT)} \tag{48}$$

The index c stands for configurations of *MQWs* , T is the temperature and k_B the Boltzmann constant.

Following still more the analogy with ferromagnetism, we guess that, if T_c is the fluid critica1 temperature, then

$$\psi(T) = 0, \quad \text{when} \quad T > T_c$$

and

$$\psi(T) \neq 0, \quad \text{when} \quad T < T_c \quad .$$

It is even possible that, when $T < T_c$, $\psi(T)$ could depend on time.

The idea that the condensate wave function should be the superfluid order pararneter is not new[41]. The novelty we are proposing is the mechanism that breaks the condensate order, i.e., the macroscopic quantum waves.

A *rough estimate of the critical temperature:* the development of the *MQWs* statistical is a very hard program. In what follows we will do a *crude* calculatíon of the critical temperature of our systern, to show that it has the same order of magnitude as that of the ^{4}He λ-point. To this purpose we resort to three simplifications:

(1) The first one corisists into neglecting the contribution of elementary excitations. In our superfluidity theory, the λ-transition is due to the existence of *MQWs*, so that, to discard quasi-particles certainly will not lead to a severe change in the critical temperature value.

(2) The second approximation is to consider an hypothetical fluid were only static *MQWs* (those for which $V=0$ and $\gamma=1$) exist. Of course this approximation also does not change the order of magnitude of the critical temperature. What do we know about the static waves? The difference in phase between two sectors separated by them is $2\,\delta(1) = a$. The energy per unit of area they have is

$$\sigma = \frac{4}{3} c\,\rho \; , \tag{49}$$

and their width Δ is shown in figure 2:

$$\Delta \simeq 2/mc \; . \tag{50}$$

(3) Finally, we compare the static *MQWs* fluid with the simple cubic 3-dimensional Ising model[42], because for this model we know an approximate relation involving σ, Δ and the critical temperature.

The simple cubic Ising model is defined by the Hamiltonian

$$- J \sum_{ij} S_i \, S_j \; ,$$

being $\{S_i = \pm 1\}$ a set of "spins" on a lattice of cubic symmetry (each site has six neigbors), and the sum Σ_{ij} is carried out over nearest neighbors only. The critical temperature of this model is[43]

$$T_I = 4.51 \; J/k_B \tag{51}$$

In figure 6 is shown a segment of a boundary surface between two blocks of opposite "spins" on the Ising lattice. This surface divides the lattice into two regions of "spins" +1 and -1, in the same way that W_0 divided the fluid condensate into sectors of wave functions $\sqrt{\rho}\,\exp(-i\lambda\rho t)$ and $-\sqrt{\rho}\,\exp(-i\lambda\rho t)$ respectively.

Δ_I is the distance between two nearest neighbors of the lattice.

In an Ising lattice there exist three types of interfaces between blocks of "spins". One of them is that of figure 6. It is such that any "spin" belonging to the boundary of a given domain has only one neighbor on the other domain, and its width is Δ_I.

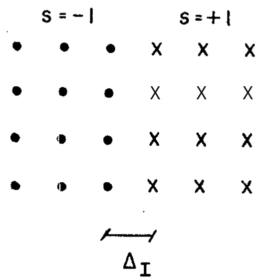

Fig.6 - Segment of the boundary between two spins blocks on an Ising lattice. Δ_I is the lattice spacing - and also the boundary width.

The other two are diagonal interfaces. For them, each "spin" on the boundary of a block has respectively 2 and 3 neighbors in the opposite block, whereas their widths are $\Delta_I/\sqrt{2}$ and $\Delta_I/\sqrt{3}$.

In the calculations presented in the sequence we shall forget the existence of the last two types of interface. Their inclusion also does not change the order of magnitude of the number we are looking for (see comment after Eq. (55)).

For the interface of figure 6, we can relate its width Δ_I to the energy per unit of area contained on it (relative to the fundamental state):

$$\sigma_I = 2J/\Delta_I^2 \ . \tag{52}$$

Thus, from Eqs. (51) and (52), we obtain an approximate relation involving the critical temperature and the width and energy per unit of area of the domains boundaries, valid for the simple cubic Ising model:

$$T_I \simeq 2.26 \ \sigma_I \ \Delta_I^2/k_B \ . \tag{53}$$

Although being not exactly the same thing, the configurations of static topological waves have a certain similarity to those of the Ising model. Then we may assume that an equation analogous to (53) should also hold (in a very approximate sense, of course) to the critical temperature of the *MQWs* system

$$T_c \simeq 2.26 \ \sigma\Delta^2/k_B \ . \tag{54}$$

Now, using Eqs. (49), (50) and (54), we find that the fluid critical temperature should be near

$$T_c \simeq 12.0\, \hbar^3 \rho / m^2 c k_B \; . \tag{55}$$

Plugging in the above relation the values of ρ_4 given by Eq. (40), and remembering that c is the sound velocity at absolute zero [39], we get $T_c \approx 2.1K$, for liquid ^4He at normal pressure.

If, instead of the interface of figure 6, we had considered in our calculetions one of the two diagonal boundaries mentioned above, our results would be respectively $T_c \simeq \sqrt{2} \times 2.1K = 3.0K$ and $T_c \simeq \sqrt{3} \times 2.1K = 3.6K$.

It is remarkable that all these numbers have the order of magnitude of the λ-point temperature, which is 2.19K.

In the remainder part of this section we shall discuss, inthe light of our proposals, how to understand a phenomenological theory and some experimental results.

We stress, however, that doing this, we are *not* trying to prove that the *MQWs* exist, but only showing that their existence, *if true*, could be consistent with the presented picture.

e) Tisza's two-fluid model

To describe some properties of Helium II, specially the thermomechanical and mechanocaloric effects[18-20], Tisza[33,44,45] postulated that, below the A-point, liquid Helium is made up of two components: the *normal fluid* and the *superfluid*. He supposed also that, whereas the normal fluid should behave like an ordinary fluid, the superfluid should have two very unusual properties:

i) its entropy would be zero; and

ii) it would flow with no resistance through channels of extremely small diameters (10^{-2}, 10^{-3} cm).

Now we will try to understand Tisza's model in the light of the theory proposed here. Due to our scarcity of knowledge about the *MQWs* statistical mechanics, our argumentation is inevitably full of (reasonable) conjectures.

To identifyin our theory the normal fluid and the superfluid, let us follow Landau[3], Kramers[46] and Lee, Huang and Yang[13]. According with these authors the superfluid is just the condensate, whereas the normal fluid component is made up by its excitations - phonons and bound excitations, wich will be studied in the next chapter.

Note that the topological waves, which belong to the condensate, provide a beautiful and simple explanation about how the superfluid could move even when the system is contained in a closed reservoir.

It is possible to get a sense of how the condensate entropy S and internal energy U depend upon the temperature by exploring againthe analogy with rnagnetic systems.

Consider a ferromagnetic model of null external field. If its entropy is given by $S_M(T)$, let us define what we call the critical range[47] of the model. It is the temperature interval $(T_c-\Delta_1, T_c+\Delta_2)$ where T_c is the critical temperature and Δ_1 and Δ_2 are defined by

$$S_M(T_c-\Delta_1) = 0.2\ S_M(\infty) \qquad \text{and} \qquad S_M(T_c+\Delta_2) = 0.8\ S_M(\infty)$$

($S_M(\infty)$ is the saturated entropy of infinite temperature).

In magnetic systems, the typical values of Δ_1 and Δ_2 are such that[48]

$$\Delta_1,\ \Delta_2 \ll T_c$$

i.e., the extent of the critical range is very small when compared with T_c.

So, below the critical range, which is very narrow, the magnetic entropy is nearly zero; within the critical range itself it changes abruptly, and almost reaches the saturation value just above that interval. In summary, the magnetic entropy $S_M(T)$ resembles a step function.

Besides that, the magnetic internal energy also grows suddenly arround the critical point-recall the specific heat divergence at T_c.

Since we believe that the *MQWs* statistical mechanics looks like the ferromagnetic one, we have already elements to understand some qualitative features of the condensate thermodynamics below and above the λ-point.

From the above considerations the following picture emerges (concerning to the properties of the *MQWs* system):

a) With the exception of a narrow region near the λ-point (the critical range), for any other temperature below T_c, the ***condensate*** entropy and internal energy are both very small;

b) above T_c these quantities become much larger than they were in the interval of item (a);

c) a corollary of statements (a) and (b) is that the number of *MQWs* (recall that they carry the condensate energy) is small for $T < T_c - \Delta_1$, and grows sudden and strongly when the system crosses the critical range from below.

Property (a) by itself could justify hypothesis (i) of Tisza's theory. Below the critical range the major contribution to the specific heat and entropy should come from the elementary excitations (normal fluid).

Note however that, within the critical range, the condensate contribution starts growing quickly to become dominant near T_c.

To understand Tisza's second hypothesis we first recall that, at absolute zero, a single *MQW* flows through a capillary without change of velocity (without resistance), because a change of velocity would violate the conservation of topological charge - of course a capillary is still a macroscopic body.

Therefore the capillary walls cannot interfere into the *MQWs* motion. Nor can the elementary excitations do it. The only mechanism we imagine to hinder the *MQWs* motion is the proper interaction betweenthem.

From properties (a) and (c) we conclude that *below the critical range* the situation must be similar to that of absolute zero (i.e., the condensate waves move without resistance), since, in this case, there are relatively few topological waves in the capillary, and, as a consequence, the interactions between them should not be enough to obstruct their motion. Besides that, this motion is certainly quite regular because the condensate entropy is small.

Nevertheless, *above the critical range,* things are supposed to be severely changed, because there, where the number of *MQWs* is very large, their frequent interactions and the high condensate chaos (too high entropy) would probably forbid any kind of regular motion.

The motion of the normal fluid component within a narrow channel, is on the other hand, always viscous, since the quasi-particles interact between themselves and with the channel walls.

It is important to stress that, according with our picture of superfluidity (and contrary to what it normally assumed), above the A-point the condensate is not fully converted into normal fluid. It merely undergoes a phase transition, going to a non-superfluid phasewhich however is not made up of elementary excitations. I.e., above the A-point we still have two fluids, but no one of them is a superfluid.

Concerning to the Helium II flow throw narrow channels, we even point out that the strong dependence of the fluid mean velocity upon the capillary diameter may be one more indication that macroscopic objects can be present in the superfluid dynamics.

f) Second Sound

Based upon the two-fluid model, Tisza[33] and Landau[3] have independently proposed that, besides the acoustic waves, a different kind of subsonic waves should also propagate in liquid Helium II. Latter on, Lifshitz[49] showed that this new wave, called second sound, would correspond to temperature oscillations.

We conjecture that our *MQWs*, being also subsonic, can play an important role in the motion of second sound.

The first experimental observation of second soundwascarried out by Peshkov[50] in 1946. By means of the standing waves technique, he measured the second sound velocity between 1.0K and the λ-point, with high accuracy.

However the pulse technique develloped by Pellam[51] and Osborne[52] came to show that, below 1.0K, the second sound behavior undergoes a severe change.

Properties of second sound in this temperature region are exhaustively discussed by Atkins[18]. We will present here only two of these properties, which are pertinent to our subsequent argumentation.

Using heat pulses of 20 or 40 μsec, Kramers *et al*[53] investigated the propagation of second sound in tubes of 0.40, 0.80 and 0.95 cm in diameter. Below 0.7K they observed that:

1) The detected signal has a shape radically different from the original one. The received pulse width, for example, is more than one thousand times the heat pulse width. This fact, in our oppinion, suggests that *the second sound carriers have a very large spectrum of velocities*.

2) The pulse shape and, in particular, its maximum velocity vary considerably when the geometry (length and diameter of the propagation tube) is modified[18]. This observation may evidently indicate that *the second sound carriers (or at Zeast a part of them) are made up by macroscopic objects*, which should be very sensitive to macroscopic changes of the system.

Therefore the topological waves, being macroscopic and having a large spectrum of velocities, are a natural candidate to carry second sound.

The major problem with this conclusion is to conciliate it with the fact that when $T \gtrsim 1.0K$ the second sound velocity is almost well defined. Nevertheless, it is possible that the existence of a relatively large number of slow topological waves (which occurs at higher temperatures) can inhibit, or even forbid, the motion of the rapid ones. A decisive check of this possibility must come from the solution of the *MQWs* statistical mechanics.

The existence of macroscopic quantum waves in Helium II can, in principle, be the cause of its high thermal conductivity (which is hundreds times greater than those of metals). What mechanism can carry more energy than a coherent macroscopic wave?

Another interesting possibility we point out is that the counterflow phenomenon, which occurs in liquid ^4He, might be due to the motion of topological waves in the condensate - since the *MQWs* momentum and velocity are opposite, the motion of matter is then opposite to the motion of energy.

4. EXCITATIONS

This chapter is devoted to study the elementary excitations of the topological waves systems.

First of all, the equation of motion and general expressions for the energy and momentum of these quasi-particles are deduced.

Our goal is to find their spectrum to confront it with liquid Helium spectrum as presented by Woods and Cowley[16,28] and Dietrich et *al.*[29].

A *MQW* has essentially two kinds of excitations: phonon-like excitations and quasi-particles bound to it. The formers have a dispersion relation identical to that of Bogoliubov's phonons.

The novelty that comes with the *MQWs* is just the set of bound quasi-particles. They have a spectrum very similar to liquid Helium spectrum.

To quantize these excitations we employ again the Bohr-Sommerfeld quantization rule.

We succeed in finding the exact wave functions for two classes of bound excitations: for those having zero frequency and for the quasi-particles of very high frequency bound within the static *MQWs*. The other excitations are treated by means of a variational approach where the trial functions are designed to produce the exact low frequency behavior.

a) Equation of Motion

The treatment of the *MQWs* excitations will be carried out in strict parallelism with what we did in subsection II B. Let us consider a system of charge Q, defined in a box of volume AL.

To study the infinitesirnal excitations of a MQW, we add to its wave function a small fluctuation, so defining the field

$$\phi = \{(V - i\gamma \ \text{tgh} \ \gamma\xi)\sqrt{\rho'} + \eta\} \exp(-i\lambda\rho' t) \tag{56}$$

where ρ' is a constant slightly different from the density $\rho = Q/V$, and η is orthogonai to W_V

$$\int (V + i\gamma \ \text{tgh} \ \gamma\xi) \ \sqrt{\rho'} \ \eta \ d\vec{x} = 0 \tag{57}$$

We recall that

$$\xi = mc(x - cVt) \tag{58}$$

and that $d\vec{x} = d\vec{s}dx$, where $\vec{s}$ and x correspond respectively to the variables transverse and longitudinal to the motion of the *MQW*. In this paper, definition (58) means solely a change of variable, not a change of referente frame.

Considering these facts and using Eqs.(56) and (57), we obtain a relation between ρ and ρ'

$$\rho AL = Q = \int d\vec{x}\phi^*\phi = \rho' AL + \int d\vec{x}\eta^*\eta - \frac{2\rho'\gamma}{mc} A \tag{59a}$$

or

$$\rho = \rho' + \frac{1}{Ar_\prime} \int d\vec{x}\, \eta^* \eta - \frac{2\rho'\gamma}{mc} \left(\frac{1}{L}\right) \tag{59b}$$

If we plug ϕ into Eq. (29), use the above equation, and retain only terms which are linear in η and of order zero in $(1/L)$, weget

$$(i/\lambda\rho)\partial_t \eta = -\frac{1}{2m^2c^2} \nabla^2_{\vec{s}} \eta - \frac{1}{2} \partial^2_\xi \eta + iV \partial_\xi \eta + \left\{1 - \frac{2\gamma^2}{(\cosh\gamma\xi)^2}\right\} \eta +$$

$$+ \left\{ V^2 - \gamma^2 + \frac{\gamma^2}{(\cosh \gamma\xi)^2} - 2iV\gamma \text{ tgh } \gamma\xi \right\} \eta^* \tag{60}$$

where $\nabla^2_{\vec{s}}$ is the transverse laplacian, and $\eta = \eta(\xi, \vec{s}, t)$, i.e., the partial time derivative $\partial_t \eta$ does *not touch* the original time dependence of ξ.

Eq. (60) is the equation of motion of the fluid elementary excitations in the presence of a macroscopic quantum wave.

Consider the transformations of space inversion $(P: x \to -x)$ time reversal T and charge conjugation C. Eq. (60) is not invariant under separate applications of these transformation. It is, however, invariant under *CPT*. This occurs because the topological wave W_V breaks any one of these symnetries, but is still invariant under the combined *CPT*.

$$W^*_V(-\xi, -t) = W_V(\xi, t)$$

(Note that $-\xi = PT\xi$).

When $|\xi| \gg 1$, at large distances from the *MQWs* plane, Eq. (60) becomes equal to the equation of motion of Bogoliubov's phonons, which we studied in chapter II. That is why the system considered here alsohas phonon-like quasi-particles, not bound in the *MQW*, but which are scattered by it. Of course, the dispersion relation of these phonons must be equal to that of the quasi-particles moving in Bogoliubov's condensate.

Nevertheless, the quasi-particle spectrum of the wave W_V is more rich than that of Ω, because Eq. (60) exhibits solutions of the bound-state type, which have no counterpart within Bogoliubov's condensate.

In order to study these new excitations it is convenient to represent the fluctuation η in the following manner

$$\eta_{\vec{k}} = \sqrt{\frac{mc}{A}}\,\{u(\xi)\exp i(\vec{k}\vec{s}-\omega t) + v(\xi)\ \exp i(-\vec{k}\vec{s}+\omega t)\}\ . \tag{61}$$

We repeat that A is the transverse section of the parallelepiped boxthat contains the *MQW.* The factor $\sqrt{mc}$ was written in Eq. (61) to simplify some expression which will appear later.

After quantization, $\eta_{\vec{k}}$ will describe a quasi-particle whose transverse momentum is $\vec{k}$.

Now, plugging $\eta_{\vec{k}}$ in **Eq. (60)**, we get a system of coupled equation for $u(\xi)$ and $v(\xi)$:

$$\{-\frac{1}{2}\,\partial_\xi^2 - \frac{2\gamma^2}{(\cosh\ \gamma\xi)^2} + 1 + iV\partial_\xi - \frac{\omega}{mc^2} + \frac{k^2}{2m^2c^2}\}\ u\ +$$

$$+\ \{V^2 - \gamma^2 + \frac{\gamma^2}{(\cosh\ \gamma\xi)^2} - 2iV\gamma\ \mathrm{tgh}\ \gamma\xi\}\ v^* = 0\ , \tag{62a}$$

and

$$\{-\frac{1}{2}\,\partial_\xi^2 - \frac{2\gamma^2}{(\cosh\ \gamma\xi)^2} + 1 + iV\partial_\xi + \frac{\omega}{mc^2} + \frac{k^2}{2m^2c^2}\}\ v\ +$$

$$+\ \{V^2 - \gamma^2 + \frac{\gamma^2}{(\cosh\ \gamma\xi)^2} - 2iV\gamma\ \mathrm{tgh}\ \gamma\xi\}\ u^* = 0\ . \tag{62b}$$

To synthesize this çystem, we define the "spinor" $\psi(\xi)$,

$$\psi(\xi) = \begin{pmatrix} u\,(\xi) \\ v^*(\xi) \end{pmatrix} \tag{63}$$

and the operator $\hat{\Sigma}(V, k)$,

$$\hat{\Sigma}(V,k) = \{ -\frac{1}{2}\partial_\xi^2 - \frac{2\gamma^2}{(\cosh\gamma\xi)^2} + 1 + \frac{k^2}{2m^2c^2} \} I +$$

$$+ \{ iV\partial_\xi \}\sigma_3 + \{ V^2 - \gamma^2 + \frac{\gamma^2}{(\cosh\gamma\xi)^2} \} \sigma_1 +$$

$$+ \{ 2V\gamma \text{ tgh } \gamma\xi \} \sigma_2 \quad , \tag{64}$$

Here, I is the identity matrix, and σ_1, σ_2 and σ_3 are the Pauli matrices.

By taking the complex conjugate of Eq. (62b), we can write the system (62) as

$$mc^2 \; \hat{\Sigma}(V,k)\psi = \omega(V,k)\sigma_3\psi \; . \tag{65}$$

So, for each value of k, we have an eigenvalue equation where the eigenvalue is just the frequence w.

Once we know an eigenfunction of Eq. (65), we are lead, through Eqs. (63) and (61), to the wave function of an excitation of W_V, which has transverse wave number $\vec{k}$.

It is important to point out that the solutions ψ which have physical relevance are those corresponding to *non-negative* values of $\vec{k}^2$. Otherwise, the quasi-particle wave functions (61) should correspond to states of infinite energy, because the integration in the transverse coordinates would diverge.

Before going on to study the solutions of Eqs. (60) and (65), let us first present the condition of quantization and general expressions for the quasi-particles energy and momentum.

b) Quantization, Momentum and Energy

In Appendix A, by using the Bohr-Sommerfeld rule, we show what is the quantization condition for the eigensolutions $\eta_{\vec{k}}$ of Eq. (61). The single quasi-particle states must obey the constraints:

$$\int d\xi \ \psi^{t}\sigma_3\psi = 1, \ \text{when } \omega > 0 \ , \tag{66a}$$

and

$$\int d\xi \ \psi^{t}\sigma_3\psi = - 1, \ \text{when } \omega < 0 \ , \tag{66b}$$

where the auxiliary "spinor" ψ is defined in Eq. (63).

From Eqs. (65) and (66) we see that

$$|\omega| = mc^2 \int d\xi \ \psi^{\dagger} \ \hat{\Sigma}(V,k)\psi \ . \tag{67}$$

Concerning to the physical excitations (for which k^2 is real and non-negative) the operator $\hat{\Sigma}(V,k)$ is hermitian. Thus the modulus of ω is consistently the expectation value of an hermitian operator.

The transverse momentum associated with the *eigensolution* (61) is

$$\vec{P}_{\text{transverse}} = - \frac{i}{mc} \int \eta^*_{\vec{k}} \ \vec{\nabla}_{\vec{s}} \ \eta_{\vec{k}} \ d\vec{s}d\xi \ . \tag{68}$$

Plugging above $\eta_{\vec{k}}$ (from Eq. (61)) and integrating in $\vec{s}$ we get:

$$\vec{P}_{\text{transverse}} = \vec{k} \int d\xi \psi^{t}\sigma_3\psi \ . \tag{69a}$$

Then,

$$\vec{P}_{\text{transverse}} = , \quad \text{when } \omega > 0 \ , \tag{69b}$$

and

$$\vec{P}_{\text{transverse}} = - \vec{k}, \quad \text{when } \omega < 0 \ . \tag{69c}$$

To change ω into $-\omega$ is equivalent to change the roles of $u(\xi)$ and $v(\xi)$ $(u(\xi) \leftrightarrow v(\xi))$ and the sign of the wave number. This means that

$$\psi_{\omega} = \begin{pmatrix} u \\ v^* \end{pmatrix} \quad \text{and} \quad \psi_{-\omega} = \begin{pmatrix} v \\ u^* \end{pmatrix}$$

are "spinors" associated with the same quasi-particle - a conclusion we could reach only by inspecting Eq. (61).

Now, we shall compute the longitudinal momentum ℓ carried by the excitations $\eta_{\vec{k}}$.

In the presence of quasi-particles the longitudinal momentum of the topological wave system will be

$$P + \ell = -i \int d\xi d\vec{s} \{ \sqrt{\rho} \ (V+i\gamma \ \mathrm{tgh}\gamma\xi) + \eta^* \} \partial_\xi \{ \sqrt{\rho} \ (V-i\gamma \ \mathrm{tgh}\gamma\xi) + \eta \} \ , \quad (70)$$

where P is the MQW proper momentum.

(*The translation mode*: since the original theory is invariant under translation and the MQW W_V is not, there exists a special solution of Eq. (60), given by

$$\eta_{\mathrm{translation}} = (\partial_\xi W_V) \exp (i\lambda\rho t) \ , \quad (71)$$

which is known as the translation mode. This solution does notcorrespond to excitations, but rather, it simply reflects the fact that the *MQWs* system can be translated as a whole in the x-direction. People, who are familiar with semiclassical methods of quantization, know that the translation mode represents an infinitesimal translation of the fluid system). To avoid this translation, we must impose that the quasi-particles wave functions be orthogonal to $\eta_{\mathrm{translation}}$ [27,32,54], i.e.,

$$\int \eta^* \eta_{\mathrm{translation}} \, d\vec{s} d\xi = 0 \ . \quad (72)$$

Once (72) is required, the integral in Eq. (70) decouples in the following manner

$$P + \ell = - i\rho \int d\vec{s} d\xi (V+i\gamma \ \mathrm{tgh} \ \gamma\xi) \partial_\xi (V-i\gamma \ \mathrm{tgh} \ \gamma\xi) - i \int d\vec{s} d\xi \ \eta^* \partial_\xi \eta . \quad (73)$$

From Eqs. (31) and (37), we recognize the first of the above integrals as being just the topological wave momentum. Thus, the quasi -particle longitudinal momentum should be

$$\ell = -i \int d\vec{s} d\xi \ \eta^* \partial_\xi \eta \ . \quad (74)$$

Inserting in (74) the function $\eta_{\vec{k}}$ of Eq. (61), and integrating in the transverse directions, we obtain the longitudinal momentum of these quasi-particles

$$\ell(V,k) = -imc \int d\xi \{ u^*(\xi)\partial_\xi u(\xi) + v^*(\xi)\partial_\xi v(\xi) \} =$$

$$= -imc \int d\xi \{ u^*(\xi)\partial_\xi u(\xi) - v(\xi)\partial_\xi v^*(\xi) \} \quad , \tag{75}$$

and finally

$$\ell(V,k) = -imc \int d\xi \psi^\dagger \sigma_3 \partial_\xi \psi \tag{76}$$

It is important to point out that the quasi-particles of the *MQW* system have not well defined longitudinal momenta (ψ is not an eigenfunction of $i\partial_\xi$). This is so because the *MQW* destroys the invariance under translation of the condensate.

The longitudinal momentum uncertainty should be larger for quasi-particles bound within the small velocity *MQWs*, since the *MQW* width diminishes with V. An excitation bound in a region of width $2/mc\gamma$ must have a momentum uncertainty of about $mc\gamma/2$.

The Hamiltonian (3) can also be written in the form

$$H = \frac{1}{mc} \int d\vec{s} d\xi \{ -\frac{mc^2}{2} \phi^*\partial_\xi^2\phi - \frac{1}{2m} \phi^* \nabla_{\vec{s}}^2\phi + \frac{\lambda}{2} (\phi^*\phi)^2 \} \ . \tag{77}$$

We repeat that we have not done a change of reference frame, but a simple change of variable. Plugging into (77) the field ϕ as given by Eq. (56), using Eq. (59b) to relate ρ' and ρ, and keeping only terms up to second order in η, we obtain the Hamiltonian of the system *MQW* + excitation:

$$H = H\{W_V\} + H_1\{W_V,\eta\} + H_V\{\eta\} \ , \tag{78}$$

where $H\{W_V\}$ is the energy associated with the wave function W_V, which was already used in section III to compute the energy carried by the *MQW*. $H_1\{W_V,\eta\}$ is a term linear in η, that we will show to vanish. And $H_V\{\eta\}$

is a Hamiltonian, quadratic in η, that describes the quasi-particles motion in the topological wave system. If $g(\xi)$ is given by

$$g(\xi) = W_V \exp(i\lambda\rho t) = (V - i\gamma \operatorname{tgh} \gamma\xi)\sqrt{\rho} \quad , \tag{79}$$

then $H_1\{W_V, \eta\}$ and $H_V\{\eta\}$ are written as follows

$$H_1\{W_V,\eta\} = \frac{1}{mc} \int d\vec{s}d\xi \;\{ - \frac{mc^2}{2} ((\partial_\xi^2 g^*)\eta + (\partial_\xi^2 g)\eta^*) + \lambda|g|^2(g\eta^* + g^*\eta)\} \quad , \tag{80}$$

and

$$H_V\{\eta\} = \frac{1}{mc} \int d\vec{s}d\xi \;\{ \frac{mc^2}{2} \partial_\xi\eta^*\partial_\xi\eta + \frac{1}{2m} \vec{\nabla}_{\vec{s}}\eta^* \vec{\nabla}_{\vec{s}}\eta +$$

$$+ \frac{\lambda}{2} (4|g|^2 \eta^*\eta + g^{*2}\eta^2 + g^2\eta^{*2} - 2\rho\eta^*\eta)\} \; . \tag{81}$$

$H_1\{W_V, \eta\}$ vanishes because we imposed that η is orthogonal to the translation mode. Since W_V is a solution of Eq. (29), it follows that $g(\xi)$ must obey the equation

$$\frac{mc^2}{2} \partial_\xi^2 g - \lambda|g|^2 g = i\, V\, mc^2\, \partial_\xi g \; . \tag{82}$$

Through a substitution of this relation into Eq. (80), we get

$$H_1\{W_V,\eta\} = -iVc \int d\xi d\vec{s}\, \{(\partial_\xi g)\eta^* + (\partial_\xi g^*)\eta\} \; . \tag{83}$$

It is clear, then, that the above integral must vanish, because $\partial_\xi g$ is proportional to the translation mode (see Eqs. (71) and (79)). Thus, the Harniltonian of the *MQWs* elementary excitations is, in fact, given solely by $H_V\{\eta\}$. Taking into account the definition of $g(\xi)$, and combining Eq. (81) with the equation of motion (60), we derive a very simple expression for the energy of quasi-particles moving in the *MQW* W_V.

$$E\{V,\eta\} = \int d\vec{s}d\xi \;\{ -icV\eta^*\partial_\xi\eta + \frac{i}{2mc} (\eta^*\partial_t\eta - \partial_t\eta^*\eta)\} \; . \tag{84}$$

To obtain the energy of the quasi-particles described by the eigenfunction $\eta_{\vec{k}}$, we plug this function in (84), and integrate over the transverse coordinate. The result is

$$E(V,k) = cV\ell(V,k) + \omega \int d\xi \{ u^*(\xi)\ u(\xi) - v^*(\xi)\ v(\xi) \} , \tag{85a}$$

or

$$E(V,k) = cV\ell(V,k) + \omega \int d\xi\ \psi^\dagger\ \sigma_3 \psi\ . \tag{85b}$$

We recall that cV is the topological wave velocity, $\ell(V,k)$ is the excitation longitudinal momentum, and ω is its frequency.

Finally, from the quantization condition (66), and from Eq. (85b), we get

$$E(V,k) = cV\ell(V,k) + |\omega| . \tag{86}$$

We have already pointed out that the longitudinal momentum of a *MQW* excitations is nof well defined. From the above equation we see that neither is its energy. The good quantum number to characterize its motion is the difference $E-cV\ell=|\omega|$. In the cannonical formulation of the *MQWs* problem (to appear) we explore this fact by working with the operator $\hat{H}-cV\hat{\mathcal{L}}$ instead of the Hamiltonian. The variational approach of sub-section 4-d is also implemented by minimizing $E-cV\ell$, not the energy.

Let us now summarize the results of this subsection: we have shown that, if ψ is the auxiliary "spinor" defined in Eq. (63), the quantization of the eigenfunction $\eta_{\vec{k}}$ (written in Eq. (61)) is implemented by conditions (66). After quantization, $\eta_{\vec{k}}$ describes the states of a single quasi-particle moving in a *MQW* of velocity cV. The transverse momentum of such an excitation is $\vec{k}$, whereas its mean longitudinal momentum and energy are given respectively by Eqs. (76) and (86).

Before concluding we define the ***mean value*** of the quasi-particle total momentum, by using the transverse momentum eigenvalue ($\vec{k}$) and the longitudinal momentum ***mean value*** ($\ell(V,k)$).

The total momentum ***mean value*** will be

$$p(V,k) = \sqrt{k^2 + \ell^2(V,k)}\ . \tag{87}$$

c) Some Special Excitation

The complete set of eigensolutions of Eq. (65) is still unknown. A variational procedure will be employed in the next subsection to treat the bound quasi-particles, that, for a given value of k, have the lowest energy.

Before this is done it is convenient to study some special solutions of (65) whose exact expressions we know.

In appendix B we present four *zero-frequency* solutions of that equation[55].

$$R = \sqrt{1-\gamma^2+\gamma^4} \tag{88a}$$

and

$$S = (1-2\gamma^2) \ . \tag{88b}$$

The solution ψ_2,

$$\psi_2 = \frac{A_2}{(\cosh\gamma\xi)} \left\{ \sqrt{R-S} \begin{pmatrix}1\\1\end{pmatrix} + i\sqrt{3}\sqrt{R+S}\ \mathrm{tgh}\ \gamma\xi \begin{pmatrix}1\\-1\end{pmatrix} \right\} \tag{89}$$

is (among those shown in Appendix B) the one which has most physical relevante. It describes a quasi-particle bound in a *MQW* of velocity cV, and whose transverse wave number is

$$k_2^2 = m^2c^2\{2R+\gamma^2-2\} \tag{90}$$

Solution (89) obeys the following strange relation (this follows from the fact that its frequency is zero).:

$$\int d\xi\, \psi_2^\dagger \sigma_3 \psi_2 = 0 \tag{91}$$

It does not satisfy any one of the quantization conditions (66), and we are lead to think that even ψ_2 has no physical meaning. Actually, what happens is that this function describes an excitation of infiniteenergy. ψ_2 could satisfy (66a) if we naively take the constant A_2 in Eq. (89) to be infinite. Plugging ψ_2 in Eqs. (76) and (86), we observe that the longitudinal momentum and the energy of this excitation are also infinite quantities when $A_2 \to \infty$. However, in this limit, the relations

$$\frac{E(V,k_2)}{\ell(V,k_2)} = cV = \frac{E(V,k_2)}{p(V,k_2)} \tag{92}$$

are a finite number (cV). Therefore we can expect that in the limit $\omega\to 0$ (that wo'ld be a singular limit) the relation between the mean energy and mean momentum of quasi-particles bound in W_V should be just cV. This is, in fact, the result of our variational computation done in the next sub-section.

In figure 7 we see the curves $E(V,k) \times p(V,k)$, when $\omega\to 0$. They are all straight lines whose slope is cV $(\gamma=\sqrt{1-V^2})$.

In appendix C we treat the quasi-particles bound within the static $MQWs$ $(V=0)$ in the region of large values of $|\omega|$. There it is shown that, in these circunstances, we have an even and an odd bound excitations. When $\omega>0$, the relations between their transverse momenta and the frequency are respectively

$$\frac{k_0^2}{2m^2c^2} = \frac{\omega}{mc^2} + 0.2193 + 0(1/\omega) \tag{93a}$$

and

$$\frac{k_1^2}{m^2c^2} = \frac{\omega}{mc^2} - 0.8423 + 0(1/\omega) \tag{93b}$$

and after quantization their "spinors" are given by

$$\psi_{k_0} = \{0.808/(\cosh\,\xi)^{\alpha}\}\binom{1}{0} \tag{94a}$$

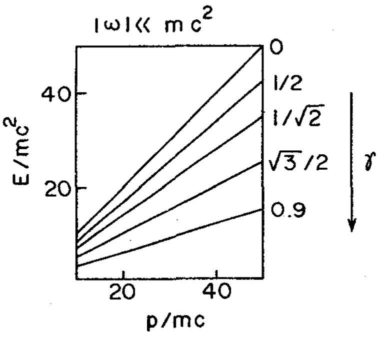

Fig.7 - Relations between the mean energy and momentum of bound quasi--particles in the small $|\omega|$ limit.

and

$$\psi_{k_1} = \{0.839 \ \text{senh} \ \xi/(\cosh \ \xi)^{\alpha}\} \begin{pmatrix}1\\0\end{pmatrix} \tag{94b}$$

where $\alpha = (\sqrt{17} - 1)/2 = 1.562$.

Working with the above equations and with Eqs. (76), (86) and (87), we conclude that when ω is a large number the energy - total momentum relations for the even and odd quasi-particles bound in the static topological waves are:

$$\frac{E}{mc^2} = \frac{1}{2}\left(\frac{p}{mc}\right)^2 - 0.2192 + 0(1/\omega) \tag{95a}$$

for the even excitations, and

$$\frac{E}{mc^2} = \frac{1}{2}\left(\frac{p}{mc}\right)^2 + 0.8423 + 0(1/\omega) \tag{95b}$$

for the odd ones.

For a given value of p, the even bound mode has less energy than the odd one.

In the small ω region we have found only one bound excitation of wave number $\vec{k}$, whereas in the large ω region, for $V=0$, we found two. Concerning to this point, two possibilities may beoccurring; 1) when $\omega=0$ there exists an yet unknown solution of Eq. (65), which is linked to the odd solution of the large ω case; or 2) Maybe the nonphysical "spinor" ψ_3 of appendix B turns to be physicai in a finite value of ω, and goes to the odd mode when $\omega\to\infty$.

Figure 8 shows a region in the (E,p) plane delimited by two parabolas. The upper boundary is the asymptotic part of the Bogoliubov's phonons dispersion relation (see Eq. (28)), and the lower boundary is the parabola of the static *MQWs* even excitations, which is given by Eq.(95a). In the next subsection we will see that, when ω is a large number. the energy-momentum relations of the excitations which are bound in the topological wave W_V $(V\neq 0,1)$, having the lowest mean energy compatible with a given $\vec{k}$, are curves that lie between these two parabolas.

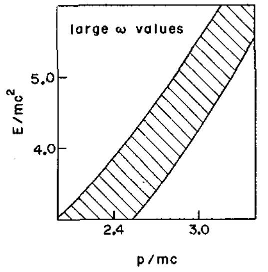

Fig.8 - Habitat of the bound quasi-particles for large values of w.

d) Variational Approach

We resort here to a variational approach to study the quasi-particles bound in a *MQW*. As we know, in the present theory, the good quantum numbers to specify a quasi-particle state are its transverse momentum $\vec{k}$, and the frequency modulus $|\omega|$. The latter is given by the difference

$$E(V,k) - cV\,\ell(V,k) \tag{96}$$

To implement the variational approach we first represent each excitation state by a trial "spinor", which depends on two parameters and satisfies manifestly the quantization condition (66). After, this, the two parameters are adjusted in such a way that $|\omega|$ attains its minimum value.

This procedure allow us to handle only those bound quasi-particles which, for a given value of $\vec{k}$, have the lowest frequency (and, in consequence, the lowest energy too). The bound excitations of larger frequente mentioned in the last subsection are missed.

Consider the excitations of a *MQW* whose velocity is cV. Our trial "spinor" for a state $|\vec{k}\rangle$ shall be

$$\psi_{\vec{k}} = \sqrt{\gamma/2}\;\frac{1}{(\cosh\gamma\xi)}\Big\{\cos(\phi/2)\begin{bmatrix}\cosh(y/2)\\ \sinh(y/2)\end{bmatrix} + i\sqrt{3}\,\sin(\phi/2)\;\mathrm{tgh}\gamma\xi\begin{bmatrix}\cosh(y/2)\\ -\sinh(y/2)\end{bmatrix}\Big\} \tag{97}$$

where ϕ and y are real numbers which will be chosen to minimize $|\omega|$. This trial "spinor" was suggested by the small set of exact solutions already known. The reader can verify that $\psi_{\vec{k}}$ defined in (97) obeys the quantization condition (66). Then, using Eq. (65) we obtain

$$|\omega| = mc^2 \int d\xi \; \psi^{\dagger}_{\vec{k}} \; \hat{\Sigma}(V,k) \; \psi_{\vec{k}} \tag{98a}$$

or

$$|\omega| = mc^2 \{A(k,\gamma,\phi)\cosh y + B(\gamma,\phi)\sinh y\} \tag{98b}$$

Here $A(k,\gamma,\phi)$ and $B(\gamma,\phi)$ are given by:

$$A(k,\gamma,\phi) = 1 + \frac{k^2}{2m^2c^2} - \frac{19}{30}\gamma^2 - \frac{8}{15}\gamma^2 \cos\phi - \frac{\gamma V}{\sqrt{3}}\sin\phi \tag{98c}$$

and

$$B(\gamma,\phi) = \frac{2}{15}\gamma^2 + (1 - \frac{22}{15}\gamma^2)\cos\phi - \frac{2\gamma V}{\sqrt{3}}\sin\phi \tag{98d}$$

From Eqs. (76) and (86) we can compute the mean longitudinal momentum and the mean energy of $\psi_{\vec{k}}$:

$$\ell(V,k) = (mc\gamma/\sqrt{3})\sin\phi \cosh y \tag{99}$$

and

$$E(V,k) = cV\,\ell(V,k) + |\omega| \tag{100}$$

The mean value of the total momentum shall be

$$P(V,k) = \{k^2 + \frac{1}{3}m^2c^2\gamma^2 \sin^2\phi \cosh^2 y\}^{1/2} \tag{101}$$

Doing a computer calculation, we are able to determine what is the pair (ϕ,y) that minimizes the frequency modulus $|\omega|$. Once this is done, we insert ϕ and y in Eqs. (98), (100) and (101) to obtain respectively the frequency itself, and the *mean* values of the energy and total momentum-all these quantities refering to a quasi-particle of transverse momentum k, which is bound in a MQW of velocity cV.

In what follows we present our results.

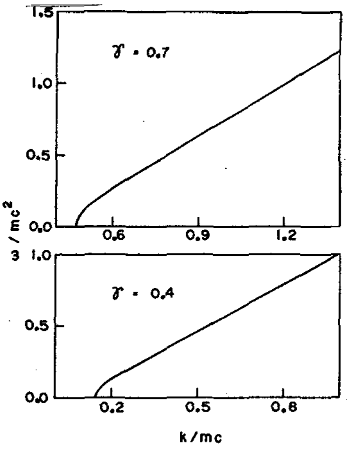

Fig.9 - Dependence of the bound excitations frequence on their transverse momenta.

In figure 9 we plot ω as a function of k for $\gamma=0.7$ and $\gamma=0.4$ $(\gamma=\sqrt{1-V^2})$. Both graphs suggest that, as said above, the point $\omega = 0$ is singular.

We see that for each *MQW* there is a lower bound on the transverse momentum *k*. The minimum k is that of Eq. (90) and corresponds to quasi-particles with infinite energy.

This fact makes the macroscopic waves not amenable to fluctuations of large extent, i.e., the planes of these waves have a certain rigidity.

Figure 10 shows the energy-momentum curves referring to five different values of γ. When $\gamma=0$ $(V=1)$ the curve is, as expected, the Bogoliubov's phonons dispersion relation (see fig. 1 and Eq. (25)). For $\gamma=1$ $(V=0)$ we have a function $E(p)$ that starts, when $\omega=0$, in the point $(E=0\ ,\ p=mc)$ and goes to a high ω behavior very similarly to that of Eq. (95a).

The three other energy-momentum relations of figure 10 are all elbow shaped graphs. They have a parabolic upper part (correspondent to large values of $|\omega|$) and a straight line lower part (referring to the small $|\omega|$ values). All energy-momentum curves, for which $\gamma\neq0$, 1, have a similar shape.

Definition: The *elbow-point* of an energy-momentum curve is that point which, in the plane $(E/mc^2, p/mc)$ is at smallest distance from the origin.

In figure 11 we show in detail the elbow-like part of the energy-momentum curve corresponding to the excitations bound within $W_{\sqrt{3}/2}$ (γ =0.5). There, we also indicate the values of k/mc associated to certain points.

Since $\vec{k}$ is a bidimensional vector the density of states which have transverse momentum square $\vec{k}^2$ is proportional to k. Let $N(k_i, k_j)$ be the number of states in the interval (k_i, k_j). This number is proportional to $(k_j^2 - k_i^2)$.

So, the numbers of states belonging to the segments (0.38, 0.45), (0.45, 0.52) and (0.52, 0.59) (that are shown in figure 11) does not differ too much

$$\frac{N(0.38mc,\ 0.45mc)}{N(0.45mc,\ 0.52mc)} = 0.856 \tag{102a}$$

and

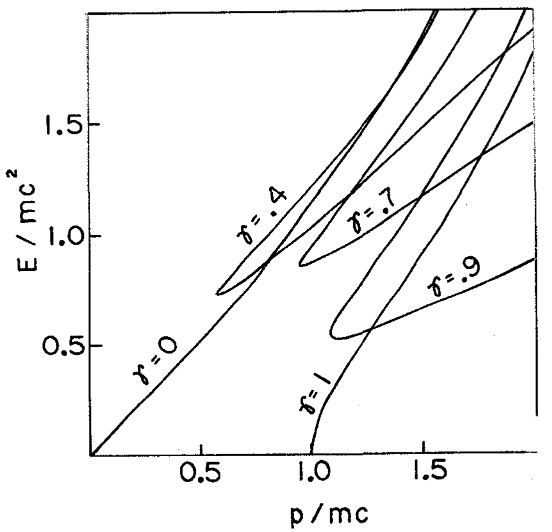

Fig.10 - Energy - momentum curves of quasi-particles bound in five different macroscopic quantum waves.

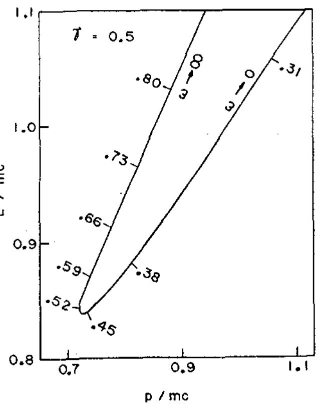

Fig.11 - The elbow region of the energy - momentum curve corresponding to $\gamma=0.5$.

$$\frac{N(0.52mc,\ 0.59mc)}{N(0.45mc,\ 0.52mc)} = 1.144 \tag{102b}$$

However the segment (0.45, 0.52) - that contains the elbow point - is much more short than the two others. Then we reach the important conclusion that *there is a high concentration of states in the neighborhood of an elbow-point.*

Figure 12 is the curve of elbow-points. The values of γ associated with some of them are indicated. In a system where exists any kind

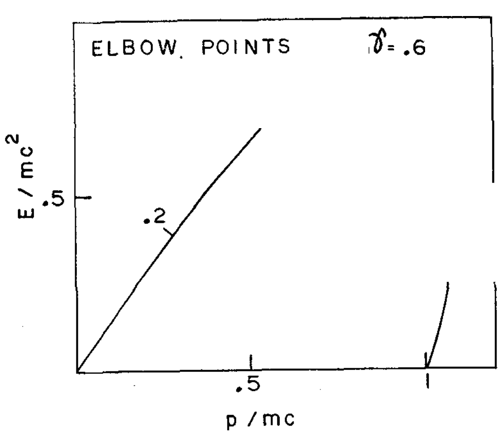

Fig.12 - Position of elbow-points.

of *MQWs*, contributing all them with the same proportion, we shall have high concentrations of states near this curve.

We repeat that figures (10) and (11) refer to the *mean* values of energy and momentum. The quasi-particles of a given topological wave are, in fact, spread around its energy momentum curve - the extent of such a spread depending upon the wave width.

e) The Spectrum

Now we present the whole excitation spectrum generated by our variational computation, and a *qualitative* confrontation of our results with the data of references (8), (28) and (29).

The development of the *MQWs* statistical mechanics — which could provide, for instance, the "numbers" of topological waves of each type ~ existent on a fluid reservoir at a given temperature — would be an essential step to try a precise quantitative comparision of the present theory with experiments. Therefore, the comparision done here must remain on a qualitative level.

We shall *suppose* that the MQWs actually exist in liquid Helium. Taking this fact for granted, what should the quasi-particle spectrum of this system look like?

Consider three different temperatures $T_1 > T_2 > T_3$, which will not be specified due to our lack of knowledge about the *MQWs* statistical mechanics.

We know that the slow *MWQs* ($\gamma \simeq 1$) carry much more energy per area than the rapid ones (see fig. 4(c)). So, the statistical weight of the former should be important at relatively high temperatures and must decrease as the temperature diminishes.

If the fluid is at a relatively high temperature (T_1), it must, in principle, contain any kind of topological wave, and, as a result of our variational calculation, its spectrum will cover the shaded area of fig. 13(a). To draw this figure, we took into account the quasi-particles of all *MQWs* . But not its energy and momentum uncertainties.

Consider now that at a lower temperature T_2 , a certain part of the slow topological waves is statistically suppresed (because they have too much energy per unit of area). In this case the fluid spectrum will look like that of figure 13(b). There we show only those excitations belonging to *MQWs* for which $\gamma \lesssim 0.9$.

As discussed above, this spectrum has a great concentration of states near the elbow-points curve (with γ varying, here, in the range $0 \lesssim \gamma \lesssim 0.9$) . Then, supposing that we excite the quasi-particles of our

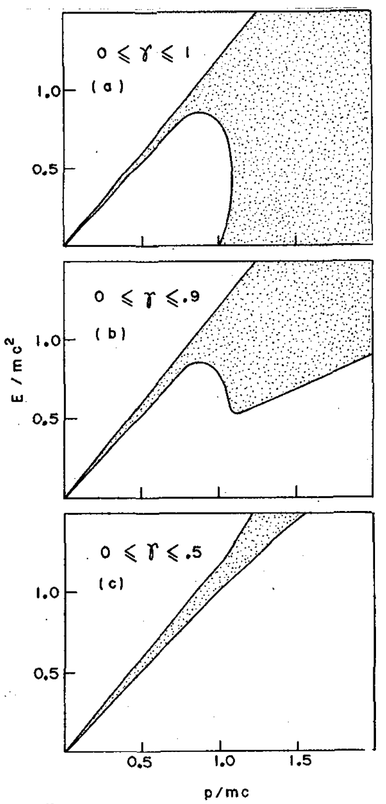

Fig.13 - Spectrum of bound quasi-particles, (a) for a relatively high temperature (T_1) where all *MQWs* contribute; (b) at an intermediate temperature (T_2) where the slower topological waves are statistically supressed; and (c) the situation at a low temperature (T_3) on which only the very rapid waves survive.

fluid (by using a neutron beam, for instance) at $T=T_2$, we can expect that, for a given momentum, the cross section of single quasi-particles production will have a peak close to the albow-points curve. This peak would be due *not to* the production amplitude, but to the final states density in phase space.

Finally, let T_3 be a very small temperature, on which practically only the topological waves having $\gamma \lesssim 0.5$ ($V \gtrsim 0.866$) can survive. In this temperature the fluid spectrum will be similar to the shaded area of figure 13(c).

These conclusions were reached without knowledge of the *MQWs* statistical mechanics, and by means of a variational approach. In our opinion, however, a more precise treatment (both quantum and statistical) should not change this qualitative picture drastically.

We emphazise again that in the variational approximation we missed a class of large frequency bound excitation, so that, the spectrum of this fluid theory is still more rich in the large energy region.

We are now in a position to compare our conclusion with the liquid ^4He spectrum obtained in neutron scattering experiments.

Figure 14(a) - which gives the ^4He spectrum at 1.1K and normal pressure - was taken almost literally from figure 6 of reference (28). The difference is that we have not cut the spectrum in the second peak 1/2 - height as done there. The boundaries of the experimental spectrum areaare actually somewhat indefinite. Nevertheless, figures 3 and 5 of reference (28) (which show some typical energy distributions at fixed values of the momentum) provide information about their position.

The similarity between figures 14(a) and 13(b) is very promising. Based upon our theoretic results we could say that, if the *MQWs* really exist in liquid Helium, at 1.1K and normal pressure a certain part of them (those having low velocities: $0.9 \lesssim \gamma < 1$) would be thermodynamically suppressed.

Figure 14a also shows the positions of the first ("phonon-roton") and second peaks of the neutron scattering cross section.

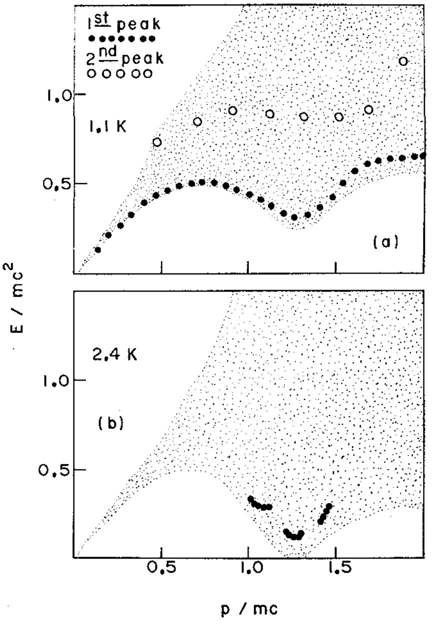

Fig.14 - (a) Spectrum of liquid ^4He at 1.1 K, from refs. (16) and (28); (b) ttie situation at 2.4 K. Here, only the region around the first peak was teken from ref. (29). The rest of the graph was inspired from the 1.1 K case (see the text).

Note that the elbow-points curve of our theory could explain *at least a part* of the first peak. This peak (which is relatively narrow) should be due to the phase space factor of the production cross section.

To see whether or not the $\lambda|\phi|^4$ theory could describe the whole "phonon-roton" peak it would require a solution of the statistical mechanics problem.

It is possible thatthe *high energy part* of the second peak (that is much more broad than the former) be an scattering amplitudepeak (Ref.56) (see preliminary discussion in the next subsection).

The current *interpretation*[8] is that, in figure 14a, everything out of the first peak should be due to multiple production process. Of course we do not partake of this view point. It is certain that multiple scattering process, specially the double-scattering ones, give their contribution to the measured cross section. However, our theoretic results indicate that (if the *MQWs* exist) the single-particle production is an important process even far from the first peak.

Figure 14b was not entirely taken from experiments. There we show what *should be* the spectrum at 2.4K and normal pressure. Under these thermodynamics conditions the available experimental data is that of reference (29), *which refers only to the neighborhood of the "roton" peak minimum*. The minimum position and the gap are respectively 1.915/Å and 0.302 meV, whereas the first peak width near the minimum is 0.38 meV.

All dots of figure 14b correspond to the first peak line (see figure 7 of reference (29)). In this case the spectrum nearly touches the momentum axis because the gap is very small and the peak width is quite large.

The remainder parts of that figure (those which are far from the "roton" minimum") *does not come from experiment*. They were drawn in analogy with the 1.1K case.

What matters however is that (and this is an experimental fact) at 2.4K the spectrum touches, or at least almost touches, the momentum axis. Therefore the situation at this temperature is very similar to the theoretic picture shown in figure 13a where, at a relatively high temperature, any kind of MQW contributes to the spectrum.

Measurements of neutron scattering in liquid Helium were never made below 1.0K. Thus, there is not experimental data to compare with our figure 13c which refers to a relatively low temperature T_3.

In this respect, our theoretic prediction is that below 1.1K the Helium II spectra will continuously change from the picture of figure 14a to be, at a certain lower temperature T_3, similar to figure 13c.

When going to even lower temperatures, approaching the absolute zero, the rapid *MQWs* will also lose their thermodynamical weight, and the spectrum would tend to be solely the set of Bogoliubov's phonons (see figure 1).

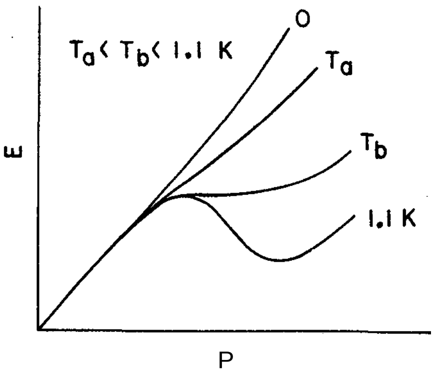

Fig. 15 - Qualitative prediction concerning to the behavior of the "phonon-roton peak" (1st peak) at temperatures below 1.0 K.

As a consequence our theory also predicts that, if $T_a < T_b < 1.1$K, the first peak would change as is shown in figure 15: when the temperature diminishes the "roton minimum" gap should grow, and, at a certain temperature T_b, the minimum disappears. Below T_b the first peak would tend toward Bogoliubov's dispersion relation.

We point out that, concerning to Landau's parametrization, the existence of "rotons" is not relevant in low temperatures. In this region the major contribution to specific heat comes from "phonons"[57].

The present experimental situation[29] shows that the "roton minimum" gap grows quite slowly when temperature goes from 1.5 to 1.1 K. This fact may lead some readers to be sceptical about our theoretic predictions. We recall, however, that a lot of the Helium II properties change suddenly below 1.0K. For example, the second sound velocity, that is almost constant in the interval (1.0K, 1.8K) starts growing quickly below 0.7K to become very high and not well defined[18].

f) Proposals for Further Investigations

Here we want to list a few problems which are still under investigation[56]. We shall also present some preliminary results referring to them. Our purpose is mainly to give an idea about how colorful the theory of superfluidity would become in case of existence of macroscopic quantum waves.

Elastic neutron scattering: - Besides exciting the *MQWs* neutrons can also be elastically scattered by them. To treat the elastic scattering we must consider the interaction Hamiltonian density

$$\beta |W_V|^2 \; N^\dagger N \tag{103}$$

where N is the neutron spinor, β is an effective ^{4}He - neutron coupling (β is a very small number, $\beta \lesssim 10^{-4}\lambda$ [58]) and $|W_V|^2$ is the matter distribution of the *MQW* :

$$|W_V|^2 = \rho \{ 1 - \frac{\gamma^2}{(\cosh \gamma\xi)^2} \} \tag{104}$$

The potential $A/(\cosh x)^2$, known as Pöschl-Teller potential is solved for example in reference (59). From the results of this reference we are able to deduce the probability of neutron elastic reflection by a *MQW* of velocity cV. Being p the momentum exchange, the reflection probability gives:

$$\frac{(\pi\beta/2\lambda)^2}{(\pi\beta/2\lambda)^2 + \sinh^2(\pi p/2\gamma mc)} \tag{105}$$

In the reference frame where the *MQW* is at rest, $p/2$ is the neutron longitudinal momentum.

Since $\pi\beta/2\lambda$ is a very small number, this probability ,corresponds to a very narrow peak around p=0. The width of such a peak is

$$(\gamma\beta/\lambda)mc \approx 10^{-4}mc$$

Then, in the (E,p) plane this *single* elastic scattering is supposed to contribute only for very small values of the momentum, being not relevant in the region where the "phonon-roton" peak starts to be observed ($p \gtrsim 0.1mc$ (Ref.28)). However, if the neutron crosses a very large number (10^4) of *MQWs*, the effect of *multiple elastic scattering* could in principle be important. The question would be to know its significance relative to the background.

Inelastic neutron scattering: using the quasi-particles wave function of our variational method, and considering the neutron currentas a classical object - i.e., taking $N^{\dagger}N$ to be $\delta(\vec{x}-\vec{v}t)$ - we studied the production of bciund quasi-particles in the limit of large $|\omega|$ values.

Our preliminary conclusion is that the *high energy part* of the second peak could be understood as a peak in the production amplitude. Note, in this respect, that the second peak width[8,28] (> *me*) is nearly the momentum uncertainty of an object bound on a typical *MQW*.

Concerning the bound excitations which, for a given value of k^2, have more energy than those considered in this paper, we should say that their possible role is still unclear. This difficulty comes from the fact that any variational computation provides a good description of the ground state, but not of the first excited state.

^3He - ^4He mixtures: - Within the spirit of the approximations considered in this paper, Helium-3 should interact with Helium-4 with the same effective coupling constant of the ^{4}He self interaction. If ϕ_3 is the ^{3}He field, the approximated interaction Hamiltonian involving both these particles would be

$$\lambda \, \phi^{\dagger} \, \phi \, \phi_3^{\dagger} \, \phi_3 \qquad (106a)$$

whereas

$$\lambda |W_V|^2 \phi_3^{\dagger}\phi_3 = \lambda\rho \left\{ 1 - \frac{\gamma^2}{(\cosh \gamma\xi)^2} \right\} \phi_3^{\dagger}\phi_3 \qquad (106b)$$

should describe the interaction of ^{3}He atoms with a topological wave.

Employing again the results of reference (59) it is very easy to see that there exist bound states of ^{3}He atoms within the *MQWs*.

Consider an ^{3}He - ^{4}He mixture. In the case where the relative density of ^{3}He is *small*, by neglecting the ^{3}He self interaction and by considering Pauli exclusion principle, we conclude that the fundamental state of the mixed system has a non vanishing "number" of *MQWs*. I.e., the presence of ^{3}He atoms would favor the formation of *MQWs*.

A brief note about Landau's parametrization: - Without a reasonable knowledge of the *MQWs* statistical mechanics, to attempt to deduce Landau's phenomenological parametrization[2,3] would be a velleity. We want however to write a few words in this respect. In our theory the fluid specific heat should have three types of contributions:

$$C = C_{ph} + \underbrace{C_b + C_{MQW}} \tag{107}$$

where C_{ph}, C_b and C_{MQW} are due respectively to phonons, to bound excitations and to the *MQWs*.

Actually C_b and C_{MQW} are not independent things[40]. Let us, however, assume Eq. (107) to hold.

When the temperature is small the *MQWs* and, as a consequence, their bound excitations are very few in number. Therefore, in this low temperature region, C_{ph} would be the leading term in Eq. (107) and the specific heat would also follow the T^3 law, in agreement with Landau's parametrization[57].

At the critical point, the analogy with magnetic system suggests that C_{MQW} is the divergent part of Eq. (107) (see chapter 3). For temperatures which are not too close the critical point, C_{MQW} is, as already discussed, probably not relevant.

Before considering what could occur in the region of intermediate temperatures we point out that the high concentration of states in the elbow - points curve plays probably an important role in the fluid statistical mechanics, as it did in the discussion about neutron scattering of the preceding subsection.

Inspecting figure 13b, which represents the quasi - particles spectrum at an intermediate temperature, we see that the major contributions to C_b should come from the area around the point ($E=0.55mc^2$; p =1.2 mc) - excitations on the cone near the origin are bound on very wide *MQWs* and have not weight in phase space due to their small transverse momenta. Therefore, if $\Delta = 0.55\ mc^2$, we could expect that a very *crude approximation* to C_b would give $C_b \simeq D \exp(-\Delta/k_BT)$, where the constant D should come from an integration on the area around the point (0.55 mc^2 ; 1.2 mc). D should depend upon the temperature and upon the (yet unknown) relative "numbers" of *MQWs* of each type.

Thus, *it is possible* that, even in the region of intermediate temperatures, the present theory can lead to a parametrization for the specific heat similar to that of Landau.

We recall that measurements of the roton curve second derivative $(1/2\mu)$ gives, depending on the nature of the experiment realized, very different results[29,60]. For example, at 1.2 ^{O}K and normal pressure, the value μ=0.160m comes from neutron scattering[29], whereas density measurements gives[60] μ = 0.5m. This discrepancy indicates that Landau's parabola may really be an effective ("mean") parametrization of a more rich spectrum spread over an area.

5. SKETCH OF AN APPARATUS TO OBSERVE THE MACROSCOPIC QUANTUM WAVES

Here we will present the sketch (a simple sketch, not a detailed design) of an apparatus to look for macroscopic quantum waves in Helium II. It is shown in figure 16(a), and consists of essentially three parts: (1) a tube of nearly 1.0 cm in diameter that contains the fluid; (2) a source of x-ray with mean wave length of a few angströms. The x-ray beam must be very narrow ($\sim 10^{-2}$ cm) and almost without spread; and (3) a sheet of emulsion sensitive to this type of x-ray which can move between two rollers at an speed of about 5,0 m/sec[61].

The x-ray beam enters into the fluid forming a small angle (~10 degrees) with the tube.

At least two types of experiments can be done with this apparatus.

1) *Search for a single macroscopic quantum wave*: this is a very complex experiment. The slow *MQWs* are characterized by a strong change in the condensate density, whereas the rapid ones are not. Hence, our x-ray beam should be mostly reflected by the former. At low temperatures[62] (~0.1 - 0.5K) these slow topological waves are supposed to be very far one froni another. Consider a particular wave which at times t_1 and t_2 is in the positions shown in figure 16(a). It reflects a part of the incident beam to the emulsion detector. Since the emulsion is moving, the

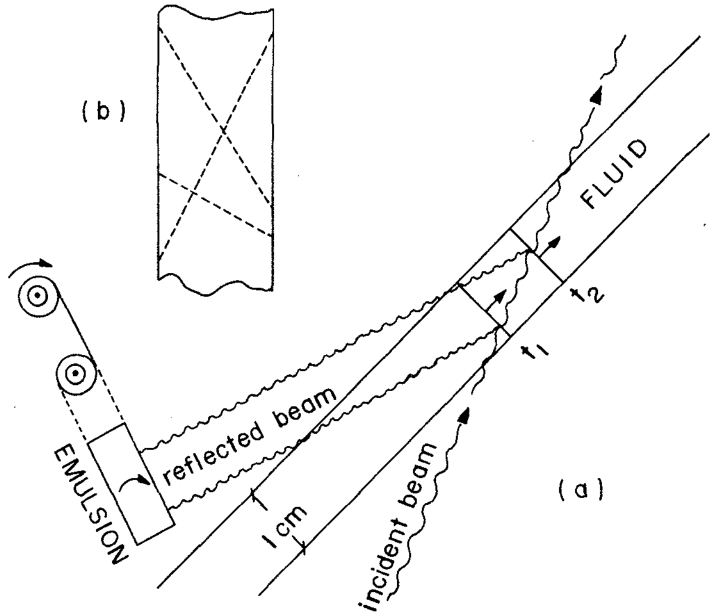

Fig.16 - (a) Experimental apparatus to observe *MQWs* (see the text); (b) the marks on a segment of emulsion that should correspond to three different *MQWs*.

reflected beam will mark it along a sloped straight line, as show in figure 16(b) - there we drew the signals of three different waves. If the background is not too perverse, we can hope to see these signals. Due to the severe restrictions upon the incident beam (narrowness and collimation) its power is certainly so small that, to look for the emulsion marks, it should be necessary to resort to microscopy techniques.

2) Scattering by second sound; our theory suggests that the *MQWs* play an important role in the motion of second sound. Then, exciting on the tube a second sound pulse (that would correspond to a group of *MQWs*), the reflected beam intensity will increase when the pulse crosses the x-ray range. This phenomenon should produce an sloped band in the emulsion film, more easily observable than the straight lines of the single topological waves.

It is much more easy to carry out this second experiment than the former one, because here the temperature may be higher (-1.0 - 2.0K, not needing *magnetic cooling*), and the x-ray beam may be not so narrow.

6. CONCLUSIONS

We showed that the old Bogoliubov's superfluidity theory exhibits some macroscopic quantum waves which were so far unknown.

The spectrum of quasi-particles bound in these waves is very similar to the liquid Helium spectrum observed in neutron scattering.

Besides that, by partitioning the condensate in blocks of phase, the states of multi - *MQWs* provide a *possible* explanation to the λ-transition.

If experiments come to confirm the existence of topological waves in liquid Helium, the matter presented here should be the starting point of a microscopic description of superfluidity.

The *MQWs* are, in essence, one-dimensional objects (though their excitetions are not). We hope that the modern techniques to study one-dimensional solitons in non relativistic models[63] should be very useful to develop a more precise treatment of them.

Finally we point out how simple the basic ideas of our scheme are. According to the strategy of simplicity we have a certain chance to be right.

ACKNOWLEDGEMENTS

We are indebted to the members of the Institute of Physics at U.S.P. (São Paulo) and to those of the Department of Physics at P.U.C. (Rio) for their participation in two seminars about this matter we gave respectively in São Paulo and Rio. We thank, in particular, H. M. Nussenzveig and J. A. Swieca for very instructive conversations, and C. C. Becerra, G.M. Carneiro, O. Dietzsch, H. Fleming, E.J.V. de Passos and J. Tiomno for discussions. We are specially grateful to Maria José Bechara, who has followed this work from its beginnings - her contributions to this paper are inestimable.

APPENDIX A – The Bohr-Sommerfeld Quantization Rule

The extension of the Bohr-Sommerfeld quantization rule to a non relativistic field theoretic model reads[30-32]: if ϕ is a periodic solution of the classical equations of motion, and if Π is its cannonical momentum, it follows that

$$\int_0^\tau dt \int d\vec{x}\, \Pi\, \partial_t \phi = 2\pi n \quad , \tag{A1}$$

where τ is the period of ϕ and n is a positive integer.

First of all we shall use this rule to quantize the Bogoliubov's phonons. To this purpose let us look for the Lagrangian which is consistent with the equation of motion (15) and with the Hamiltonian $H\{\eta\}$ (see Eq.(17b)). This Lagrangian must be

$$L\{\eta\} = \int d\vec{x}\, i\, \eta^* \partial_t \eta - H\{\eta\} \quad . \tag{A2}$$

Therefore the momentum of η is $i\eta^*$, so that the Bohr-Sommerfeld quantization of the excitation field is given by the expression:

$$\int_0^\tau dt \int d\vec{x}\, i\eta^* \partial_t \eta = 2\pi n \quad . \tag{A3}$$

In (A3), η is a periodic solution of Eq.(15).

Being $\omega=2\pi/\tau > 0$, the substitution of the eigensolution $\eta_{\vec{k}}$ (see Eq.(21)) in (A3) gives

$$A^2 = n/2\omega \quad . \tag{A4}$$

For a given n we have the state of n identical excitations. To describe a single quasi-particle we must take $n=1$, or

$$A^2 = 1/2\omega \quad . \tag{A5}$$

The role of Eq.(A3) is similar to that of the normalization condiction of ordinary quantum mechanics, in the sense that it fixes the modulus of the amplitude A. Note that the condensate wave function Ω was not quantized because this system contains an infinite number of particles. This is equivalent to the Bogoliubov's procedure of making $a_0 \approx \sqrt{N_0} \approx a$ [2].

In the presence of a *MQW* the quasi-particles Hamiltonian $H_V\{\eta\}$ (see Eq. (81)) is not a good quantity to characterize their states. Nor the longitudinal momentum is. However we can formulate the quasi-particles dynamics by using the "transformed Hamiltonian"

$$R_V\{\eta\} = H_V\{\eta\} - cV\ell\{\eta\} \quad . \tag{A6a}$$

The Lagrangian compatible with $R\{\eta\}$ and with the equation of motion (60) is

$$L_V\{\eta\} = \frac{1}{mc}\int d\vec{s}d\xi \; i\eta^*\partial_t\eta - R_V\{\eta\} \quad . \tag{A6b}$$

Taking the fundamental coordinates to be s and ξ (ξ has not dimension of length!) , we conclude that the momentum of η is $i\eta^*/mc$. Thus, to the fluctuation η, the Bohr-Sommerfeld rule implies that:

$$\int_0^\tau dt \frac{1}{mc}\int d\vec{s}d\xi \; i\eta^*\partial_t\eta = 2\pi n \quad . \tag{A7}$$

To quantize the excitations of Eq. (61) it is necessary to treat separately the cases $\omega>0$ and $\omega<0$ (see, however, the comment after Eqs. (69)).

Consider first that $\omega=2\pi/\tau>0$. Plugging $\eta_{\vec{K}}$, given by Eq. (61), in (A7) and integrating over transverse coordinates and over time, we obtain:

$$\int d\xi \, \{u^*(\xi)u(\xi) - v^*(\xi)v(\xi)\} = n \quad , \tag{A8a}$$

or, using Eq. (63).

$$\int d\xi \; \psi^t \sigma_3\psi = n \quad . \tag{A8b}$$

When $\omega=-2\pi/\tau<0$, it can easily be verified that

$$\int d\xi\psi^t\sigma_3\psi = -n \quad . \tag{A9}$$

Then, for states of a single quasi-particles we have

$$\int d\xi\psi^{\dagger}\sigma_3\psi = 1, \quad \text{when} \quad \omega>0 \ , \tag{A10a}$$

or

$$\int d\xi\psi^{\dagger}\sigma_3\psi = -1, \quad \text{when} \quad \omega<0 \tag{A10b}$$

As discussed in subsection 4-B, any spinor of negative ω corresponds in fact to another one of positive ω - both them describing the same quasi-particle.

The topological wave is a moving object, so that the fluid system is not strictly periodic. To the reader who wonders that this fact could forbid the use of the Bohr-Sommerfeld rule we say that: before formulating the model, we can go to the reference frame where the *MQW* is at rest. There, the periodicity is manifest and we may use the rule, coming back to the fluid system later on. The result of such a round trip operation is the same as that presented above.

APPENDIX B — Zero-Frequency Solutions

Here we list four zero-frequency solutions of Eq.(65), whose exact expressions we know. One of them corresponds to quasi-particles *bound within* the *MQWs*. Two others are related to symmetries of the $\lambda|\phi|^4$ theory - the translation and gauge modes. And the last solution has no physical relevance.

Let us first define the numbers

$$R = \sqrt{1-\gamma^2+\gamma^4} \ , \tag{A11a}$$

and

$$S = (1 - 2\gamma^2) \ . \tag{A11b}$$

Any one of those solutions is associated to a unique value of the transverse momentum square (k^2).

They can be written as

$$\psi_m = A_m \begin{pmatrix} r_m(\xi) \\ r^*_m(\xi) \end{pmatrix} \quad . \qquad \text{(A 12)}$$

Where A_m is a *real* constant.

To the translation mode, which had been considered in subsection 4-8, we have

$$r_0(\xi) = \frac{i}{(\cosh\gamma\xi)^2} , \qquad \text{(A13a)}$$

and

$$k_0^2 = 0 \qquad \text{(A13b)}$$

The gauge mode, that is a consequence of the gauge invariance of the model, is

$$r_1(\xi) = A_1(\gamma \text{ tgh } \gamma\xi + iV) , \qquad \text{(A14a)}$$

and

$$k_1^2 = 0 . \qquad \text{(A14b)}$$

The solution which has physical importance is given by

$$r_2(\xi) = A_2 \frac{1}{(\cosh\gamma\xi)} \{ \sqrt{R-S} + i\sqrt{3}\sqrt{R+S} \text{ tgh } \gamma\xi \} . \qquad \text{(A15a)}$$

Its transverse momentum square is

$$k_2^2 = m^2c^2 \{ 2R+\gamma^2-2 \} . \qquad \text{(A15b)}$$

Finally we present a solution without physical meaning

$$r_3(\xi) = A_3 \frac{1}{(\cosh\gamma\xi)} \{ \sqrt{R+S} - i\sqrt{3}\sqrt{R-S} \text{ tgh } \gamma\xi \} \qquad \text{(A16a)}$$

For r_3 the value of the wave number square is negative

$$k_3^2 = -m^2c^2 \{ 2R+\gamma^2-2 \} . \qquad \text{(A16b)}$$

That is why it lacks any physical significance.

APPENDIX C – The Large Frequency Behavior of Quasi-Particles Bound in the Static MQW

For the static *MQWs* the operator $\hat{\Sigma}$ is

$$\hat{\Sigma}(0,k) = (-\frac{1}{2}\partial^2_\xi - \frac{2}{(\cosh\xi)^2} + 1 + \frac{k^2}{2m^2c^2})I + (\frac{1}{(\cosh\xi)^2} - 1)\sigma_1 . \tag{A17}$$

Thus, when $V=0$, Eq. (65) can be put in the form

$$\{\hat{O}_0(\omega) + \hat{O}_I\}\psi = -\frac{k^2}{2m^2c^2}\psi , \tag{A18}$$

where

$$\hat{O}_0(\omega) = (-\frac{1}{2}\partial^2_\xi - \frac{2}{(\cosh\xi)^2} + 1) - \frac{\omega}{mc^2}\sigma_3 , \tag{A19a}$$

and

$$\hat{O}_I = \{\frac{1}{(\cosh\xi)^2} - 1\}\sigma_1 . \tag{A19b}$$

We are now considering that ω is a fixed parameter and that $-k^2/2m$ is the eigenvalue.

To solve the eigenvalue equation (A18) in the large ω limit, we can treat $\hat{O}_I$ as a perturbation to $\hat{O}_0(\omega)$.

It is very easy 'to obtain the bound-state solutions of the zero order equation

$$\hat{O}_0(\omega)\psi^0 = -\frac{k^2}{2m^2c^2}\psi^0 . \tag{A20}$$

We will present here these solutions, showing that the corrections to $k^2/2m^2c^2$, which comes the perturbetion $\hat{O}_I$, can be neglected when ω is large.

The operator

$$- \frac{1}{2} \partial_\xi^2 - \frac{2}{(\cosh\xi)^2} + 1 \, , \tag{A21}$$

has two discrete eigenvalues[59]:

$$\beta_0 = - 0.2192 \quad , \tag{A22a}$$

which corresponds to the even fundamental state; and

$$\beta_1 = 0.8423 \quad , \tag{A22b}$$

that is associated with an odd excited state.

Beside that it has a continuous set of eigenvalues in the interval

$$1 \leq \beta \leq \infty \; . \tag{A23}$$

Since a_3 is a diagonal matrix, the eigensolutions of Eq. (A20) must be of one among two types: "up" or "down", given respectively by

$$\psi_\beta^\uparrow = \begin{pmatrix} s_\beta(\xi) \\ 0 \end{pmatrix} \quad , \tag{A24a}$$

and

$$\psi_\beta^\downarrow = \begin{pmatrix} 0 \\ r_\beta(\xi) \end{pmatrix} \quad . \tag{A24b}$$

where $s_\beta(\xi)$ and $r_\beta(\xi)$ are such that

$$(- \frac{1}{2} \partial_\xi^2 - \frac{2}{(\cosh\xi)^2} + 1 - \frac{\omega}{mc^2}) \; s_\beta = - \frac{k^2}{2m^2c^2} \, s_\beta \; , \tag{A25a}$$

$$(- \frac{1}{2} \partial_\xi^2 - \frac{2}{(\cosh\xi)^2} + 1 + \frac{\omega}{mc^2}) \; r_\beta(\xi) = - \frac{k^2}{2m^2c^2} \, r_\beta(\xi) \; . \tag{A25b}$$

Considering the spectrum of operator (A21) and the above equation, we conclude that, when ω is positive (negative), $\psi^\uparrow(\psi^\downarrow)$ will be related to a physical excitation $\eta_{\vec{k}}$, and $\psi^\downarrow(\psi^\uparrow)$ to an unphysical one, because k^2 must always be positive.

Without loss of generality, let us take ω to be positive.

In order zero in $\hat{O}_I$ the transverse momentum squares of the even and odd bound-state quasi-particles are, for a given value of w, respectively

$$\frac{k_0^2}{2m^2c^2} = \frac{\omega}{mc^2} + 0.2192 \ , \qquad \text{(A26a)}$$

and

$$\frac{k_1^2}{2m^2c^2} = \frac{\omega}{mc^2} - 0.8423 \ . \qquad \text{(A26b)}$$

In first order of perturbation theory the corretions to $k_0^2/2m^2c^2$ and $k_1^2/2m^2c^2$ vanish, because

$$<\psi_\beta^\uparrow|\hat{O}_I|\psi_\beta^\uparrow> = 0 = <\psi_\beta^\downarrow|\hat{O}_I|\psi_\beta^\downarrow> \ , \qquad \text{(A27)}$$

since the operator $\hat{O}_I$ couples a states $\psi^\uparrow$ only to states $\psi^\downarrow$.

Being $\psi_{\beta_0}^\uparrow$ the eigensolution of (A20) whose eignevalue is (A26a), the second order corretions to this eigenvalue will be

$$- \int d\beta \frac{|<\psi_{\beta_0}^\uparrow|\hat{O}_I|\psi_\beta^\downarrow>|^2}{\{(2\omega/mc^2) + \beta - \beta_0\}} = O(1/\omega) \ . \qquad \text{(A28)}$$

This integration runs over the continuum interval (A23).

The odd discrete state $\psi_{\beta_1}^\downarrow$ does not contribute in (A28) because $\psi_{\beta_0}^\uparrow$ has even parity.

The reader should convince himself that, since there is no direct coupling between two "up" states, the higher order corretions to (A26a) are also always of order $O(1/\omega)$:

$$\frac{k_0^2}{2m^2c^2} = \frac{\omega}{mc^2} + 0.2192 + O(1/\omega) \ . \qquad \text{(A29a)}$$

By means of an analogous procedure, we deduce that the even discret eigenvalue is

$$\frac{k_1^2}{2m^2c^2} = \frac{\omega}{mc^2} - 0.8423 + O(1/\omega) \ . \qquad \text{(A29b)}$$

The eigensolutions of (A20), correspondent to the eigenvalues (A29a) and (A29b), are respectively[59]

$$\psi^{\dagger}_{k_0} = \{A_0/(\cosh\xi)^{\alpha}\} \binom{1}{0} \qquad \text{(A30a)}$$

and

$$\psi^{\dagger}_{k_1} = \{A_1 \sinh\xi/(\cosh\xi)^{\alpha}\} \binom{1}{0} \qquad \text{(A30b)}$$

where a = $(\sqrt{17} - 1)/2$ and A_0 and A_1 must be determined by the quantization conditions (66).

FOOTNOTES AND REFERENCES

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37. See, for instance, ref. (35), pg. 497.

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56. I.Ventura; work under development.

57. See, for instance, ref. (18) pg. 63.

58. We can estimate β by comparing the scattering lengths of the process ^4He ^4He $\rightarrow$ ^4He ^4He and neutron ^4He $\rightarrow$ neutron ^4He. Whereas in the former case the scattering length is about one Angstrom, in the latter it isnear

3.0 × 10^{-15}m - see, for example, reference (11) and also A.W.McReynolds, Phys. Rev. 84, 969 (1951).

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