The Green's Function of a Relativistic Free-Electron in a Uniform Magnetic Field

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A closed representation is obtained for the Green's function of the Dirac equation of a free-electron in a uniform magnetic field.

Deduz-se uma representação fechada para a função de Green para a equação de Dirac para um eletron livre na presença de um campo magnético uniforme.

1. DERIVATION OF THE GREEN'S FUNCTION

The present note contains a simple derivation of the Green's function for a Dirac particle in a uniform magnetic field.

The Green's function satisfies the following inhomogeneous Dirac equation

\[(i\gamma^\mu \pi^\mu - \mu)S(x,y) = \delta^4(x-y)\]  \hspace{1cm} (1)

where \(\pi^\mu = \partial^\mu - e A^\mu\) is the energy-momentum four vector;

\[A = (0, -1/2 Hx, 1/2 Hx_1, 0)\]

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is the electromagnetic four vector potential; \( \mathbf{H} \) is the uniform magnetic field in the z-direction and \( \mu \) is the electron mass.

We can make the following Ansatz

\[
S(x,y) = (i\gamma_\mu \pi^\mu + \mu) \mathcal{F}(x,y)
\]

and as \( \mathcal{A} \) does not depend on \( x_0 \) and \( x_3 \), we may write \( \mathcal{F}(x,y) \) as a Fourier transform of a transversal Green's function

\[
\mathcal{F}(x,y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dp_0 dp_3 \exp[\frac{i}{2}p_0 (x_0 - y_0) - ip_3 (x_3 - y_3)] \mathcal{K}(\hat{s}, \hat{s}', p_0, p_3)
\]

which satisfies the following differential equation

\[
\left[ \mathcal{P}_0^2 - \mathcal{P}_3^2 - 2\omega \mathcal{S}_3 - \mu^2 - \omega s^2 + \mathcal{V}_s^2 - 2\omega \mathcal{S}_3 \right] \mathcal{K}(\hat{s}, \hat{s}', p_0, p_3) = \delta(\hat{s}' - \hat{s})
\]

with \( \hat{s} \equiv (x_1, x_2) \) and \( \omega = e \mathcal{H}/2 \). Assuming \( \mathcal{K}(\hat{s}, \hat{s}'; p_0, p_3) \) diagonal on \( \sigma_3 \) we may write \( \sigma_3 \mathcal{K} = \lambda \mathcal{K} \) with \( \lambda = \pm 1 \).

As in the non-relativistic case (Bellandi, Zimerman-1975) we can firstly seek for \( \mathcal{K}(\hat{s}, \hat{s}'; p_0, p_3) \) in the series form

\[
\mathcal{K}(\hat{s}, \hat{s}'; p_0, p_3) = \frac{1}{(2\pi)^3} \sum_{m=-\infty}^{\infty} \mathcal{K}^m(\hat{s}, \hat{s}'; p_0, p_3) \exp[im(\phi - \phi')]
\]

where \( m \) are the eigenvalues of \( L_3 \); \( s \) and \( \phi \) are the polar coordinates in the \((x_1, x_2)\) plane.

The partial Green's function \( \mathcal{K}^m(s, s'; p, p) \) satisfies the following differential equation

\[
\left[ \frac{d^2}{ds^2} + \frac{1}{s} \frac{ds}{ds} - \frac{m^2}{s^2} - \omega s^2 + \lambda - 2\omega a \right] \mathcal{K}^m(s, s'; p_0, p_3) = (s s')^{-1/2} \delta(s - s')
\]

with \( \lambda = \mathcal{P}_0^2 - \mathcal{P}_3^2 - 2\omega s - \mu^2 \). A solution of this equation is (we put \( \omega = 1 \))

\[
\mathcal{K}^m(s, s'; p_0, p_3) = \Gamma(1 + \frac{|m| + m - \lambda/2}{2}) (ss')^{-1} \frac{\lambda - m}{2} \mathcal{F}(s) \frac{\lambda - m}{2} \mathcal{F}(s)
\]

(7)
Here $\llangle$ and $\gg$ are the lesser and greater of $s^I$ and $s^I$ respectively and $M$ and $W$ are the Whittaker functions. Replacing the product of the Whittaker function in Eq. 7 by an integral representation in the complex plane (Buchholz, 1969) we can perform the summation in Eq. 5 and the final expression for the transversal Green's function is

$$K(\vec{s}, \vec{s}'; p_0, p_3) = \frac{1}{2\pi} \Gamma\left(\frac{1}{2} - \frac{\lambda}{4}\right) \exp\left[\frac{i}{2} \frac{\lambda}{4} |\vec{s} \times \vec{s}'|\right] \frac{1}{|\vec{s} - \vec{s}'|} u^{\frac{\lambda}{4}} \left(\lambda |\vec{s} - \vec{s}'|^2\right).$$

Since this function is an analytical function, Eq. 8 is valid for arbitrary values of $\lambda$. This expression must be interpreted as a $4 \times 4$ matrix, and for a defined value of $s$ we must write $1/2(1 + s^3)K; \Sigma^3 = \sigma^{12}$.

REFERENCES