Revista Brasileira de Física, Vol. 8, Nº 3, 1978

The Use of Exterior Forms in Einstein's Gravitation Theory

W. THIRRING *and* R. WALLNER
Institut für Theoretische Physik, Universität Wien

Recebido em 3 de Maio de 1978

We use Cartan's calculus to reformulate the general variational principle and conservation laws in terms of exterior forms. In applying this method to Einstein's gravitation theory, we do not only benefit from the great economy of Cartan's formalism but also gain a deeper understanding of fundamental results already known. So the existence of superpotential-forms may be deduced from $d\circ d \equiv 0$ and as a consequence the vanishing of total energy and momentum in a closed universe is affirmed in a more general way. Simple expressions for the sundry superpotentials are obtained quite naturally. As a byproduct, Einstein's equations are rewritten in a form where the coderivative of a 2-form (the superpotential-form) is a current, and therefore resembles the inhomogeneous Maxwell equations. In passing from the Lagrangian to the Hamiltonian 4-form, we immediately enter the ADM formalism without lengthy calculations.

Usamos o cálculo de Cartan para reformular o princípio variacional geral e as leis de conservação em termos de formas exteriores. Aplicando esse método à teoria de gravitação de Einstein, não somente nos beneficiamos da grande economia do cálculo de Cartan como também adquirimos uma compreensão mais profunda de resultados fundamentais já conhecidos. Assim, a existência de formas superpotenciais pode ser deduzida de $d\circ d \equiv 0$ e, como consequência, a anulação da energia e do momento totais em um universo fechado é estabelecida de uma maneira mais geral. Expressões simples para os superpotenciais específicos são obtidas de uma maneira natural. Como subproduto, as equações de Einstein são reescritas de uma maneira semelhante às de Maxwell inomogêneas, em que a coderivada de uma 2-forma (a forma superpotencial) é uma corrente. Passando da 4-forma Lagrangeana à Hamiltoniana, chegamos imediatamente ao formalismo ADM sem muitos cálculos.

1. INTRODUCTION

Frequently, exterior forms are believed to play an essential role in physics (as well as in mathematics), because they are quantities closely connected with integration and are therefore well suited for expressing the physical laws of nature. But they also provideavery elegant and economic calculation technique, summarized in Cartan's formalism which makes the physical laws not only nicer but also easier to survey. Moreover, in view of the de Rham cohomology, they are expected to procure one of the best ways to join physics to the topology of the underlying manifold.

Although Cartan's formalism comes into effect more and more, some of its properties are still unknown to physicists or are not expressed in the most convenient manner. We therefore summarize the essential facts in Appendix A, where our notation is explained.

Appendices B and C are only concerned with some explicit calculations dropped in the main text.

In Section 2, we review the variational principle in terms of exterior forms and apply it to Einstein's gravitation theory. The connection of conservation laws and superpotential-forms is exhibited in Section 3. Briefly, the existence of superpotential-forms is a consequence of the fundamental identity $d\circ d \equiv 0$ (or "the boundary of a boundary is zero": see Ref.1). Usually, for pure gravitation one starts with the contracted Bianchi identities[2]

$$D * J^{\mu} := d * J^{\mu} + \omega^{\mu}{}_{\nu} \wedge * J^{\nu} \equiv 0 \,, \qquad (1.1)$$

where $J^{\mu} := T^{\mu\nu} e$ corresponds to the energy-momentum tensor or Einstein-tensor respectively, by the Einstein equations. Because this covariant local law (1.1) does not, in general, lead to globally conserved quantities (Refs.3,4,12), one tries to find a 1-form $t^{\mu} \in E_1$, so that

$$d * t^{\mu} \equiv \omega^{\mu}{}_{\nu} \wedge * J^{\nu} \qquad (1.2)$$

Obviously, $*t^{\mu}$ is unique only up to closed 3-forms. Furthermore, from (1.2) , it is clear that t^Y does not transform homogeneously under change of the basis (2.3) (below) , so that the corresponding tensor $t^{\mu\nu}(t^{\mu} := t^{UV} e_{\nu})$ will be called a pseudotensor. Howeuer, once such a t^Y has been found, we have as a consequence the conservations laws

$$d * (J^{\mu} + t^{\mu}) \equiv 0 , \tag{1.3}$$

with conserved currents $J^{\mu} + t^{\mu}$. But from the identity (1.3), one infers the existence of an exact 3-form, say $-d * S^{\mu}$, so that

$$* J^{\mu} + *t^{\mu} = - d * S^{\mu} . \tag{1.4}$$

But from (1.4) one derives not only the conservation law (1.3), but the stronger statement (S a 3-dimensional submanifold C M^4)

$$\int_S * J^{\mu} + * t^{\mu} = - \int_{\partial S} * S^{\mu} , \tag{1.5}$$

whence the total energy and momentum of a spacelike hypersurface may be expressed as a surface integral. Moreover, if S is compact and withaut boundary ($\partial S = \emptyset$) and S^{μ} is globally defined on S, then

$$\int_S * J^{\mu} + * t^{\mu} = 0, \tag{1.6}$$

that is, the total energy and momentum in a closed universe are zero. This shows that, for instance, the "energy" concept has basically a topological background and may be correlated with topological invariants. In particular, we will see that the global existence of S^{μ} depends on the topological properties of S, as will *be* discussed in Section 5, where we sketch our topological assumptions and justifications. To conclude, we deal in Section 4 with a simple derivation of the ADM-Hamiltonian in terms of exterior forms.

2. THE VARIATIONAL PRINCIPLE

Let $\phi_A \in E_{p_A}(H^4)$, $A = 1,2,\ldots,N$, denote a set of p-forms which completely describe a physical system, where N is the number of algebraically independent fields. Let us further assume that the equations of our theory may be derived from an action integral

$$I = \int_{A \subseteq M^4} L(\phi_A, d\phi_A) \ , \quad L \in E_4 \tag{2.1}$$

by the variational principle

$$\delta I = \delta \int_A L = 0 \tag{2.2}$$

(compact A). Because of the invariant "form" (in a double sense) of the Lagrangian $L \in E_4$ under niere change of the basis,

$$e^\mu \rightarrow \bar{e}^\mu = A^\mu{}_\nu \, e^\nu \quad (A^\mu{}_\nu \in E_0) \, , \tag{2.3}$$

the theory will be already covariant. The second type of gauge transformation besides (2.3) which will be dealt with, is the (say, arbi-, trarily small) coordinate transformation ($\xi = \xi^\mu e_\mu \in E_1$).

$$x^\mu \rightarrow \bar{x}^\mu = x^\mu + \ell(\xi) x^\mu = x^\mu + \xi^\mu \ . \tag{2.4}$$

Frequently, we will also assume (2.3) to be infinitesimal,

$$A^\mu{}_\nu = \delta^\mu{}_\nu + a^\mu{}_\nu \ , \quad | \, a^\mu{}_\nu(p) \, | \ll 1 \quad (\forall p \in M^4) \tag{2.5}$$

and therefore use

$$\delta e^\mu = a^\mu{}_\nu \, e^\nu \ . \tag{2.6}$$

Note, that if we are not dealing with a coordinate basis (i.e., $e^\mu = dx^\mu$) (2.6) and (2.4) are independent one of each other. As usual[4], we introduce the "total variation"

$$\bar{\delta}\phi_A := \delta\phi_A - \ell(\xi)\phi_A \ , \tag{2.7}$$

which commutes with ordinary differentiation when applied to functions Moreover, δ as well as $\bar{\delta}$ comute with the exterior differentiation because of the basis-invariant property of d. Now, (2.2) together with (2.7) lead to the identity

$$\delta L = \bar{\delta} L + \ell(\xi) L \equiv 0 \quad , \tag{2.8}$$

or, after carrying out the variation,

$$\bar{\delta}\phi_A \wedge * E^A + d * t(\xi) \equiv 0 \quad , \tag{2.9}$$

where

$$* E^A := \frac{\partial L}{\partial \phi_A} - (-)^{p_A} d \frac{\partial L}{\partial d\phi_A} \tag{2.10}$$

are the usual Euler-Lagrange expressions and

$$* t(\xi) := i(\xi) L + \bar{\delta}\phi_A \wedge \frac{\partial L}{\partial d\phi_A} \tag{2.11}$$

the canonical pseudo-energy-momentum current, "pseudo" because of the possibly inhomogeneous transformations property of ϕ_A. According to P. G. Bergmann[5], we refer to weakly (i.e. modulo the field equations $* E^A$ $=0$) conserved $*t(\xi)$ as to the generator (generating 3-form) of the infinitesimal transformations $\bar{\delta}$. This point of view stems from analytical mechanics, where the constants of motion of a system are identicalwith the generators of infinitesimal canonical transformations.

Now we turn to Einstein's general relativity. According to the traditional view[6], we regard the basis 1-forms $\{e^\mu\} \subset E_1$ as playing the role of gravitational potentials and the connection form $\omega^\alpha{}_\beta \in E_1$, defined by

$$\begin{aligned} de^\alpha &= - \omega^\alpha{}_\beta \wedge e^\beta \quad , \\ dg_{\alpha\beta} &= g_{\mu\alpha}\, \omega^\mu{}_\beta + g_{\mu\beta}\, \omega^\mu{}_\alpha =: \omega_{\alpha\beta} + \omega_{\beta\alpha} \end{aligned} \tag{2.12}$$

as corresponding to the field strengths. Because of the inhomogeneous transformation property of $w^a{}_B$ under (2.3),

$$\bar{\omega}^{\mu}{}_{\sigma} A^{\sigma}{}_{\tau} = - dA^{\mu}{}_{\tau} + A^{\mu}{}_{\sigma} \omega^{\sigma}{}_{\tau} , \qquad (2.13)$$

we should rather use the curvature 2-forms $R_{\alpha\beta} \in E_2$

$$R_{\alpha\beta} := d\omega_{\alpha\beta} - \omega_{\mu\alpha} \wedge \omega^{\mu}{}_{\beta} , \qquad (2.14)$$

where the inhomogeneous terms drop out, and construct out of it an invariant Lagrangian 4-form. The most general expression in four dimensions leading to second order equations in the $g_{\alpha\beta}$ is[7]

$$L = \Lambda\varepsilon + R_{|\alpha\beta|} \wedge *e^{\alpha\beta} + \alpha\, R_{|\alpha\beta|} \wedge R^{\alpha\beta}$$

$$+ \beta \left[R_{|\alpha\beta|} \wedge *e^{\alpha\beta} \wedge *(R_{|\sigma\tau|} \wedge *e^{\sigma\tau}) - R_{\alpha} \wedge *R^{\alpha} + R_{|\alpha\beta|} \wedge *R^{\alpha\beta}\right] . \qquad (2.15)$$

If we neglect (for simplicity) the cosmological term $\Lambda\varepsilon$ and observe that the coefficients of a and β (being essentially exact, see Appendix B.d) equate the corresponding Euler-Lagrange expressions (2.10) identically to zero (Refs.7,9,24), we are left with the free (pure geometrical) Einsteinian Lagrangian .

$$L(e_{\mu}, de_{\mu}) = \frac{1}{2} R^{\alpha}{}_{\beta} \wedge * e_{\alpha}{}^{\beta} . \qquad (2.16)$$

An independent variation of e_{μ}, $g_{\alpha\beta}$ and $\omega^{\alpha}{}_{\beta}$ leads to[6] (see Appendix B.a)

$$\bar{\delta}L = \frac{1}{2} (\bar{\delta}\, \omega^{\alpha}{}_{\beta} \wedge *e_{\alpha}{}^{\beta}) + \frac{1}{2} \bar{\delta}\omega^{\alpha}{}_{\beta} \wedge D * e_{\alpha}{}^{\beta} - \frac{1}{2} \bar{\delta}e_{\mu} \wedge * G^{\mu} , \qquad (2.17)$$

where $G^{\mu} := G^{\mu\nu} e_{\nu}$ is the Einstein-form, its components forming the Einstein-tensor (see (3.15)). The identity (2.9) then becomes

$$- \frac{1}{2} \bar{\delta}e_{\mu} \wedge *G^{\mu} + d * t(\xi) = 0 , \qquad (2.18)$$

where

$$*t(\xi) := i(\xi)\, L + \bar{\delta}\, \omega^{\alpha}{}_{\beta} \wedge \frac{\partial L}{\partial d\omega^{\alpha}{}_{\beta}} \, . \qquad (2.19)$$

By the Bianchi identities

$$DR_{\alpha\beta} \equiv 0 \, , \qquad (2.20)$$

the variational derivative L with respect to the basis obeys the contracted Bianchi identities

$$D * G^{\mu} \equiv 0 \, , \qquad (2.21)$$

which can also be deduced from the invariance property of (2.1), (see Ref.8) .

3. CONSERVATION LAWS AND SUPERPOTENTIAL-FORMS

Let $\{e^{\mu}\}$ be a coordinate basis $(e^{\mu} = dx^{\mu})$. Then by $\bar{\delta} e^{\mu} = d\xi^{\mu} - di(\xi)\, dx^{\mu} = 0$, we obtain (see Ref. 9 for instance)

$$\begin{aligned} \bar{\delta} e_{\mu} &= \bar{\delta} g_{\mu\nu} e^{\nu} = - (\xi_{\mu;\nu} + \xi_{\nu;\mu}) e^{\nu} \, , \\ \bar{\delta}\omega^{\alpha}{}_{\beta} &= \bar{\delta}\Gamma^{\alpha}{}_{\beta\gamma} e^{\gamma} = \frac{1}{2} g^{\alpha\tau} (\bar{\delta} g_{\tau\beta;\gamma} + \bar{\delta} g_{\tau\gamma;\beta} - \bar{\delta} g_{\beta\gamma;\tau}) e^{\gamma} \, , \end{aligned} \qquad (3.1)$$

where $\bar{\delta} g_{\tau\beta;\gamma} := (\bar{\delta} g_{\tau\beta})_{;\gamma}$ of course. By (3.1), the canonical pseudo-current $*t(\xi)$ then becomes (see Appendix B.b)

$$*t(\xi) = - * G(\tau) + \frac{1}{2}\, d * d\xi , \qquad (3.2)$$

where $G(\xi) := \xi^{\mu} G_{\mu}$, $\xi^{\mu} \in E_0$ and $G_{\mu} \in E_1$. Although $*t(\xi)$ is only weakly conserved, the expression

$$*J(\xi) := * G(\xi) + * t(\xi) = \frac{1}{2}\, d * d\xi \qquad (3.3)$$

is a strongly conserved quantity, that is, irrespective of any field equation, $*J(\xi)$ obeys

$$d * J(\xi) \equiv 0 \,, \tag{3.4}$$

and therefore we have associated a strongly conserved quantity with an infinitesimal coordinates transformation. However, by (2.18), this is unique only up to closed 3-forms, so we can also define another pseudo-current $*\tilde{t}(\xi)$ by

$$*\tilde{J}(\xi) := * G(\xi) + *\tilde{t}(\xi) = d * d\xi. \tag{3.5}$$

Evaluating the righthand side of (3.5) in components, one obtains at once

$$d * d\xi = (\xi_{\mu;\nu} - \xi_{\nu;\mu})^{;\mu} * e^{\nu} \,. \tag{3.6}$$

The components are known as forming Komar's generalized energy flux vector $E^{\lambda}(\xi)$ (besides a factor -2, which comes from another choice of units; see Ref.10. Let ξ^{μ} be constant in the e_{μ}-basis (this is possible because of the arbitrariness of ξ in (2.4)), (3.5) becomes

$$* \tilde{J}_{\mu} = * G_{\mu} + *\tilde{t}_{\mu} = d * de_{\mu}. \tag{3.7}$$

We call

$$V_{\mu} := d e_{\mu} \qquad (\in E_2) \tag{3.8}$$

the "Møller-form", because its components in a coordinate basis

$$V_{\mu}{}^{\nu\rho} = (g_{\mu\beta;\alpha} - g_{\mu\alpha;\beta})g^{\alpha\nu} g^{\beta\rho} \tag{3.9}$$

are $1/\sqrt{-g}$ times the original Møller-potential $\chi_{\mu}{}^{\nu\rho}$ (Refs. 11, 2),

$$\chi_{\mu}{}^{\nu\rho} = \sqrt{-g}\, V_{\mu}{}^{\nu\rho} \,. \tag{3.10}$$

In general, the superpotential-forms may be introduced as follows. From

(3.1), the symmetry of $G^{\mu\nu}$ and by the contracted Bianchi identities (2.21), one obtains out of (2.18) the identity

$$d[* G(\xi) + * t(\xi)] \equiv 0 . \tag{3.11}$$

But from the identity one infers the existence of an exact 3-form, say $-d * S(\xi)$, which allows us to write

$$*G(\xi) + *t(\xi) = - d * S(\xi), \tag{3.12}$$

where, without loss of generality we choose $S(\xi) \in E_2$ to be linear in ξ, that is $S(\xi) := \xi^\mu S_\mu$, $S_\mu \in E_2$.

Let for a moment ξ^μ be constant again. Then, from (3.12) and (3.2), it follows that

$$* J_\mu := * G_\mu + * t_\mu = - d * S_\mu \tag{3.13}$$

Thus we see, that S_μ is actually the analogy of the so called "superpotentials" (Refs.4,5,12), wherefore we call S_μ a "superpotential-form". Note, that the existence of this "superpotentials" follows from the fundamental identity $d o d \equiv 0$.

Instead of calculating S_μ explicitly from (3.12) by (3.2), we use a more direct and more elegant way to find it out[8]:

The simplest Lagrangian connecting geometry and matter is (8π times gravitational constant =: $l := c$)

$$L = R_{|\alpha\beta|} \wedge * e^{\alpha\beta} + L\,(\text{matter}) , \tag{3.14}$$

whereby we obtain (see (2.17))

$$* G_\mu = - R_{|\alpha\beta|} \wedge * e^{\alpha\beta}{}_\mu = \frac{\delta L(\text{matter})}{\delta e^\mu} =: * J_\mu , \tag{3.15}$$

where J_μ corresponds to the energy-momentum current.

We can rewrite (3.15) as an equation like the inhomogeneous Maxwell equation where the coderivative of a 2-form is a current. To this end, we write the curvature form as in (2.14) and rewrite the first term in (3.15) like this

$$d\omega_{\alpha\beta} \wedge * e^{\alpha\beta}{}_{\mu} = d(\omega_{\alpha\beta} \wedge * e^{\alpha\beta}{}_{\mu}) + \omega_{\alpha\beta} \wedge d * e^{\alpha\beta}{}_{\mu}. \qquad (3.16)$$

Retaining only the exact form on the left, we obtain[8]

$$d * S_{\mu} = - *(J_{\mu} + t_{\mu}) \leftrightarrow \Delta S_{\mu} = J_{\mu} + t_{\mu} , \qquad (3.17)$$

where

$$S_{\mu} := \frac{1}{2} i(\omega_{\alpha\beta})\, e^{\alpha\beta}{}_{\mu} \qquad (\in E_2) \qquad (3.18)$$

and

$$*t_{\mu} := - \frac{1}{2} \omega_{\alpha\beta} \wedge (\omega_{\mu\nu} \wedge * e^{\alpha\beta\nu} + \omega^{\beta}{}_{\nu} \wedge * e^{\alpha\nu}{}_{\mu}) . \qquad (3.19)$$

Comparing (3.17) with (3.13), we rediscover the superpotential-form S_V. Because $*t_{\mu}$ consists of pure geometric terms, the interpretation of (3.17) is that the currents of energy ($\mu=0$) and momentum ($\mu=1,2,3$) have a contribution of matter (J) and one of gravitation (t). Note, that from (3.15) up to this stage we are not concerned with any coordinate basis. However, if we examine the superpotential-form (3.18) in such a coordinate basis, its components can be readily calculated (see Appendix B.c)

$$S_{\mu}{}^{\nu\rho} = \frac{1}{2(-g)} g_{\mu\tau} [(-g)(g^{\nu\beta} g^{\tau\rho} - g^{\rho\beta} g^{\tau\nu})] . \qquad (3.20)$$

This is $(-1/\sqrt{-g})$ times the well known von Freud expression for the superpotential (Refs. 13, 12, 2):

$$U_{\mu}{}^{\nu\rho} = - \sqrt{-g}\, S_{\mu}{}^{\nu\rho} \qquad (3.21)$$

out of which several pseudotensors may be constructed. For instance, the

Landau-Lifschitz-forn $*t^{\mu}_{LL}$ is obtained by extracting $1/\sqrt{-g}$ out of $d*S_{\mu}$:

$$d * S_{\mu} = \frac{1}{\sqrt{-g}} d * \sqrt{-g}\ S_{\mu} - \omega^{\alpha}{}_{\alpha} \wedge * S_{\mu} . \tag{3.22}$$

Therefore, we have

$$\Delta \sqrt{-g}\, S_{\mu} = - \Delta U_{\mu} = \sqrt{-g}\ (J_{\mu} + t_{\mu}) , \tag{3.23}$$

$$* t^{\mu}_{LL} := * t_{\mu} - \omega^{\alpha}{}_{\alpha} \wedge * S^{\mu} \tag{3.24}$$

(where $t^{\mu} := g^{\mu\nu}$ t), or in components corresponding to a coordinate basis:

$$(-g)(T_{\mu}{}^{\nu} + t_{LL\ \mu}{}^{\nu}) = [(-g)\ S_{\mu}{}^{\nu\rho}]_{,\rho} . \tag{3.25}$$

Although it is obvious from (3.25) (compare Landau-Lifschitz[14]), we prove in Appendix C the equivalence of t^{μ}_{LL} to the Landau-Lifschitz pseudo--tensor. There we calculate the components $t_{LL}{}^{\mu\nu}$ because a) we had never seen it explicitly given and *b*) to show the splendour of the modern Cartan formalism and the miseries of the classical tensor calculus.

In a coordinate basis, $*t^{\mu}_{LL}$ leads to a symmetric energy-rnomentum expression, so it will be a good (but not necessary, see Ref.15) candidate for a suitable angular momentum expression. Further note,. that $* t^{\mu}$ and $* t^{\mu}_{LL}$ coincide in a basis of constant $\sqrt{-g}$ (because $\omega^{\alpha}{}_{\alpha} = dg/2g$).

Writing S_{μ} in the form (B.18) (see Appendix B)

$$S_{\mu} = - de_{\mu} - \frac{1}{2} i(e^{\alpha}) de_{\alpha\mu} \tag{3.26}$$

and keeping only the first term on the lefthand side of (3.17), we get in a coordinate basis

$$d * de_{\mu} = * J_{\mu} + * \tilde{t}_{\mu} \longleftrightarrow \Delta de_{\mu} = - J_{\mu} + \tilde{t}_{\mu}, \tag{3.27}$$

$$* \tilde{t}_{\mu} := * t_{\mu} - \frac{1}{2} d * i(e^{\alpha}) de_{\alpha\mu} , \tag{3.28}$$

which lead us again to the Møller-form $V_\mu = de_\mu$. Note, that in this way Einstein's equations are cast in exactly the same form as Maxwell's equations $\lambda\, dA = J$, A = potential, J = current.

Now, from equation (3.27) we read off that transformations (2.3), which do not change the time direction e_0, leave $\tau_0 := J_0 + t_0$ unchanged, that is, $\tau_0{}^\mu$ transforms like a 4-vector under these transformations. In particular, $\tau_0{}^0$ and $\tau_0{}^i$ behave like a scalar and 3-vector under arbitrary spatial transformations and are therefore well suited for a consistent interpretetion as localized energy and momentum density (Refs. 11, 10). This outstanding property of the Møller expression has led A. Komar to construct his "generalized energy-flux vector", which we already deduced from the variational principle (equations (3.3) and (3.6)).

We remark again the striking analogy to the Maxwell equations, where ξ corresponds to the potential A. If ξ refers to a rigid time translation, we return to the Møller case.

Sundry expressions like (3.17) which basically rested on the existence of superpotential-forms by $d \circ d \equiv 0$, have the important property that one can not only deduce the conservation law (1.3) but also the stronger statement (1.5), that is, the possibility of expressing total energy and momentum of a spacelike hypersurface as surface integrals and that, for instance, they vanish in a closed universe.

4. THE HAMILTONIAN

In this Section, we shall deal with the free case ($J^\mu = 0$). Therefore, the term "energy" only refers to the energy of the gravitational field, not matter.

Starting from a Lagrangian 4-form in (2.1), one usually changes to the Hamiltonian formalism through a Legendre-transformation

$$\dot{\phi}_A \rightarrow \frac{\partial L}{\partial \dot{\phi}_A} =: \pi^A \,, \tag{4.1}$$

by which one passes from the "configuration space" to the "phase space", with points given by (ϕ_A, π^A). But to carry this out in our formalism, one has to know what the dot in (4.1) (the "time-derivative") stands for, that is, to single out a time direction. Let us specialize to general relativity and recall that our basic variables are not the metric coefficients but rather the basis 1-forms e_μ, where the time direction may be (locally) chosen to be represented by e^0 or e_0. To make this choice transparent, we pass over to a Gaussian ("comoving" or "synchronous", see Ref.1) basis $\{\bar{e}^\mu\}$, where the splitting of time and space directions becomes more graphic. This basis may be defined by

$$(\bar{g}^{\alpha\beta}) = \begin{pmatrix} -1 & 0 \\ 0 & \bar{g}^{ik} \end{pmatrix}, \quad (\bar{g}_{\alpha\beta}) = \begin{pmatrix} -1 & 0 \\ 0 & \bar{g}_{ik} \end{pmatrix}. \tag{4.2}$$

In terms of this basis $\{\bar{e}^\mu\}$, the metric $g = \bar{e}_\mu \otimes \bar{e}^\mu$ splits into

$$g = -\bar{e}_0 \otimes \bar{e}^0 + \bar{e}_k \otimes \bar{e}^k , \tag{4.3}$$

where

$$^3g := \bar{e}_k \otimes \bar{e}^k \tag{4.4}$$

corresponds to the "projection operator" orthogonal to $\bar{e}^0$ (Ref. 3). If there is a hypersurface $S \subset M^4$, such that $\bar{e}^0|_S = 0$, then $^3g|_S$ may be interpreted as the "first fundamental form" (not form in the sense $\in E_p$) on that hypersurface.

Because of the simple decomposition of the nietric in (4.2), one immediately obtains for the connection 1-forms

$$\begin{aligned} d\bar{g}_{00} &= \bar{\omega}_{00} = 0 , \\ d\bar{g}_{0k} &= \bar{\omega}_{0k} + \bar{\omega}_{k0} = 0 \leftrightarrow \bar{\omega}_{0k} = -\bar{\omega}_{k0} , \\ d\bar{g}_{ik} &= \bar{\omega}_{ik} + \bar{\omega}_{ki} = {}^3d\bar{g}_{ik} + \langle d\bar{g}_{ik}, \bar{e}_0\rangle \bar{e}^0 , \end{aligned} \tag{4.5}$$

where ${}^3d$ means exterior derivative with respect to $E_p(S)$. Before decomposing the curvature form $R_{\alpha\beta}$ in the same way, let us assume (for simplicity) $\{\bar{e}^\mu = dy^\mu\}$ to be a coordinate basis, $\{y^\mu\}$ being the Gaussian *coordinate* system. Let S be given by y^0 = constant (a hypersurface of proper time simultaneity). Then $\bar{e}^0$ is the unit hypersurface orthononal and the "second fundamental form" of S (or "exterior curvature") is defined by

$$K := \frac{1}{2}\, \ell(\bar{e}^0)\ {}^3g =: K_i \otimes \bar{e}^i , \tag{4.6}$$

where $K_i = K_i{}^j\, \bar{e}_j$ is given by (recall $\ell(\bar{e}_0)\bar{e}^k = 0$)

$$K_i := -\frac{1}{2}\, \ell(\bar{e}_0)\bar{e}_i = \omega_{0i} , \tag{4.7}$$

will be called "exterior curvature form" (although "exterior connection form" would be a more appropriate terminology). Note, that by (4.7), K_i represents the (proper-) time derivative of the $\bar{e}_i$'s off the hypersurface S. Therefore, we shall decompose the Lagrangian (2.16) in tens of $\{\bar{e}_k, K_i\}$. This can easily be done by using the Gauss - Codazzi equations, which we write like this:

$$\bar{R}_{00} \equiv 0 ,$$

$$\bar{R}_{0i} = \bar{D}K_i = {}^3\bar{D}K_i + (K_{ik,0} + K_i{}^j\, K_{jk})\bar{e}^{0k} , \tag{4.8}$$

$$\bar{R}_{ik} = {}^3\bar{R}_{ik} + K_i \wedge K_k + (K_{\ell k|i} - K_{\ell i|k})\bar{e}^{0\ell} ;$$

where ${}^3\bar{D}$ and ${}^3\bar{R}_{ik}$ are the covariant exterior derivative and its corresponding curvature form with respect to $E_p(S)$, the vertical bars denoting the component notation of ${}^3\bar{D}$.

Inserting (4.8) into (2.16) leads to

$$L = \bar{D}K_k \wedge * \bar{e}^{0k} + \frac{1}{2}\,({}^3\bar{R}_{ik} + K_i \wedge K_k) \wedge * \bar{e}^{ik}, \tag{4.9}$$

or, by

$$\bar{D}K_k \wedge * \bar{e}^{0k} = d(K_k \wedge * \bar{e}^{0k}) - K_i K_k \wedge * \bar{e}^{ik} , \qquad (4.10)$$

$$L = d(K_k \wedge * \bar{e}^{0k}) + \frac{1}{2} ({}^3\bar{R}_{ik} - K_i \wedge K_k) \quad * \bar{e}^{ik} . \qquad (4.11)$$

The main advantage of (4.11) is, that the time derivatives K_i are singled out and one can compute the "conjugate momentum" of $\bar{e}_k$ as

$$* \pi^k := \frac{\partial L}{\partial \ell(\bar{e}_0)\dot{\bar{e}}_k} = - \frac{1}{2} \frac{\partial L}{\partial K_k} = \frac{1}{2} K_i \wedge * \bar{e}^{ki}, \qquad (4.12)$$

or in component notation ($\pi^k \in E_1$)

$$\pi^{ik} = \frac{1}{2} (g^{ik} \text{ tr } K - K^{ik}), \qquad (4.13)$$

where tr $K := K^i{}_i$. Therefore, we obtain the Hamiltonian 4-form

$$H := L + \ell(\bar{e}_0)\bar{e}_k \wedge * \pi^k , \qquad (4.13)$$

or, when (4.11) and (4.12) are inserted into (4.13)

$$H = - d(K_k \wedge * \bar{e}^{0k}) - \frac{1}{2} ({}^3\bar{R}_{ik} + K_i \wedge K_k) \wedge * \bar{e}^{ik} , \qquad (4.14)$$

which by (4.9) once more reduces to

$$H = d(K_k \wedge * \bar{e}^{k0}) + \bar{e}_0 \wedge * \bar{G}^0 , \qquad (4.15)$$

where $\bar{G}^0$ is the zeroth Einstein-form, written in the Gaussian basis $\{\bar{e}^\mu\}$:

$$* \bar{G}^0 = ({}^3R_{ik} + K_i \wedge K_k) \wedge * \bar{e}^{ik0} - 2(K_i{}^k{}_{|k} - \text{tr } K_{|i}) * \bar{e}^i . \qquad (4.16)$$

If we express the Gaussian basis $\{\bar{e}^\mu\}$ in term of any basis $\{\bar{e}^\mu\}$ according to (2.3), there are only four functions N^μ sufficient to determine $A^\mu{}_\nu$, which may be written as

$$A^\mu{}_\nu = \delta^\mu{}_\nu + N^\mu \delta^0{}_\nu - \delta^\mu{}_0 \delta^0{}_\nu \qquad (4.17)$$

or explicitly

$$\bar{e}^0 = Ne^0, \ \bar{e}_0 = (1/N)(e_0 - N^k e_k) \ ,$$
$$\bar{e}^k = e^k + N^k e^0 \ , \ \bar{e}_k = e_k \ , \tag{4.18}$$

where $N := N^0$ and $N^i (i = 1,2,3)$ are the lapse and shift functions (see Refs. 1, 16). Clearly, $\{e'\}$ is Gaussian iff $N = 1$ and $N^i = 0$ for all i. By (4.18), the metric coefficients in terms of $\{e', N^\mu\}$ are written as

$$(g^{\mu\nu}) = \begin{pmatrix} -\frac{1}{N^2} & \frac{1}{N^2} N^k \\ \frac{1}{N^2} N^i & \bar{g}^{ik} - \frac{1}{N^2} N^i N^k \end{pmatrix} , \ (g_{\mu\nu}) = \begin{pmatrix} -N^2 + N^i N_i & N_k \\ N_i & \bar{g}_{ik} \end{pmatrix} \tag{4.19}$$

where the indices of N^k are raised and lowered by the components of ^3g. Usually, (4.19) is the starting point of a canonical treatment of general relativity.

For instance, (4.7) reads in terms of $\{e^\mu, N^\mu\}$ (using (2.13))

$$K_i = \bar{\omega}_{0i} = - \bar{\omega}^0{}_i = - N \, \omega^0{}_i \leftrightarrow K_{ik} = - N\Gamma^0{}_{ik} \ , \tag{4.20}$$

which leads to the well known equation[1]

$$K_{ik} = \frac{1}{2N} (N_{i|k} + N_{k|i} - g_{ik,0}) \ , \tag{4.21}$$

where the extrinsic curvature is expressed in terms of the ADM lapseand shift functions.

the homogeneous transformation property of G^μ (see (3.15)) and by (4. ,, we obtain immediately for the Hamiltonian in terms of $\{e', N^\mu\}$ the expression

$$H = d(NK_k \wedge * e^{k0}) + N^\mu H_\mu \ , \tag{4.22}$$

$$H_0 := \frac{1}{N} e_0 \wedge * G^0 = \frac{1}{N} * G^0{}_0 \tag{4.23a}$$

$$H_k := - e_k \wedge * G^0 = - * G^0{}_k . \tag{4.23b}$$

By (4.16), H_μ may be also easily expressed in terms of K^i and ^3R (see Ref. 1). We omit it as well as the further canonical treatment, since the line is clear and follows closely the ADM formulation of gravity (Refs.1,16). Writing the Lagrangian (4.11) in the basis $\{e^\mu\}$, it turns out that there are no $\dot{N}^\mu$ involved and therefore no canonical momenta to N^μ do appear. This leads to the primary constraints $\partial L/\partial \dot{N}^\mu = 0$ which indicate that the N^μ play the role of mere Lagrangian multipliers.The fact that the primary constraints hold for any space-time point leads directly to the more serious secondary or dynamical constraints $H_\mu = 0$, which indicate that the conjugate variables cannot be arbitrary on an initial hypersurface, that is, not all of the canonical variables are "true", i.e., dynamical. But this is exactly what has to be expected since, bythe general covariance of the theory, the Lagrangian must be singular, which leads to constraints when passing to a canonical formalism. Finally, the rest of the Hamiltonian field equations may be derived from varying (4.11) with respect to $\bar{e}_k$ and π^k. Because of (4.23a,b) they reproduce the remaining Einstein equetions $G^i{}_k = 0$.

Remark. As is well known, the exact 4-form $d(K_k \wedge * e^{k0})$ does not alter the dynamical equations but does change the definition of energy (see e.g. Ref.17), and it is still an open question whether the full Lagrangian (2.15) has to be considered (see Appendix B.d).

5. TOPOLOGICAL REMARK

Of course, the considerations made in Section 4 depend on the existence of a "Cauchy-splitting" of space-time, i.e., if space-time may be expressed in the form of a topological product of a 3-hypersurface with the real line, such that each member of the family of hypersurfaces is space like. This is the case if we demand space-time to be globally hyperbolic[3].

The existence of superpotential-forms implies the possibility of expressing the total energy and momentum of a spacelike hypersurface in terms of surface integrals (Section 3). But, as already remarked by C. W. Misner[18], closed surfaces cannot be covered by a single set of singularity-free coordinates, so that the total energy and momentum definitions involving pseudotensors may become problematic. This would also be the case in our formalism, if we were dealing only with holonomic (= coordinate) bases, which refer to a suitable covering of the manifold by charts. However, our various total energy and momentum definitions are more generally established on any suitable form-basis and are therefore only concerned with the parallelizability (or at least with orientability) of the manifold. So for instance, the torus $S^1 \times S^1$ (or the sphere S^2) is parallelizable (orientable, respectively), though it cannot be covered by one non-singular chart. We therefore conclude that the surface integrals like (1.5) with (3.18) make sense if the surface is parallelizable. Parallelizability is known to be closely connected with the Euler-Poincaré-characteristic of the underlying manifold. So, every compact oriented manifold of odd dimension has vanishing Euler-Poincaré-characteristic and is therefore parallelizable (and hence orientable[19]). For that reason, we also believe that the argument concerning a vanishing total energy/momentum in a closed universe makes sense indeed.

In general, in the compact as well as in the noncompact case, the main question we are left with is orientability of space, since every 3-manifold, if oriented, is also parallelizable (according to a theorem of E. Stiefel[20]). Since several physical facts (entropy theorem, expansion of the universe) induce us to believe that space-time is time-orientable, then by the CPT theorem it is also space-orientable[3]. If then, as we assume, space-time is globally hyperbolic, it is itself parallelizable[21] and our formalism is globally defined. Moreover, we were also allowed to use a global orthonormal basis and are therefore immediately bound up with the spinor structure of space-time[21].

APPENDIX A

Let M^n be an n-dimensional differentiable manifold and $E_p(M^n)$ the module of exterior differential p-forms over $E_0(M^n) := C^\infty(M^n;R)$, the set of smooth mappings from M^n to the real numbers R. The direct sum of $E_p(M^n)$, $p = 0,1,\dots, n$ is then widen to a (graded) algebra by the componentwise extended *exterior product*

$$\wedge \ : E_p \wedge E_q \to E_{p+q} \ .$$

Out of a l-basis $(e^a) \subset E_1$, $\alpha = 0,1,\dots, n-1$, we construct a $\binom{n}{p}$ - basir $\{e^{\alpha_1\dots\alpha_p}\} \subset E_p$ by

$$e^{\alpha_1\dots\alpha_p} := e^{\alpha_1} \wedge \dots \wedge e^{\alpha_p} = p!\ e^{[\alpha_1} \otimes\dots\otimes e^{\alpha_p]}, \qquad (A.1)$$

where $\otimes$ denotes tensor products and square brackets antisymmetrisation, i.e.

$$[\alpha_1\dots\alpha_p] := \frac{1}{p!} \sum_{\sigma \in \Sigma_p} (-)^\sigma \alpha_{\sigma(1)} \cdots \alpha_{\sigma(p)} \qquad (A.2)$$

(Σ_p, the p-th permutation group). Thus, any p-form $\omega \in E_p$ can be written as

$$\omega = \omega_{|\alpha_1\dots\alpha|} e^{\alpha_1\dots\alpha_p}, \quad \omega_{[\alpha_1\dots\alpha_p]} = \omega_{\alpha_1\dots\alpha_p} \qquad (A.3)$$

Vertical bars demand summation over $\alpha_1 < \alpha_2 < \dots < \alpha_p$.

Differentiation is represented by the *exterior derivative* d: $E_p \to E_{p+1}$, defined by

$$\omega \to d\omega := d\omega_{|\alpha_1\dots\alpha_p|}\ e^{\alpha_1\dots\alpha_p}, \qquad (A.4)$$

where we define df to be the ordinary differential of $f \in E_0$. So, when applied to the coordinate functions $x^\mu \in E_0$ ($\mu = 0,1,\dots, n-1$), we get a special kind of basis $e^\mu = dx^\mu$, called *coordinate* (or *natural*) basis. In general, the describing features of d are:

1) $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2 \qquad (\omega_1, \omega_2 \in E_p)$,

2) $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-)^p \omega_1 \wedge d\omega_2$ (A.5)

3) $d \circ d \equiv 0$.

A p-form α is *closed* iff $d\alpha = 0$; it is exact iff there is a $\beta \in E_{p-1}$ such that $\alpha = d\beta$. Of course, every exact form is closed, but the converse is only true on a starshaped region (or open ball) of M^n (Poincaré's lemma).

If the manifold admits of a pseudo-Riemannian metric , we may use it to define a *scalar product* in E_1, $<,> : E_1 \times E_1 \to E_0$ by

$$<e^\alpha, e^\beta> : = g^{\alpha\beta} = g(e^\alpha, e^\beta). \tag{A.6}$$

As dual basis (cobasis) $\{e_\alpha\}$ to $\{e^\alpha\}$ *with respect* to g (or $<,>$, respectively), we shall use

$$e_\alpha : = g_{\alpha\beta}\, e^\beta \ (\in E_1), \quad g_{\alpha\beta}\, g^{\beta\gamma} = \delta_\alpha{}^\gamma . \tag{A. 7}$$

Note, that we are therefore dealing with dual basis-forms and not with dual basis-vector fields (say, ∂_μ in a coordinate basis).. Because for every vector field $\xi \in \mathcal{J}(M^n)$ there is a unique adjoint (with respect to g) 1-form $\tilde{\xi} \in \mathcal{J}^*(M^n) = E_1$ by

$$\tilde{\xi} : = g(\xi, .) \longleftrightarrow \tilde{\xi}_\mu = g_{\mu\nu}\, \xi^\nu, \tag{A.8}$$

and it is therefore equivalent to deal with basis and cobasis 1-forms or with basis forms and cobasis vector fields or with basis and cobasis vector fields, the latter frequently used within the *vierbein* (or tetrad) formalism. However, making use of forms is usually more practical by reason of the Cartan-formalism, which we are now going to summarize.

There is a natural extension of $<,>$ introduced by the *inner product* (or contraction) $\rfloor = E_1 \times E_p \to E_{p-1}$ $(E_{-1} = \emptyset)$

$$(\alpha,\omega) \to i(\alpha)\omega := \omega(\alpha,\ldots) \,, \tag{A.9}$$

so that for instance

$$i(e^\alpha)e^\beta = \langle e^\beta, e^\alpha\rangle = \langle e^\alpha, e^\beta\rangle = g^{\alpha\beta} \,.$$

We list the main properties of i ($\alpha,\beta \in E_1$; $h,f \in E_0$):

1) $i(f\alpha + h\beta) = f\, i(\alpha) + h\, i(\beta)$,
2) $i(\alpha)(\omega_1 + \omega_2) = i(\alpha)\omega_1 + i(\alpha)\omega_2 \qquad (\omega_1,\omega_2 \in E_p)$,
3) $i(\alpha)(\omega_1 \wedge \omega_2) = i(\alpha)\omega_1 \wedge \omega_2 + (-)^p\, \omega_1 \wedge i(\alpha)\omega_2, \quad (\omega_1 \in E_p,\ \omega_2 \in E_q)$
4) $i(\alpha) \circ i(\alpha) \equiv 0$,
5) $i(e^{\alpha_1\ldots\alpha_p})\omega = p!\ \ (e^{\alpha_p}) \circ \ldots \circ i(e^{\alpha_1})\omega \quad (\omega \in E_q,\ q \geq p)$.

(A.10)

The last property allows us to define an extended inner product for forms, $i(\alpha)\omega$, $\alpha \in E_p$, $\omega \in E_q$, $p \leq q$, but this ought to be handled carefully. Note that property 3) is only valid in this form $\alpha \in E_1$. By the extended inner product we obtain an extended scalar product in E_p:

$$\langle e^{\alpha_1\ldots\alpha_p}, e_{\beta_1\ldots\beta_p}\rangle := i(e_{\beta_p}) \circ \cdots \circ i(e_{\beta_1})\, e^{\alpha_1\ldots\alpha_p}$$

$$= \frac{1}{p!}\ (e_{\beta_1\ldots\beta_p})e^{\alpha_1\ldots\alpha_p} \,. \tag{A.11}$$

Thus, by the linearity of i, we get

$$\langle \alpha, \beta\rangle = \alpha_{|\mu_1\ldots\mu_p|}\, \beta^{\mu_1\ldots\mu_p} \qquad (\alpha, \beta \in E_p) \,. \tag{A.12}$$

(A. 11) is often called "*p-th permutation tensor*" and is denoted by

$$\delta^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_p} := \langle e^{\alpha_1\ldots\alpha_p}, e_{\beta_1\ldots\beta_p}\rangle = \frac{1}{p!}\ \delta^{\alpha_1}_{[\beta_1}\ \delta^{\alpha_2}_{\beta_2}\ldots\ \delta^{\alpha_p}_{\beta_p]} \,, \tag{A.13}$$

where the last equation is easily proved by (A.11).

Exterior derivative and contraction define the *Lie-derivative*

$$\ell : E_1 \times E_p \to E_{p+1} \ , \quad \ell(\alpha,\omega) =: \ell(\alpha)\omega \ ,$$

where

$$\ell(\alpha) := i(\alpha)d + d\,i(\alpha) , \tag{A.14}$$

so that the following formulas are valid ($\alpha \in E_1$, $\omega_1 \in E_p$, $\omega_2 \in E_q$):

$$1)\ \ell(\alpha)\left[\omega_1 \wedge \omega_2\right] = \ell(\alpha)\omega_1 \wedge \omega_2 + \omega_1 \wedge \ell(\alpha)\omega_2 \ ,$$

$$2)\ \ell(\alpha)d = d\ell(\alpha) \ , \tag{A.15}$$

$$3)\ \ell(\alpha) \circ i(\alpha) = i(\alpha) \circ \ell(\alpha) \ .$$

A convenient notation is the *covariant exterior derivative* D, $D : E_p \to E_{p+1}$ corresponding to a connection ∇ of M^n. So for instance, in a coordinate basis, D meets d but the ordinary derivative " , " exchanged for the covariant derivative " ; ". Thus, in general, D equals d when applied to forms without free index, otherwise property 3) in (A.15) must be replaced by

$$DD(\ldots)^{\mu\nu\ldots} = R^{\mu}{}_{\sigma} \wedge (\ldots)^{\sigma\nu\ldots} + R^{\nu}{}_{\sigma} \wedge (\ldots)^{\mu\sigma\ldots} + \ldots \ ,$$

$$(\ldots)^{\ldots} \in E_p \ , \ p = 0,1,\ldots, n-1 \ , \tag{A.16}$$

where $R^{\alpha}{}_{\beta}$ are the curvature forms of the connection ∇. If D corresponds to the unique Levi-Civita-connection (∇ metric-compatible and torsion-free), then

$$Dg_{\alpha\beta} = 0 \ , \tag{A.17}$$

$$De^{\mu} = de^{\mu} + \omega^{\mu}{}_{\nu} \wedge e^{\nu} = \mathrm{Tor}^{\mu} = 0 \ . \tag{A.18}$$

Roughly speaking, one may think of D as the antisymmetric part of ∇. Note, that whereas d is basis-invariant, D is not.

With D, the ordinary Lie-derivative may be extended to the *covariant Lie-derivative* $L : E_1 \times E_p \to E_p$, $L(\alpha,\omega) =: L(\alpha)\omega$, defined by

$$L(\alpha) := i(\alpha)D + Di(\alpha) \; . \qquad \text{(A.19)}$$

For instance, when applied to forms without free index, $L(e_\mu)$ is quite the analog of ∇_{∂_μ}. Property 2) of (A.15) is no longer valid for L. We remark that ℓ and L may be extended by use of the extended inner product, but we do not need them further and therefore they will be omitted.

Let M^n be *orientable*, i.e. there exists a continuous nowhere vanishing $\varepsilon \in E_n(M^n)$. Because dim $E_n(p) = 1$ for all $p \in M^n$, a connected manifold is orientable iff $E_n(p)$ has two components, each component forming an equivalence class of n-forms, called *orientation*. We are concerned with the *canonical n-form* ("metric volume element" or "generalized Levi-Civita tensor"):

$$\varepsilon := \sqrt{(-)^s g}\, e^0 \wedge e^1 \wedge \ldots \wedge e^{n-1} \qquad (\in E_n) \; ,$$
$$g := \text{Det}\,(g_{\alpha\beta}) \quad ; \quad (-)^\sigma := \text{signature of g.} \qquad \text{(A.20)}$$

Inner product together with ε provide a linear isomorphism $*$, $* : E_p \to E_{n-p}$ (Hodgestar-operator), definedby

$$\omega \to *\omega := \frac{1}{p!}\, i(\omega)\varepsilon \qquad (\text{all } \omega \in E_p), \qquad \text{(A.21)}$$

so that e.g.

$$* e^{\alpha_1 \ldots \alpha_p} = \varepsilon^{\alpha_1 \ldots \alpha_n} e_{|\alpha_{n+1} \ldots \alpha_n|} \; , \qquad \text{(A.22)}$$

where, by (A.21), $\varepsilon^{a_1 \ldots a_n} = * e^{a_1 \ldots a_n}$ are the components of ε with respect to the basis $\{e^{a_1 \ldots \alpha_n}\}$. Sometimes it is convenient to denote the metric volume element and its components with respect to an orthonormal basis differently, say ε and $\varepsilon_{\alpha_1 \ldots \alpha_n}$ (the latter also denoted by $[\alpha_1 \ldots \alpha_n]$, see Ref.1). Note that,, for instance,

$$\varepsilon_{01\ldots n-1} = (-)^s\, \varepsilon^{01\ldots n-1} = 1, \tag{A.23}$$

$$\varepsilon^{\alpha_1\ldots\alpha_n} = \frac{1}{\sqrt{(-)^s g}}\, \varepsilon^{\alpha_1\ldots\alpha_n} = -\frac{1}{\sqrt{(-)^s g}}\, \varepsilon_{\alpha_1\ldots\alpha_n} = |g|^{-1}\, \varepsilon_{\alpha_1\ldots\alpha_n}, \tag{A.24}$$

$$\delta^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_p} = -\varepsilon^{\alpha_1\ldots\alpha_p|\mu_{p+1}\ldots\mu_n|}\, \varepsilon_{\beta_1\ldots\beta_p\,\mu_{p+1}\ldots\mu_n} \tag{A.25}$$

We also list some important features of $*$:

1) $* \circ * = (-)^{(n-p)p+s}\ \mathrm{id}_{E_p}$,

2) $\alpha \wedge * \beta = \beta \wedge * \alpha$, $* \alpha \wedge \beta = * \beta \wedge \alpha$ $(\alpha, \beta \in E_p)$, (A.26)

3) $\alpha \wedge * \beta = (-)^{(n-q)(p+q)}\ \frac{1}{p!}\ * i(\alpha)\beta$, $(\alpha \in E_p,\ \beta \in E_q,\ 1 \le p \le q)$

4) $* \varepsilon = (-)^s$, $* 1 = \varepsilon$,

5) $(* \alpha, * \beta) = (-)^s \langle\alpha,\beta\rangle$, ("isometry").

Star-operator and exterior derivative allow for the definition of the *coderivative* $\delta : E_p \to E_{p-1}$ $(E_{-1} := \emptyset)$,

$$\delta := (-)^{n(p+1)+s} * d * , \tag{A.27}$$

as well as the *Laplace-Beltrami operator* $\Delta : E_p \to E_p$,

$$\Delta := \delta d + d \delta . \tag{A.28}$$

We summarize some important properties of δ and Δ:

1) $\delta \circ \delta \equiv 0$,

2) $\delta * = (-)^p * d$, $* \delta = (-)^{p+1}\ d *$,

3) $d \lambda * = * \lambda d \quad , * d \lambda = \lambda d * ,$ (A.29)

4) $d\Delta = \Delta d \quad , \quad \lambda\Delta = \Delta \lambda \quad ,$

5) $*\Delta = \Delta* .$

Integration on an n-dimensional orientable manifold M^n is defined via pull-backs to the model-space (e.g., R^n in our case). In R^n, one integrates p-forms over p-chains $\sigma^{(p)}$, the elements of a module, which is freely generated by the set of p-cubes or p-simplexes. In order to transfer these structure into the manifold M^n, one defines singular p--cubes or p-simplexes, that is, p-cubes or p-simplexes in R^n together with differentiable and orientation preserving mappings ϕ of R^n into M^n.

Let $|\sigma^{(p)}| := \phi(\sigma^{(p)}) \subseteq M^n$ denote the support of $\sigma^{(p)}$, and carr (a) the carrier (or also support) of a $\in E_p(M^n)$, i.e. the closure of the set of points $\in M^n$ outside of which a is equal to zero. Then, since ϕ is continuous, $|\sigma^{(p)}| \cap \mathrm{carr}(\alpha)$ is compact (there exists a finite "volume") and we define the integral of a over $\sigma^{(p)}$ by

$$\int_{\phi(\sigma^{(p)})} \alpha = \int_{\sigma^{(p)}} \phi^* \alpha , \tag{A.30}$$

where ϕ^* is the usual pull-back of forms $(\phi^* : E_p(M^n) \to E_p(R^n))$.

After suitable cubulating or triangulating the rnanifold M^n, one defines the integral over singular p-chains by linear extension of (A.30). Conversely, given an integral $\int_A$ a , $A \subseteq M^n$, one triangulates or cubulates A before calculating an integral like (A.30). We recall the most important theorem in integration theory, namely the theorem of Stokes (A = compact submanifold of suitable dimension):

$$\int_A d\alpha = \int_{\partial A} \alpha . \tag{A.31}$$

In general, integrals of forms with noncompact carriers within the support of a singular chain are to be understood as improper integrals (with sufficiently rapidly vanishing components outside a suitable region), which lead directly to the notion of *de Rham-currents* (distribution-type forms).

In the simplest case, they are defined as linear functionals on E_p. For instance, a linear mapping $E_p(M^n) \to E_p$ (at $\bar{x} \in M^n$) may be defined by

$$\omega \to \int_{M^n} \delta_{\bar{x}} \wedge \omega = \omega_{|\bar{x}} , \tag{A.32}$$

where $\delta_{\bar{x}} \in E_{p|\bar{x}} \otimes E_{n-p}$, the (Dirac) *δ-distribution p-form,* reproduces the value of a p-form w at $\bar{x} \in M^n$. It can be written in a coordinate basis $\bar{e}^\mu = d\bar{x}^\mu$ at. $\bar{x}$ (see Ref. 8) as

$$\delta_{\bar{x}} = (-)^{p(n-p)} \bar{e}^{|\alpha_1 \ldots \alpha_p|} \otimes * e_{\alpha_1 \ldots \alpha_p} \delta^n(x-\bar{x}) . \tag{A.33}$$

For further references on differential geometry and integration theory see Ref.23.

APPENDIX B .

a) To obtain the field equations, suppose for a moment that L depends also on the metric coefficients:

$$L = L(g_{\alpha\beta}, e_\nu, \omega^\alpha{}_\beta, d\omega^\alpha{}_\beta) = \frac{1}{2} R^\alpha{}_\beta \wedge * e_\alpha{}^\beta . \tag{B.1}$$

Varying all arguments of L independently, one readily calculates

$$\delta R^\alpha{}_\beta = d\delta\omega^\alpha{}_\beta + \delta(\omega^\alpha{}_\mu \wedge \omega^\mu{}_\beta) ,$$

$$\tag{B.2}$$

$$\delta * e_\alpha{}^\beta = \delta g_{\kappa\lambda} \left[\delta^\kappa{}_\alpha * e^{\lambda\beta} - \frac{1}{2} g^{\kappa\lambda} * e_\alpha{}^\beta\right] + \delta e_\kappa \wedge * e_\alpha{}^{\beta\kappa} ,$$

wherefore we obtain

$$\delta L = \frac{1}{2}\, d(\delta\omega^{\alpha}{}_{\beta} \wedge * e_{\alpha}{}^{\beta}) + \frac{1}{2}\, \delta\omega^{\alpha}{}_{\beta} \wedge D * e_{\alpha}{}^{\beta}$$

$$- \delta e_{\kappa} \wedge * G^{\kappa} + \frac{1}{2}\, \delta g_{\kappa\lambda}\, E^{\kappa\lambda} , \tag{B.3}$$

where

$$*G^{\kappa} := - R_{|\alpha\beta|} \wedge * e^{\alpha\beta\kappa} , \tag{B.4}$$

$$E^{\kappa\lambda} := R^{\kappa}{}_{\beta} \wedge * e^{\lambda\beta} - g^{\kappa\lambda}\, R_{|\alpha\beta|} \wedge * e^{\alpha\beta} \tag{B.5}$$

But from (B.4) and (B.5) one has

$$e^{\lambda} \wedge * G^{\kappa} = E^{\kappa\lambda} , \tag{B.6}$$

that is, the components of (B.4) and (B.5) are equal (the Einstein tensor $G^{\kappa\lambda}$) and, therefore, variation of e_{μ} and $g_{\alpha\beta}$ leads to the same field equations. Thus, one could also vary the basis with constant scalar product $g_{\alpha\beta} = \langle e_{\alpha}, e_{\beta} \rangle$ (e.g., an orthonormal basis, see Ref.8).

Incidently, equating (B.3) to zero, the Palatini principle leads to $D * e_{a}{}^{\beta} = 0$, and therefore to the usual correlation of basis and connection forms (see (2.12)). If the variation in (B.3) is $\bar{\delta}$, then (B.3), together with (B.6), gives (2.17).

b) From (2.16) and (2.14), one immediately obtains

$$\frac{\partial L}{\partial d\omega^{\alpha}{}_{\beta}} = \frac{1}{2} * e_{\alpha}{}^{\beta} . \tag{B.7}$$

Further, we calculate

$$i(\xi) L = i(\xi) R_{|\alpha\beta|} \wedge * e^{\alpha\beta} + R_{|\alpha\beta|} \wedge *(e^{\alpha\beta} \wedge \xi)$$

$$= *R(\xi) - *G(\xi) , \qquad R(\xi) = \xi^{\mu} R_{\mu} , \tag{B.8}$$

and

$$\frac{1}{2}\bar{\delta}\omega^{\alpha}{}_{\beta} \wedge * e_{\alpha}{}^{\beta} = \frac{1}{4}\left(\bar{\delta}g_{\tau\beta;\gamma} + \bar{\delta}g_{\tau\gamma;\beta} - \bar{\delta}g_{\beta\gamma;\tau}\right)e^{\gamma} \wedge * e^{\tau\beta}$$

$$= \frac{1}{2}\bar{\delta}g_{\gamma\tau;\beta}\, e^{\gamma} \wedge * e^{\tau\beta}$$

$$= -\frac{1}{2}\left(\xi_{\gamma;\tau\beta} - \xi_{\tau;\gamma\beta}\right)e^{\gamma} \wedge * e^{\tau\beta}$$

$$= - * R(\xi) + \frac{1}{2}\left(\xi_{\gamma;\tau} - \xi_{\tau;\gamma}\right)^{;\gamma} * e^{\tau}$$

$$= - * R(\xi) + \frac{1}{2}\, d * d\xi \; . \qquad \text{(B.9)}$$

The last set of equations is accompanied by the following steps: (1) insert (3.1); (2) equate the first term to zero and sum the last two because of the symmetry/antisymmetry contraction; (3) insert (3.1); (4) insert $\xi_{\alpha;\mu\nu} = \xi_{\alpha;\nu\mu} + \xi^{\rho} R_{\rho\alpha\mu\nu}$ and $e^{\gamma} \wedge * e^{\tau\beta} = g^{\gamma\beta} * e^{\tau} - g^{\gamma\tau} * e^{\beta}$; (5) compare the last terrn to (3.6).

The sum of (B.8) and (B.9) gives then (3.2).

c) In order to analyze the expression (3.18), we compute

$$i(\omega_{\alpha\beta})e^{\alpha} \wedge e^{\beta}{}_{\mu} = - i(e_{\alpha})de^{\alpha} \wedge e_{\mu} + \omega^{\alpha}{}_{\alpha} \wedge e_{\mu} \; ,$$

$$i(\omega_{\alpha\beta})e^{\beta} \wedge e^{\alpha}{}_{\mu} = i(e^{\alpha})de_{\alpha} \wedge e_{\mu} + \omega^{\alpha}{}_{\alpha} \wedge e_{\mu} \; , \qquad \text{(B.10)}$$

$$i(\omega_{\alpha\beta})e_{\mu} \wedge e^{\alpha\beta} = i(e_{\mu})de^{\alpha} \wedge e_{\alpha} - de_{\mu} \; ,$$

so that

$$S_{\mu} = \frac{1}{2}\left[i(e_{\mu})de^{\alpha} \wedge e_{\alpha} - i(e_{\alpha})de^{\alpha} \wedge e_{\mu} - i(e^{\alpha})\,(de_{\alpha} \wedge e_{\mu})\right] . \qquad \text{(B.11)}$$

In a coordinates basis $e^{\alpha} = dx^{\alpha}$, this reduces to

$$S_\mu = -\frac{1}{2}\, i(e^\alpha)\,(de_\alpha \wedge e_\mu). \tag{B.12}$$

If we employ

$$i(e_\alpha)de^\alpha = i(e^\alpha)de_\alpha - \langle dg_{\alpha\beta}, e^\beta\rangle e^\alpha + 2\omega^\alpha{}_\alpha \tag{B.13}$$

$$i(e^\alpha)de_\alpha = -\omega^\alpha{}_\alpha - e_\alpha \Delta e^\alpha . \tag{B.14}$$

(in a coordinate basis, (B.13) = 0) in (B.12), we obtain

$$S_\mu = -\frac{1}{2}\left[\frac{1}{g}(g de_\mu - dg \wedge e_\mu) + \langle dg_{\alpha\beta}, e^\beta\rangle\, e^\alpha{}_\mu\right] \tag{B.15}$$

(note, that $dg = 2g\, \omega^\alpha{}_\alpha$). With

$$\begin{aligned} g de_\mu &= g\, g_{\mu\tau}\, g^{\rho\beta}\, g^{\tau\nu}{}_{,\beta}\, e_{\nu\rho}\,, \\ dg \wedge e_\mu &= g_{\mu\tau}\, g_{,\beta}\, g^{\nu\beta}\, g^{\tau\rho}\, e_{\nu\rho}, \end{aligned} \tag{B.16}$$

we get

$$g de_\mu - dg \wedge e_\mu = g_{\mu\tau}\left[g\, g^{\rho\beta}\, g^{\tau\nu}{}_{,\beta} - g_{,\beta}\, g^{\nu\beta}\, g^{\tau\rho}\right] e_{\nu\rho}$$

$$= \frac{1}{2}\, g_{\mu\tau}\left[g(g^{\nu\tau}\, g^{\rho\beta} - g^{\rho\tau}\, g^{\nu\beta})\right]_{,\beta}\, e_{\nu\rho} - g\langle dg_{\alpha\beta}, e^\beta\rangle e^\alpha{}_\mu\,,$$

and therefore

$$S_\mu = \frac{1}{2}\,\frac{1}{2(-g)}\, g_{\mu\tau}\left[g(g^{\nu\tau}\, g^{\rho\beta} - g^{\rho\tau}\, g^{\nu\beta})\right]_{,\beta}\, e_{\nu\rho}. \tag{B.17}$$

Because of $S_\mu = (1/2)\, S_\mu{}^{\nu\rho}\, e_{\nu\rho}$, this gives equation (3.20) for $S_\mu{}^{\nu\rho}$. To construct the Møller-form V_μ, we only rewrite (B.12):

$$S_\mu = -de_\mu - \frac{1}{2}\, i(e^\alpha)de_{\alpha\mu} . \tag{B.18}$$

d) Consider the characteristic Pontrjagin- and Euler-forms, respectively (apart from a constant, see Ref.22)

$$R_{|\alpha\beta|} \wedge R^{\alpha\beta} = \frac{1}{2} R^{\alpha}{}_{\beta} \wedge R_{\alpha}{}^{\beta} \; , \tag{B.19}$$

$$R_{|\alpha\beta|} \wedge R_{|\gamma\delta|} \; \varepsilon^{\alpha\beta\gamma\delta} = \frac{1}{4} R_{\alpha\beta} \wedge R_{\gamma\delta} \; \varepsilon^{\alpha\beta\gamma\delta} \; , \tag{B.20}$$

where (B.19) is the coefficient of a in (2.15). We rewrite (B.20) by using (A.22) and $R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}$:

$$R_{|\alpha\beta|} \quad R_{|\gamma\delta|} \wedge * \, e^{\alpha\beta\gamma\delta} = R_{|\alpha\beta|} \wedge \; e^{\mu\nu} \wedge * \, (R_{|\mu\nu|} \quad e^{\alpha\beta}) . \tag{B.21}$$

Using (A.26), we compute (note that $R^{\alpha} := i(e^{\alpha}) R_{\alpha\beta}$ corresponds to the Risci-tensor by $R^{\alpha} = R^{\alpha\beta} \; e_{\beta}$; "Ricci-form") the following:

$$R_{|\alpha\beta|} \wedge e^{\mu\nu} \wedge * (R_{|\mu\nu|} \wedge e^{\alpha\beta})$$

$$= R_{|\alpha\beta|} \wedge * \, i(e^{\nu}) \; i(e^{\mu}) \left[R_{|\mu\nu|} \wedge e^{\alpha\beta}\right]$$

$$= R_{|\alpha\beta|} \wedge * \, i(e^{\nu}) \; \left[\frac{1}{2} R_{\nu} \wedge e^{\alpha\beta} + R_{|\mu\nu|} \wedge i(e^{\mu}) e^{\alpha\beta}\right]$$

$$= R_{|\alpha\beta|} \wedge * \left[\frac{1}{2} R \; e^{\alpha\beta} - R_{\nu} \wedge i(e^{\nu}) e^{\alpha\beta} + R^{|\mu\nu|} \; \delta^{\alpha\beta}_{\mu\nu}\right]$$

$$= R_{|\alpha\beta|} \wedge * \, e^{\alpha\beta} \wedge * \, (R_{|\mu\nu|} \wedge * \, e^{\mu\nu}) - R_{\alpha} \wedge * \, R^{\alpha} + R_{|\alpha\beta|} \wedge * \, R^{\alpha\beta} \tag{B.22}$$

so that (B.20) is exactly the coefficient of β in (2.15). But (B.19) and (B.20) are exact (or "essentially exact" in the latter case, see below) in terms of the connection 1-forms $\omega^{\alpha}{}_{\beta}$ and the basis e^{μ}: insert (2.14) into (B.19) to get (use (2.20))

$$R^{\alpha}{}_{\beta} \wedge R^{\beta}{}_{\alpha} = d(\omega^{\alpha}{}_{\beta} \wedge R^{\beta}{}_{\alpha}) - w^{a}{}_{\beta} \wedge \omega^{\beta}{}_{\gamma} \wedge R^{\gamma}{}_{\alpha} \,, \qquad \text{(B.23)}$$

and with the help of

$$\omega^{\alpha}{}_{\beta} \wedge \omega^{\beta}{}_{\gamma} \wedge \omega^{\gamma}{}_{\delta} \wedge \omega^{\delta}{}_{\alpha} = - \omega^{\delta}{}_{\alpha} \wedge \omega^{\alpha}{}_{\beta} \wedge \omega^{\beta}{}_{\gamma} \wedge \omega^{\gamma}{}_{\delta} = 0 \,, \qquad \text{(B.24)}$$

$$\omega^{\alpha}{}_{\beta} \wedge \omega^{\beta}{}_{\gamma} \wedge R^{\gamma}{}_{\alpha} = \frac{1}{3} \; d(\omega^{\alpha}{}_{\beta} \wedge \omega^{\beta}{}_{\gamma} \wedge \omega^{\gamma}{}_{\alpha}) \,, \qquad \text{(B.25)}$$

immediately obtain

$$R^{\alpha}{}_{\beta} \wedge R^{\beta}{}_{\alpha} = d[d\omega^{\alpha}{}_{\beta} \wedge \omega^{\beta}{}_{\alpha} + \frac{2}{3} w^{a}{}_{\beta} \wedge \omega^{\beta}{}_{\gamma} \wedge \omega^{\gamma}{}_{a}] \,. \qquad \text{(B.26)}$$

Likewise, insert (2.14) into (B.20) to get (use $D[R^{\alpha}{}_{\beta} \wedge * e_{\gamma\alpha}{}^{\beta\delta}] = 0$)

$$R_{\alpha\beta} \wedge R_{\gamma\delta} \wedge * e^{\alpha\beta\gamma\delta}$$

$$= d(\omega^{\alpha}{}_{\beta} \wedge R^{\gamma}{}_{\delta} \wedge * e_{\gamma\alpha}{}^{\beta\delta}) - \omega^{\alpha}{}_{\sigma} \wedge \omega^{\sigma}{}_{\beta} \wedge R^{\gamma}{}_{\delta} \wedge * e_{\gamma\alpha}{}^{\beta\delta} \,. \qquad \text{(B.27)}$$

This can also be rewritten with the help of

$$\omega^{\alpha}{}_{o} \wedge \omega^{\sigma}{}_{\beta} \wedge R^{\gamma}{}_{\delta} \wedge * e_{\gamma\alpha}{}^{\beta\delta}$$

$$= \frac{1}{3} \, d(\omega^{a}{}_{\beta} \wedge \omega^{\gamma}{}_{\tau} \wedge \omega^{\tau}{}_{\delta} \wedge * e_{\gamma\alpha}{}^{\beta\delta}) - \frac{1}{3} A(\omega) \,,$$

where

$$A(\omega) := d\omega^{\alpha}{}_{\beta} \wedge \omega^{\gamma}{}_{\delta} \wedge d * e_{\gamma\alpha}{}^{\beta\delta} -$$

$$- \omega^{\alpha}{}_{\sigma} \wedge \omega^{\sigma}{}_{\beta} \wedge \omega^{\gamma}{}_{\tau} \wedge \omega^{\tau}{}_{\delta} \wedge * e_{\gamma\alpha}{}^{\beta\delta}$$

$$= (d\omega^{\alpha}{}_{\beta} + \frac{1}{2} \omega^{\alpha}{}_{\sigma} \wedge \omega^{\sigma}{}_{\beta}) \wedge \omega^{\gamma}{}_{\delta} \wedge d * e_{\gamma\alpha}{}^{\beta\delta} \qquad \text{(B.28)}$$

(the last equation may be obtained by $D * e_{\gamma\alpha}{}^{\beta\delta} = 0$). Note, that $A(\omega)$ vanishes in a basis of constant $\langle \bar{e}^{\alpha}, \bar{e}^{\beta} \rangle = g^{\alpha\beta}$, because $d * e^{\alpha\beta\gamma\delta} = d\varepsilon^{\alpha\beta\gamma\delta} = 0$ (see (A.22) and (A.23)).

Equation (B.27) now reads

$$R_{\alpha\beta} \wedge R_{\gamma\delta} \wedge * e^{\alpha\beta\gamma\delta}$$

$$= d \left[(d\omega^{\alpha}{}_{\beta} \wedge \omega^{\gamma}{}_{\delta} + \frac{2}{3} \omega^{\alpha}{}_{\beta} \wedge \omega^{\gamma}{}_{\tau} \wedge \omega^{\tau}{}_{\delta}) \wedge * e_{\gamma\alpha}{}^{\beta\delta} \right] + \frac{1}{3} A(\omega) . \tag{B.29}$$

As mentioned above (Section 5, see also Ref. 8), to obtain the field equations we are allowed to use an orthonormal basis throughout, so that the last term in (B.29) does not contribute.

The coefficients of a and ß in (2.15) are now show to be exact (the latter at least in a constant, e.g. orthonormal, basis) and therefore donot alter the field equations (as may be seen from

$$\delta \int_A R_{|\alpha\beta|} \wedge R^{\alpha\beta} = 0 , \tag{B.30}$$

$$\delta \int_A R_{|\alpha\beta|} \wedge R_{|\gamma\delta|} \varepsilon^{\alpha\beta\gamma\delta} = 0 , \tag{B.31}$$

which follow from the fact that the variation ofthe variables vanishesat ∂A; see also Ref, 24 for a classical treatment of this topic).

Remark. Although in any nonspecialized basis the Euler-form (B.20) fails to be exact by $1/3\, A(\omega)$, even then it can be shown that it does not contribute to the field equations. To this end, let us abbreviate (B.29) by

$$R_{\alpha\beta} \wedge R_{\gamma\delta} \wedge * e^{\alpha\beta\gamma\delta} = d\, B(\omega) + \frac{1}{3} A(\omega) . \tag{B.32}$$

Since (8.32) is invariant under (2.3), we get

$$d B(\omega) + \frac{1}{3} A(\omega) = d\, B(\bar{\omega}) + \frac{1}{3} A(\bar{\omega}) , \tag{B.33}$$

where $\bar{\omega}^{\alpha}{}_{\beta}$ corresponds to the basis $\{\bar{e}^{\mu}\}$. Explicitly, this can be proven by use of (2.13) and

$$\bar{\omega}^{\alpha}{}_{\beta} \wedge * \bar{e}_{\alpha}{}^{\beta} = \omega^{\alpha}{}_{\beta} \wedge * e_{\alpha}{}^{\beta} - B^{\alpha}{}_{\beta} \wedge * e_{\alpha}{}^{\beta} ,$$

$$d\bar{\omega}^{\alpha}{}_{\beta} \wedge * \bar{e}_{\alpha}{}^{\beta} = d\omega^{\alpha}{}_{\beta} \wedge * e_{\alpha}{}^{\beta} + D B^{\alpha}{}_{\beta} \wedge * e_{\alpha}{}^{\beta} , \tag{B.34}$$

$$\bar{\omega}^{\alpha}{}_{\sigma} \wedge \bar{\omega}^{\sigma}{}_{\beta} \wedge * \bar{e}_{\alpha}{}^{\beta} = \omega^{\alpha}{}_{\sigma} \wedge \omega^{\sigma}{}_{\beta} \wedge * e_{\alpha}{}^{\beta} - DB^{\alpha}{}_{\beta} \wedge * e_{\alpha}{}^{\beta} ,$$

where

$$B^{\alpha}{}_{\beta} := (A^{-1})^{\alpha}{}_{\sigma}\, dA^{\sigma}{}_{\beta} . \tag{B.35}$$

Now take the basis $\{\bar{e}^{\mu}\}$ as constant in the above manner; then $A(\bar{\omega})$ in (B.33) vanishes and we are léft with

$$\frac{1}{3} A(\omega) = d[\bar{B}(\bar{\omega}) - B(\omega)] , \tag{B.36}$$

or explicitly

$$A(\omega) = - d[(2d\omega^{\alpha}{}_{\beta} + D B^{\alpha}{}_{\beta}) \wedge B^{\gamma}{}_{\delta} \wedge * e_{\gamma\alpha}{}^{\beta\delta}] . \tag{B.37}$$

But from (B.34) one has $\delta \int A(\omega) = 0$, since δe^{μ} as well as $\delta\omega^{\alpha}{}_{\beta}$ transform homogeneously (as tensorial quantities). For the latter, this can be seen immediately if one rewrites (2.13) as follows:

$$\bar{\omega}^{\mu}{}_{\sigma} = \omega^{\mu}{}_{\sigma} - D A^{\mu}{}_{\tau} (A^{-1})^{\tau}{}_{\sigma} . \tag{B.38}$$

APPENDIX C

In order to prove the equivalence of the Landau-Lifschitz 3-form to the Landau-Lifschitz pseudo-energy tensor given in Ref.14, we rewrite (3.24) like this (the constant $8\pi k$ reintroduced):

$$* t^{\alpha}_{LL} = - \frac{1}{16\pi k} \omega_{\mu\nu} \wedge [\omega^{\alpha}{}_{\beta} \wedge * e^{\mu\nu\beta} + \omega^{\nu}{}_{\beta} \wedge * e^{\alpha\mu\beta} - \omega^{\sigma}{}_{\sigma} \wedge * e^{\alpha\mu\nu}] . \tag{C.1}$$

With the help of

$$d * e^{\alpha\mu\nu} = - \omega^{\alpha}{}_{\beta} \wedge * e^{\beta\mu\nu} - \omega^{\mu}{}_{\beta} \wedge * e^{\alpha\beta\nu} - \omega^{\nu}{}_{\beta} \wedge * e^{\alpha\mu\beta}$$

$$= \varepsilon^{\alpha\mu\nu\delta} de_{\delta} - \omega^{\sigma}{}_{\sigma} \wedge * e^{\alpha\mu\nu} , \tag{C.2}$$

(C.1) immediately reduces to (see Ref.8)

$$* t^{\alpha}_{LL} = \frac{1}{16\pi k} \varepsilon^{\alpha\beta\gamma\delta} (\omega_{\beta\gamma} \wedge w^{\nu}{}_{\delta} \wedge e_{\nu} - \omega_{\mu\beta} \wedge \omega^{\mu}{}_{\gamma} \wedge e_{\delta}) \tag{C.3}$$

or, in a more compact notation,

$$* t^{\alpha}_{LL} = \frac{1}{8\pi k} e^{\alpha\beta\gamma [\mu} \omega^{\delta]}{}_{\beta} \wedge \omega_{\mu\gamma} \wedge e_{\delta} . \tag{C.4}$$

We express (C.4) in a coordinate basis $\{e^{\mu} = dx^{\mu}\}$ and obtain for the components $(*t^{\kappa}_{LL} = t^{\kappa\lambda}_{LL} * \theta)$

$$16\pi k t^{\kappa\lambda} = 2g^{\lambda\sigma} \delta^{\kappa\rho\tau[\gamma}_{\sigma\mu\nu\theta} \Gamma^{\theta]}{}_{\rho}{}^{\mu} \Gamma_{\gamma\tau}{}^{\nu} , \tag{C.5}$$

using the Christoffel-symbols $\Gamma_{\mu\nu\rho} := \langle\omega_{\mu\nu}, e_{\rho}\rangle$ (equation (A.25) andthe identity $e_{\rho\sigma\delta} \equiv \varepsilon_{\rho\sigma\delta\lambda} * e^{\lambda}$ will be helpful to get (C.5)).

Using the i-elations

$$\delta^{\alpha_1 \ldots \alpha_p | \mu_{p+1} \ldots \mu_n |}_{\beta_1 \ldots \beta_p \mu_{p+1} \ldots \mu_n} \equiv \delta^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_p} \tag{C.6}$$

$$\delta^{\kappa\rho\tau\delta}_{\sigma\mu\nu\theta} \equiv \delta^{\kappa\rho\tau}_{\sigma\mu\nu} \delta^{\gamma}_{\theta} - \delta^{\kappa\rho\tau}_{\sigma\mu\theta} \delta^{\gamma}_{\nu} + \delta^{\kappa\rho\tau}_{\sigma\nu\theta} \delta^{\gamma}_{\mu} \cdot \delta^{\kappa\rho\tau}_{\mu\nu\theta} \delta^{\gamma}_{\sigma} , \tag{C.7}$$

$$\delta^{\kappa\rho\tau}_{\sigma\mu\nu} \equiv \delta^{\kappa\lambda}_{\sigma\mu} \, \delta^{\rho\tau}_{\lambda\nu} + \delta^{\kappa}_{\nu} \, \delta^{\rho\tau}_{\sigma\mu} \, , \tag{C.8}$$

we compute

$$16\pi k \, t^{\kappa\lambda}_{\rm LL} = g^{\lambda\sigma} \Big[- \delta^{\kappa\rho\tau}_{\sigma\mu\nu} \, \Gamma^{\nu}{}_{\rho}{}^{\mu} \, \Gamma_{\gamma\tau}{}^{\gamma}$$

$$+ \delta^{\kappa\rho\tau}_{\sigma\nu\theta} \, \Gamma^{\theta}{}_{\rho}{}^{\gamma} \, \Gamma_{\gamma\tau}{}^{\nu}$$

$$- \delta^{\kappa\rho\tau}_{\mu\nu\theta} \, \Gamma^{\theta}{}_{\rho}{}^{\mu} \, \Gamma_{\sigma\tau}{}^{\nu} \Big]$$

$$= g^{\lambda\sigma} \Big[\delta^{\kappa\rho\tau}_{\sigma\mu\nu} \, (\Gamma^{\nu}{}_{\cdot\rho}{}^{\gamma} \, \Gamma_{\gamma\tau}{}^{\mu} - \Gamma^{\nu}{}_{\rho}{}^{\mu} \, \Gamma_{\gamma\tau}{}^{\gamma}) - \delta^{\kappa\rho\tau}_{\mu\nu\gamma} \, \Gamma^{\gamma}{}_{\rho}{}^{\mu} \, \Gamma_{\sigma\tau}{}^{\nu} \Big]$$

$$= g^{\lambda\sigma} \, \delta^{\gamma\kappa}_{\sigma\mu} \, \delta^{\rho\tau}_{\nu\gamma} \, (\Gamma^{\nu}{}_{\rho\theta} \, \Gamma^{\theta}{}_{\tau}{}^{\mu} - \Gamma^{\nu}{}_{\rho}{}^{\mu} \, \Gamma^{\theta}{}_{\tau\theta})$$

$$+ g^{\lambda\sigma} \, \delta^{\gamma\nu}_{\sigma\mu} \, (\Gamma^{\kappa}{}_{\gamma\theta} \, \Gamma^{\theta}{}_{\nu}{}^{\mu} - \Gamma^{\kappa}{}_{\gamma}{}^{\mu} \, \Gamma^{\theta}{}_{\nu\theta})$$

$$+ \delta^{\kappa\mu}_{\gamma\nu} \, \delta^{\rho\tau}_{\theta\mu} \, \Gamma^{\theta}{}_{\rho}{}^{\gamma} \, \Gamma^{\lambda}{}_{\tau}{}^{\nu} + \delta^{\theta\gamma}_{\mu\nu} \, \Gamma^{\kappa}{}_{\theta}{}^{\nu} \, \Gamma^{\lambda}{}_{\gamma}{}^{\mu} \, .$$

with the help of

$$g^{\lambda\sigma} \, \delta^{\gamma\kappa}_{\sigma\mu} \equiv g^{\lambda\gamma} \, \delta^{\kappa}{}_{\mu} - g^{\lambda\kappa} \, \delta^{\gamma}{}_{\mu} \equiv 2 g^{\lambda[\gamma} \, \delta^{\kappa]}{}_{\mu} \, , \tag{C.9}$$

we obtain *at* last

$$16\pi k \, t^{\kappa\lambda}_{\rm LL}$$

$$= 2 g^{\lambda[\gamma} \, \delta^{\kappa]}{}_{\mu} (\Gamma^{\nu}{}_{\nu\theta} \, \Gamma^{\theta}{}_{\gamma}{}^{\mu} - \Gamma^{\nu}{}_{\gamma\theta} \, \Gamma^{\theta}{}_{\nu}{}^{\mu}$$

$$- \Gamma^{\nu}{}_{\nu}{}^{\mu} \, \Gamma^{\theta}{}_{\gamma\theta} \quad \Gamma^{\nu}{}_{\gamma}{}^{\mu} \, \Gamma^{\theta}{}_{\nu\theta})$$

$$+ 2 g^{\lambda[\gamma} \, g^{\nu]\mu} \, (\Gamma^{\kappa}{}_{\gamma\theta} \, \Gamma^{\theta}{}_{\nu\mu} - \Gamma^{\kappa}{}_{\gamma\mu} \, \Gamma^{\theta}{}_{\nu\theta})$$

$$+ 2g^{\lambda[\gamma} g^{\nu]\mu} (\Gamma^{\theta}{}_{\theta\gamma} \Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\theta}{}_{\mu\gamma} \Gamma^{\lambda}{}_{\theta\nu})$$

$$+ 2g^{\theta[\mu} g^{\nu]\gamma} \Gamma^{\kappa}{}_{\theta\nu} \Gamma^{\lambda}{}_{\gamma\mu}$$

$$= (g^{\lambda\gamma} g^{\kappa\mu} - g^{\lambda\kappa} g^{\gamma\mu}) (2\Gamma^{\nu}{}_{\gamma\mu} \Gamma^{\theta}{}_{\nu\theta} - \Gamma^{\nu}{}_{\gamma\theta} \Gamma^{\theta}{}_{\mu\nu} - \Gamma^{\nu}{}_{\gamma\nu} \Gamma^{\theta}{}_{\mu\theta})$$

$$+ g^{\lambda\gamma} g^{\mu\nu}(\Gamma^{\kappa}{}_{\gamma\theta} \Gamma^{\theta}{}_{\mu\nu} + \Gamma^{\kappa}{}_{\mu\nu} \Gamma^{\theta}{}_{\gamma\theta} - \Gamma^{\kappa}{}_{\nu\theta} \Gamma^{\theta}{}_{\gamma\mu} - \Gamma^{\kappa}{}_{\gamma\mu} \Gamma^{\theta}{}_{\nu\theta})$$

$$+ g^{\kappa\gamma} g^{\mu\nu}(\Gamma^{\lambda}{}_{\gamma\theta} \Gamma^{\theta}{}_{\mu\nu} + \Gamma^{\lambda}{}_{\mu\nu} \Gamma^{\theta}{}_{\gamma\theta} - \Gamma^{\lambda}{}_{\nu\theta} \Gamma^{\theta}{}_{\gamma\mu} - \Gamma^{\lambda}{}_{\gamma\mu} \Gamma^{\theta}{}_{\nu\theta})$$

$$+ g^{\mu\gamma} g^{\nu\theta}(g^{\lambda}{}_{\gamma\nu} \Gamma^{\kappa}{}_{\mu\theta} - \Gamma^{\lambda}{}_{\gamma\mu} \Gamma^{\kappa}{}_{\nu\theta})$$

$$= 16\pi k \; t_{LL}^{\lambda\kappa} \; . \qquad (C.10)$$

The last equation and therefore the symmetry of $t_{LL}^{\kappa\lambda}$ in the coordinate basis seems to be obvious. We finally arrived at the Landau-Lifschitz expression given in (Ref.14), which can be readily verified, setting $\lambda \to i$, $\kappa \to k$, $\gamma \to \ell$, $\mu \to m$, $\nu \to n$ and $\pi \to p$.

Remark. Of course, it would have been much easier to write out (C.5) explicitly as it stands, but some of the above manipulations (like index-rearrangings frequently involved) are made to facilitate the comparison with the original Landau-Lifschitz expression. Nevertheless, it shows once more two attributes of the Cartan-formalism employed in (3.24) - its expediency and beauty.

Acknowledgement. The authors would like to thank Doz. H.K. Urbantke for stimulating discussions and valuable hints.

REFERENCES

1. Misner C.W., K.S. Thorne, J.A. Wheeler, *Gravitation*, Freeman (1973).
2. Weber J., *General Relativity and Gravitational Waves*, Interscience (1961).
3. Hawking S.W., G.F.R. Ellis, The *Large Scale Structure of the Universe*, Cambridge (1973).
4. Goldberg J.N., Phys. Rev. 111, 315 (1958).
5. Bergmann P.G., R.Schiller, Phys. Rev. 89, 4 (1953), Bergmann P.G., Phys. Rev. 112, 287 (1958).
6. Trautman A., *On the Structure of the Einstein-Cartan Equations* IFT /72/13, Warszawa (1972), Trautman A., "Theory of Gravitation", in The *Physicist's Conception of Nature*, J. Mehra ed., Reidel (1973).
7. Lovelock D., H. Rund, "Variational Principles in the General Theory of Relativity", in J.R. Vanstone: *Differential Topology, Differential Geometry and Applications*, Proceedings of the Thirteenth Biennial Seminar of the Canadian Mathematical Congress, 1, (1972), and Jahresberichte der Deutschen Mathematiker-Vereinigung 74, No. 1/2 (1972).
8. Thirring W., *Klassische Feldtheorie*, Springer (1978).
9. De Witt B.S., The *Dynamical Theory of Groups and Fields*, Gordon and Breach (1965).
10. Komar A., Phys. Rev. 113, 934 (1959), Phys. Rev. 127, 1411 (1962), Phys. Rev. 129, 1837 (1963), J. Math. Phys. 11, 820 (1970).
11. Møller C., Ann. Phys. 4, 347 (1958), Ann. Phys. 12, 118 (1961).
12. Trautman A., Conservation Laws in General Relativity, in L.Witten, *Gravitation, an Introduction to Current Research*, Wiley (1962).
13. Von Freud P., Ann. Math. 40, 417 (1939).
14. Landau L.D., E.M. Lifschitz, *Klassische Feldtheorie*, Akademie-Verlag, Berlin (1966), or, *The Classical Theory of Fields*, Pergamon Press (1975).
15. Bergmann P.G., R. Thomson, Phys. Rev. 89, 400 (1953).
16. Arnowitt R.L., S. Deser, C.W. Misner, " The Dynamics of General Relativity", in L. Witten, see Ref. 12 (1962).
17. Regge T., C. Teitelboim, Ann. Phys. 88, 286 (1974), De Witt B.S., Phys. Rev. 160, 1113 (1967).
18. Misner C.W., Phys. Rev. 130, 1590 (1963).
19. Steenrod N., *The Topology of Fiber Bundles*, Princeton (1951).

20. Stiefel E., "Richtungsfelder und Fernparallelismus in Mannigfaltigkeiten", Comm. Math. Helv. 8, 3 (1936).

21. Geroch R., J. Math. Phys. 9, 1739 (1968); *11*, 343 (1970).

22. Kobayashi S., K. Nomizu, *Foundations of Differential Geometry*, Interscience (1963/69).

23. Flanders H., *Differential Forms*, Academic Press (1963); Spivak M., *Calculus on Manilfolds*, Benjamin (1965); Holman H., H. Rummler, *Alternierende Differentialformen*, BI (1972); Goldberg S. I., *Curvature and Homology*, Academic Press (1962); Greub W., S. Halperin, R. Vanstone, *Connections, Curvature and Cohomology*, Academic Press (1972/73/74).

24. Bach R., Math. Zeitschr. 9, 110 (1921), Weitzenböck R., Sitzb. d. mathem.-naturw. Kl., Abt. IIa, 130, 15 Wien (1921), Juettner F., Math. Ann. 87, 270 (1922), Lanczos C., Ann. of Math. 39, 842 (1938).