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Solid Neutron Matter: The Energy Density in the Relativistic Harmonic Approximation

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A relativistic expression for the energy density as a function of particle density for solid neutron matter is obtained using Dirac's equation with a truncated harmonic potential. Ultrabaric and superluminous effects are not found in our approach.

Usando a equação de Dirac com um potencial harmônico truncado, obtemos uma expressão relativística para a densidade de energia como função da densidade de partículas da matéria neutrônica sólida. Na nossa aproximação, não encontramos efeitos ultrabáricos ou superluminosos.

1. INTRODUCTION

In the past few years special attention has been devoted to the problem of the possible solidification of the core of a neutron star[1]. Several attempts to calculate the equation of state of the dense neutron matter have been published. In our opinion the most plausible one are those of Barnerjee *et al.*[2], Canuto and Chitre[3], Pandharipande[4] and Guyer and Takemori[5].

Banerjee *et al.*[2] proposed a solid-body model for the dense matter at zero temperature. The neutrons are assumed to form a body-centered cubic lattice, with a lattice parameter A. This approach is based on the assumption that when the nuclear forces become sufficiently repulsive, a possible minimum energy state can be achieved by keeping the neutrons as far apart as possible by localizing them at lattice sites. The calculations are done in the harmonic approximation using the classical Debye model with Reid's soft core potential as the interaction between neutrons. All calculations were performed in a non-relativistic approximation.

Canuto and Chitre[3], used a non-relativistic quantum-mechanical treatment to calculate the energy of the neutron lattice. They assumed that the neutrons oscillate harmonically around their equilibrium positions at the lattice sites. The characteristic frequency ω of the harmonic oscillator and the spread of the wave function of the particle are assumed to depend on the lattice distance A. The frequency ω is obtained by the Hartree method taking the two body nucleon-nucleon potential as Reid's phenomenological soft-core potential. Correlations between pairs of oscillators are taken into account. In addition, a FCC structure for the neutron lattice was used to obtain the lowest possible value for the energy of the partictes. They considered densities up to 8×10^{15} g/cm^3 and have shown that only for densities larger than 1.5×10^{15} g/cm^3 the lattice is stable.

On the other hand, Pandharipande[4] and Takemori and Guyer[5] analysed the possibility of solidification using several versions of Reid's interaction. They conclude that for none of these a liquid-solid phase transition is found for densities up to 5×10^{15} g/cm^3.

In our opinion, the problem of solification remains unsolved and we can only speculate about this point.

If solidification is possible, relativistic effects will be significant for densities higher than 2×10^{15} g/cm^3, since the region of confinement of the neutron (see Canuto and Chitre[3]) will be smeller than 0.8 fermi, comparable to the neutron Compton wavelength, $\lambda = \hbar/mc$ which is about 0.2 f'ermi.

In the present work our purpose is, assuming that the core of the neutron star is solid, to obtain an equation of state for densities higher than 10^{15} g/cm^3 taking into account relativistic effects.

We are only interested in constructing a simple model which is capable to show the general features of a relativistic approach. We do not in-tend to obtain an exact equation of state using many-body techniques and accurate experimental results for neutron-neutron interaction[1-5].

With this in mind it is enough to consider the most simple model for a crystal which is Einstein model: each neutron is bound elastically to its own equilibrium position, each oscillating about its own origin. The motion of each neutron will affect the motion of the neighbors, but we neglected this coupling. It seems to be justified since the vibrational frequencies, due to the confinement of neutrons at these densities, are extremely high. We consider each neutron in the lattice to be a three-dimentional harmonic oscillator with frequency ω [3].

In the absence of field theoretical methods applicable to the relativistic region we propose to study the energy spectrum of the neutron by means of a static potential acting as the fourth component of a four vector in Dirac's equation. However, as well known, the Klein Paradox or its manifestations are present in the relativistic approach: when the potential energy becomes larger and larger with the distance, creation and annihilation processes become increasingly important. Consequently we get instabilities in the system and the solidification could be questionable.

To avoid these difficulties it is necessary to assume that neutron-neutron potential is soft[6]. So, the potential energy $V(r)$ for each neutron cannot increase indefinitely, so we will put $V(r) = 1/2\, m\omega^2 r^2$ (Ref.3) for $r \leq a$ and $V(r) = V_0 = 1/2\, m\omega^2 a^2$ for $r \geq a$ (the parameter $\underline{a}$, as will be seen later, is given by $\Delta \cong 2a$).

Considering this potential $V(r)$, we evaluate in section (2) the relativistic eigenvalues for the bound states of the neutron using Dirac's equation.

We note that in our simple model, V_0 is only a parameter adjusted to permit the existence of neutron bound states. In that manner the ground state energy $\underline{e_0}$ of the neutron becomes the minimum compatible with a given lattice configuration[3].

Our model, with a static potential, was inspired by Walecka[7] who has shown that the vector mesons play a dominant role in the nucleon-nucleon interaction at high densities and that the vector field A_μ can be replaced by $\vec{A} = 0$ and $A_4 = i\phi$. In our paper the total potential of the system formed by N neutron is

$$\frac{1}{2} \sum_{i \neq j=1}^{N} \phi(\vec{r}_1 - \vec{r}_j) ,$$

where $\vec{r}_i$ denotes the position of the i-th neutron, is approximated by the potential energy of N independent harmonic oscillators.

It must be stressed that the soft core approximation will not be rigorously consistent with our scheme if more mesons than vector mesons are needed to describe the soft-core. However, up to now, this is an open question[3].

A similar procedure was adopted by Cazzola et *al.*[8] but, instead of the harmonic cell, they have used a square well. Our description, involving a harmonic potential seems to be more realistic than the previous treatment since the nucleon-nucleon interaction, as required by the experimental facts, is soft.

2. SOLUTIONS OF DIRAC'S EQUATION

Let us first solve Dirac's equation for the harmonic potential.

Considering the radial Dirac's equation[9,10] and assuming that $\vec{A} = 0$ and $V = 1/2\ Kr^2$ (see section 1). we have:

$$\frac{dR_a}{dr} = \frac{\chi}{r} R_a + \left\{\frac{mc^2 - E}{\hbar c} + \frac{V}{\hbar c}\right\} R_b \qquad (1)$$

$$\frac{dR_b}{dr} = -\frac{\chi}{r} R_b + \left\{\frac{mc^2 + E}{\hbar c} - \frac{V}{\hbar c}\right\} R_a \qquad (2)$$

where R_a and R_b are the small and large components, respectively, and $\chi = -(L+1)$ if $j = \ell + 1/2$ and $\chi = +L$ if $j = L - 1/2$.

Defining $k \equiv mc^2$, $\xi \equiv (m\omega/\hbar)^{1/2}\, r$ and $E \equiv \eta\, \hbar\omega + mc^2$, equations (1) and (2) become:

$$\frac{dR_a}{d\xi} = \frac{\chi}{\xi} R_a + (\varepsilon_- + A\,\xi^2) R_b \qquad (3)$$

$$\frac{dR_b}{d\xi} = -\frac{\chi}{\xi} R_b + (\varepsilon_+ - A\,\xi^2) R_a \qquad (4)$$

where $\varepsilon_- = -\eta\varepsilon$, $\varepsilon_+ = \eta\varepsilon + 2/\varepsilon$, $A = \varepsilon/2$ and $\varepsilon = (\hbar\omega/mc^2)^{1/2}$. One easily verifies that for $\varepsilon \to 0$ (Ref.11) the non-relativistic limit is obtained, namely, $R_a \to 0$ and R_b obeys the radial equation for the non-relativistic harmonic oscillator:

$$\left[\frac{d^2}{d\xi^2} - \frac{\ell(\ell+1)}{\xi^2} - \xi^2 + 2\eta\right] R_b = 0$$

Equations (3) and (4) can be solved by expanding R_a and R_b into power series. For $\chi = +\ell$ $(j = \ell - 1/2)$ we have:

$$R_a = \xi^{\ell} \frac{\varepsilon}{2} \sum_{m=0}^{\infty} A_m\, \xi^{2m} \qquad (5)$$

and

$$R_b = \xi^{\ell+1} \sum_{m=0}^{\infty} B_m\, \xi^{2m} \qquad (6)$$

where $A_0 = 1$,

$$A_m = \frac{1}{2m}\left[\frac{\varepsilon_+\varepsilon_-\, A_{m-1}}{2\ell + 2m - 1} + A\, A_{m-2}\left(\frac{\varepsilon_+}{2\ell + 2m - 3} - \frac{\varepsilon_-}{2\ell + 2m - 1}\right) - A^2\, \frac{A_{m-3}}{2\ell + 2m - 3}\right]$$

for $m \geq 1$, with $A_{-m} = 0$ and

$$B_m = \frac{\varepsilon}{2(2\ell + 2m + 1)} (\varepsilon_+ A_m - A A_{m-1})$$

For $\chi = -(\ell + 1)$ $(j = \ell + 1/2)$ we have:

$$R_a = \xi^{\ell+2} \sum_{m=0}^{\infty} A_m \xi^{2m} \tag{7}$$

and

$$R_b = \xi^{\ell+1} \sum_{m=0}^{\infty} B_m \xi^{2m} \tag{8}$$

where

$$A_0 = \frac{\varepsilon_-}{2\ell + 3}, \qquad A_1 = \frac{1}{2\ell + 5} \left[\frac{\varepsilon_+ \varepsilon_-^2}{2(2\ell + 3)} + A\right],$$

$$B_1 = \frac{1}{2} \frac{\varepsilon_+ \varepsilon_-}{(2\ell + 3)}, \quad B_2 = \frac{1}{4} (\varepsilon_+ A_1 - A A_0),$$

$$A_2 = \frac{1}{2\ell + 7} (\varepsilon_- B_2 + A B_1),$$

$$A_m = \frac{1}{2\ell + 2m + 3} \left[\frac{\varepsilon_+\varepsilon_-}{2m} A_{m-1} + A A_{m-2} \left(\frac{\varepsilon_+}{2m - 2} - \frac{\varepsilon_-}{2m}\right) - \frac{A^2 A_{m-3}}{2m - 2}\right]$$

for $m \geq 3$

and

$$B_m = \frac{1}{2m} (\varepsilon_+ A_{m-1} - A A_{m-2}) \qquad \text{for } m \geq 1$$

It is in general not possible to write R_a and R_b in closed form. This can be done only for $\varepsilon = 0$.

Figures 1, 2, 3 and 4 are a plot of $R_a(\xi)/\xi$ and $R_b(\xi)/\xi$ as function of ξ and ε.

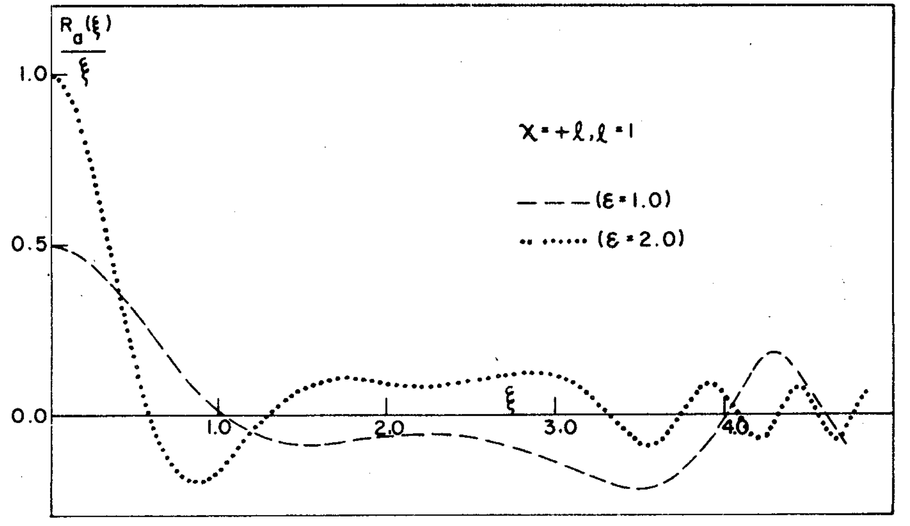

Fig.1 - $R_a(\xi)/\xi$ as a function of ξ and ε, for $\chi = +\ell = 1$ and $\eta = 5/2$.

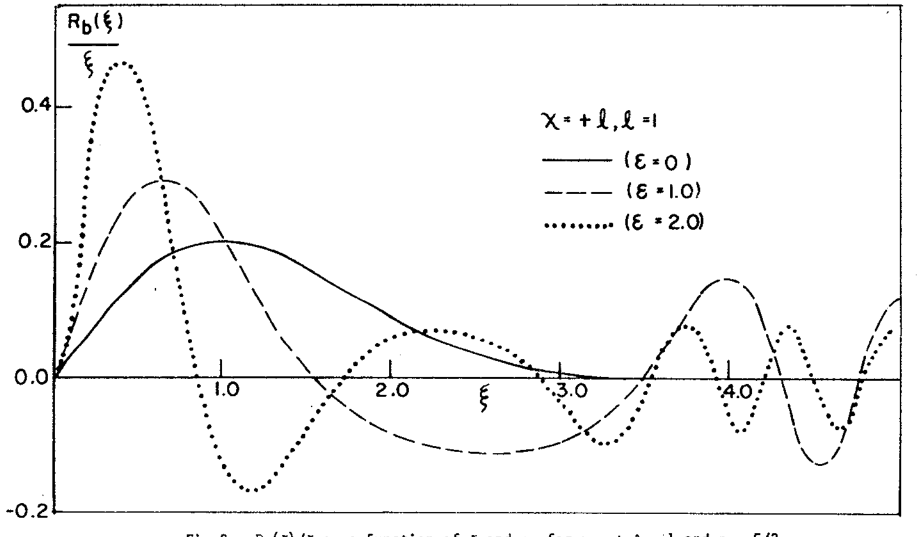

Fig.2 - $R_b(\xi)/\xi$ as a function of ξ and ε, for $\chi = +\ell = 1$ and $\eta = 5/2$.

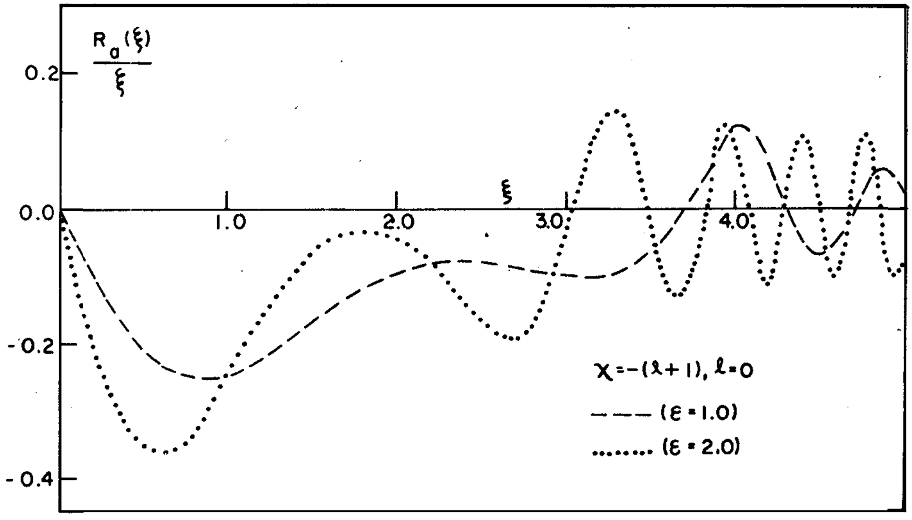

Fig.3 - R $(\xi)/\xi$ as a function of 5 and ε, for $\chi=-(\ell+1)=-1$ and $\eta=3/2$.

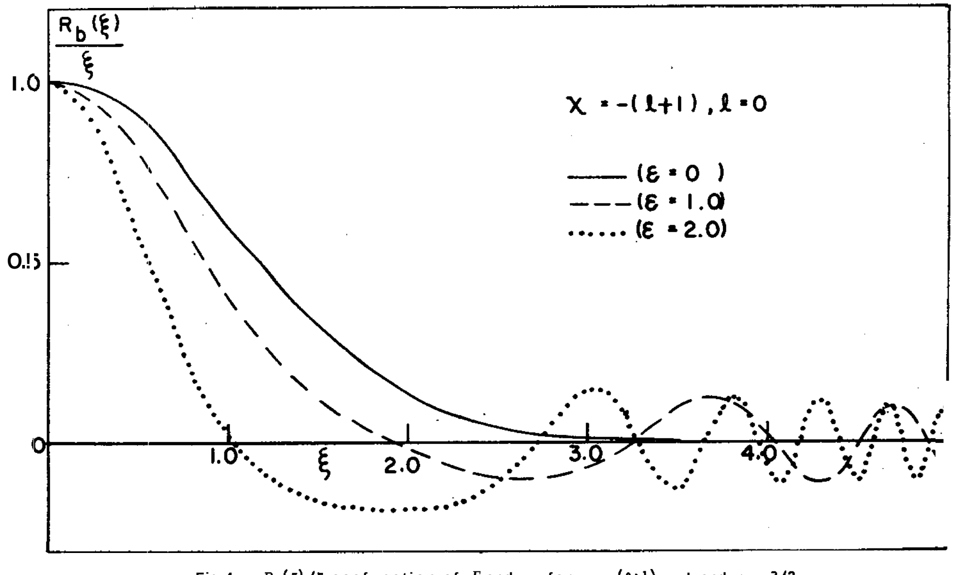

Fig.4 - $R_b(\xi)/\xi$ asafunction of E and ε, for $\chi=-(\ell+1)=-1$ and $\eta=3/2$.

Figures 1 and 2 show the case $\chi = + L$ with $\ell = 1$ and $\eta = 5/2$; figures 3 and 4 show the case $\chi = -(\ell+1)$ with $\ell = 0$ and $\eta = 3/2$.

For $\varepsilon = 0$ (non-relativistic limit) we obtâin from equation (6) that $R_b = \xi \exp(-\xi^2/2)$ for $\chi = -(\ell+1) = -1$ and from equation (8) that $R_b = \xi^2 \exp(-\xi^2/2)$ for $\chi = +\ell = +1$, which are the radial wavefunctions for the harmonic oscillator for the states $1s$ and $1p$, respectively. From equations (5) and (7) we see that $R_a = 0$, in both cases, if $\varepsilon = 0$.

For $\varepsilon > 0$ and for large values of ξ (let us say $\xi \geq \xi_c$, where ξ_c is a critical value) we see from equations (3) and (4) that $R_a \doteq \cos(A\ \xi^3/3 + \theta)$ and $R_b \doteq -\sin(A\ \xi^3/3 + \theta)$ which means that the neutron cannot be bound by the harmonic potential. This effect is known as Klein's paradox[9,12,13]: when the potential becomes of the same order of magnitude as the nucleon mass the vacuum polarization becomes important and the one particle description is no longer correct. That is, for $\xi \geq \xi_c$ Dirac's equation gives unsatisfactory results. We shall show that with a truncated harmonic potential at $r = a$, corresponding to $\xi_a = \varepsilon\, a/\lambda$ bound states do exist. Since in all cases considered in this paper, we verified that $\xi_c \gg \xi_a$, we have a guarantee that Dirac's equation can be used satisfactorily.

The radial eigenfunction for the constant potential $V(r) = V_0$ can be found from equations (1) and (2). It is known that the eigenfunctions that correspond to bound states are:

$$R_a = \sqrt{e - v_0 - 1}\ \xi\ h^{(1)}_{\ell'}\ (iq\xi/\varepsilon)\ \tilde{\omega}\ \varepsilon(e - v_0)$$

and (9)

$$R_b = \sqrt{e - v_0 + 1}\ \xi\ h^{(1)}_{\ell}\ (iq\xi/\varepsilon)$$

where $e = E/mc^2$, $v_0 = V_0/mc^2$, $\ell' = \ell + 1$ for $\chi = -(\ell + 1)$, $\ell' = \ell - 1$ for $\chi = \ell$, $\varepsilon(e - v_0)$ is the function signal of $e - v_0$, $\tilde{\omega}$ the parity of the state, $h^{(1)}_{\ell}$ the spherical Hankel functions of first order and $q = \sqrt{1 - (e - v_0)^2}$ is real.

The complete eigenfunctions ψ_a^{int} and ψ_b^{int} for $r \leq a$ are:

$$\psi_a^{int} = i\, C_{int} \frac{R_a}{\xi} Y^m_{\ell' j}$$
$$\psi_b^{int} = C_{int} \frac{R_b}{\xi} Y^m_{\ell j} \tag{10}$$

For $r \geq$ a we have:

$$\psi_a^{ext} = i\, C_{ext} \sqrt{e - v_0 - 1}\, h^{(1)}_{\ell'} (iq\xi/\varepsilon)\, \tilde{\omega}\, \varepsilon(e - v_0)\, Y^m_{\ell' j}$$
$$\psi_b^{ext} = C_{ext} \sqrt{e - v_0 + 1}\, h^{(1)}_{\ell} (iq\xi/\varepsilon)\, Y^m_{\ell j} \tag{11}$$

where R_a and R_b for $\chi = + \ell$ are given by equations (5) and (6), respectively, and for $\chi = -(\ell+1)$ by (7) and (8), respectively, C_{int} and C_{ext} are normalization constants and $Y^m_{\ell j}$ the function of the total angular momentum (j,m), formed by the composition of a spin 1/' with the spherical harmonics of order R.

For r = a, corresponding to $\xi_a = \varepsilon a/\lambda\!\!\!^-$, we must have $\psi_a^{int} = \psi_a^{ext}$ and $\psi_b^{int} = \psi_b^{ext}$. This gives:

$$\frac{R_a(\varepsilon a/\lambda\!\!\!^-)}{\sqrt{e - v_0 - 1}\, h^{(1)}_{\ell'} (iqa/\lambda\!\!\!^-)} = \frac{R_b(\varepsilon a/\lambda\!\!\!^-)}{\sqrt{e - v_0 - 1}\, h^{(1)}_{\ell} (iqa/\lambda\!\!\!^-)}\, \varepsilon(e - v_0)\tilde{\omega} \tag{12}$$

From this relation we can determine the energy levels e in a graphical way.

3. DENSITY OF ENERGY AND COMMENTS

In this section we obtain the density of energy $\underline{u}$ as a function of the density ρ of the neutron matter at zero temperature. According to the model suggested in section (1), we have $u = n\, e_0$, where n is the neutron density and e_0 is the ground state energy of one neutron for a given lattice distance Δ.

To calculate e_0 we proceed as follows: we fix $\underline{a}$ in equation (12) and look for a potential V_0 that gives the minimum value for the ground state energy $\underline{e}$ of the neutron. This minimum value is taken to be e_0.

In all cases analysed here the wavefunctions are concentrated essentially in the harmonic well. So, the lattice distance Δ is given approximately by $\Delta \cong 2a$.

Since $V_0 = (1/2)\, m\omega^2 a^2$ the characteristic frequency ω depends on Δ and consequently on the matter density as occurs in the model of Canuto and Chitre [3].

We considered only two cases: 1) $j = 1/2$, $\ell = 1$ ($\ell' = 0$) and 2) $j = 1/2$, $\ell = 0$ ($\ell' = 1$). These cases are the two lowest energy states having the lowest centrifugal barrier.

Following Canuto and Chitre [3], we considered the FCC structure as the favorite for the solidified phase. Another structure, like BCC, gives higher values for the energy.

In Figure 5 our results for the energy density u (erg/cm^3), as a function of the density ρ_0 (g/cm^3) $= n\, m$ (Ref.14), are compared with those of Banerjee *et al.* and of Canuto and Chitre. The energy density for $j = 1/2$ and $\ell = 1$ ($\ell' = 0$) will be indicated by $u_1(\rho_0)$ and for $j = 1/2$ and $\ell = 0$ ($\ell' = 1$) by $u_2(\rho_0)$.

We observe that only for densities less than 4×10^{15} g/cm^3 the energy density $u_{CC}(\rho_0)$ obtained by Canuto and Chitre is lower than $u_2(\rho_0)$. Our results $u_1(\rho_0)$ coincide with $u_{CC}(\rho_0)$ for $\rho_0 \lesssim 10^{15}$ g/cm^3 and are lower than $u_{CC}(\rho_0)$ for $\rho_0 > 10^{15}$ g/cm^3. For densities higher than 40×10^{15} g/cm^3, $u_2(\rho_0)$ become lower than $u_1(\rho_0)$.

We have shown in section (2) that when $\varepsilon \to 0$ the non-relativistic approach can be used. On the other hand, as ε becomes of the order of, or

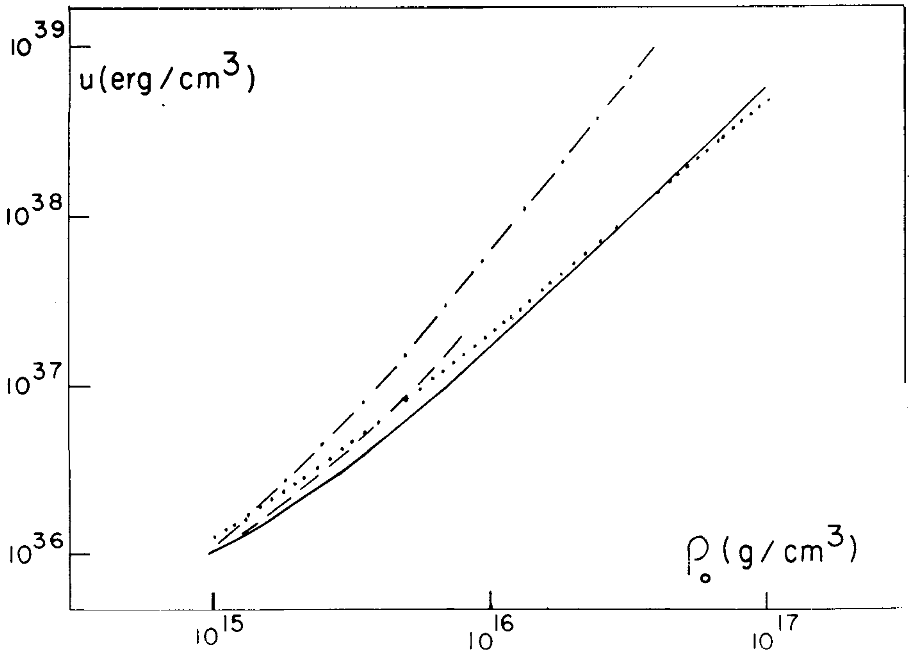

Fig.5 - The energy density u as a function of the density ρ_0. The computed values are indicated by: Banerjee et *al.* (— . —); Canuto and Chitre (— — —); our u_1 (—) and our u_2 (. . .).

greater thar 1, the relativistic treatment must be used. Of course, ε increases as ρ_0 increases. For our conditions, ε runs from 0.5 up to 2.2, meaning that ω assumes values ranging from 3.7×10^{23} rd/s to 7.7×10^{24} rd/s.

For $\rho_0 \gtrsim 5.0 \times 10^{15}$ g/cm^3, we can put, approximately, $u_1 = 10^{13.25}\, \rho_0^{1.50}$ and $u_2 = 10^{15.45}\, \rho_0^{1.36}$. So, for these densities, the pressure defined by

$$P = \left[\rho_0^2 \frac{\partial}{\partial \rho_0} (u/\rho_0)\right]_{T=0}$$

is given by $P_1 = 0.50\ u_1$ and $P_2 = 0.36\ u_2$, respectively. This means that the sound velocity c_s divided by the light velocity c is, in case 1), given by $(c_s/c)^2 \cong 0.50$ and, in case 2), by $(c_s/c)^2 \cong 0.36$.

An important result of our simple model is that $(c_s/c)^2$ is of the order 1/3, rather than 1 (Ref.1) . Therefore, no superluminous or ultrabaric effects appear in this model.

A rough estimate of e_0 in the non-relativistic limit gives $e_0 \sim \hbar^2/ma^2 + mc^2$. The energy $\hbar^2/ma^2$ is essentially the kinetic energy of one particle confined in a region of radius a that can be obtained by using the uncertainty relations. Thus, inside this region, the shape of the potential does not play a significant role in determining u (Ref.15). It is the kinetic energy due to the uncertainty relations, which competes against the externally applied pressure to determine the equilibrium configuration of the solid. In this way, we must expect that, for low densities, our results for u tend to be similar to those obtained by Banerjee et *al.*[2] and Canuto and Chitre[3] as is shown in figure 5.

Also in the relativistic limit, if we assume that the neutron matter is solid, we believe that the energy density u is essentially governed by the uncertainty relations. The absence of ultrabaric and superluminous effects probably is due to the softness of the potential.

These arguments, in some sense, give a support for choosing the simple model for the solid assumed in section 1.

As a final remark, since the neutron is in a bound state we must have $V_0 > \hbar^2/ma^2 = (\lambda\!\!\!^{-}/a)^2\, mc^2$, where $\lambda\!\!\!^{-}$ is the Compton wavelength of the neutron. As in our conditions $a \gtrsim \lambda\!\!\!^{-}$ it is expected that $V_0 \sim mc^2$. Indeed, we verified that V_0 varies from 0.35 BeV up to 6.0 BeV when ρ_0 goes from 10^{15} g/cm^3 up to 10^{17} g/cm^3. When $\rho_0 > 5 \times 10^{15}$ g/cm^3 V_0 becomes larger than 4 BeV which seems to be the maximum value for the potential energy between two nucleons[6]. So, one could interpret this by saying that for $\rho_0 > 5 \times 10^{15}$ g/cm^3 there would be no physical conditions to confine the neutron inside the cell and, consequently, solidification becomes questionable. However, due to the crudeness of our model this argument must be taken with care.

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