Revista Brasileira de Física, Vol. 8, Nº 3, 1978

Bound States af a Power-Law Confinement Potential*

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Recebido em 16 de Março de 1978

We study the spectroscopy of potentials of the type $V(r) = Kr^n + C (n>0)$ with special reference to the decay widths and hyperfine splitting of the narrow resonances in the context of the charmonium model. The bound state spectra and other observables are not very sensitive to the exponent n, and one cannot account for the observed values of the above mentioned quantities. We have also examined the consequence of changing the standard assumption of assigning $\psi'(3684)$ to be the first radial excitation. This allows one to obtain better agreement for the hyperfine splitting but introduces some difficulties elsewhere.

Estudamos a espectroscopia de potenciais do tipo $V(r) = Kr^n + C$ $(n>0)$ dando ênfase às larguras de decaimento e à separação hiperfina das ressonâncias estreita., no contexto do modelo do charmônio. O espectro dos estados ligados e outros observáveis não são muito sensíveis ao expoente n e não conseguimos reproduzir todos os valores observados das quantidades acima mencionadas. Examinamos tambem as consequências de mudar a hipótese usual de ser $\psi'(3684)$ a primeira excitação radial. Isso nos permite obter melhor concordância para a separação hiperfina, o que todavia dá lugar a outras dificuldades.

1. INTRODUCTION

The color-gauge theory of strong interactions has attracted a lot of attention in recent years[1]. In this framework, one can have a mechanism of

* Work supported in part by CNPq, Brazil.

quark-confinement through the fact that the gauge coupling of the quarks to color gluons increases with separation. On the other hand, this coupling is small at short distances and may account for the observed Bjorken scaling. The "new particles" have also been analysed using this "asymptotically free gauge theory". Here, the large masses of the fundamental constituents involved may reduce the effective coupling constant to such an extent that pertubative calculatiops become meaningful[2].

The properties of $\psi(3095)$, $\psi'(3684)$ and other narrow resonances have encouraged people to study the spectroscopy of a high mass fermion pair (generically called "charmed quark"), bound by a phenomenological confining potential which becomes small at small distances[3]. The non relativistic treatment iis justified by the supposedly weak binding. The case of a linear potential and a combination of linear and Coulomb potentialshas already been studied extensively[4]. Taking $\psi(3095)$ and $\psi'(3684)$ respectively as the $c\bar{c}$ bound states 1^3S_1, and the first radial excitation 2^3S_1, permits one to deterrnine the potential strength parameter K and the constant C, foi- arbitrary values of the charmed quark mass m_c which is treated as a variable parameter, and then one predicts other excited levels and attempts to identify some of these with the experimentally observed states. A linear potential leads to a mass difference between 1^3S_1 and 1^1S_0 states (ΔM hyperfine splitting) which is'by a factor of three too small compered with the possible experimental observation[5].

In the present work, we study the spectroscopy of a more general potential, namely $V(r) = Kr^n + C (n>0)$ with special reference to the hyperfine splitting and leptonic decay width of the narrow resonances. Our results, given in Tables 2, 5, 6 and 7, show the welcome feature that qualitatively the bound state spectrum is not very sensitive tu the positive exponent n of the potential. Next, we abandon the assumption that ψ'(3864) is the first radial excitation of $\psi(3095)$ and using as input the mass of ψ, (m_ψ), and the leptonic decay width of ψ, $\Gamma(\psi\to\ell\bar{\ell})$, also as an input, one is able to increase ΔM but it still remains smaller than the experimental result. The first radial excitation goes up to about 4 GeV.

2. MASSES OF HIGHER RESONANCES

The radial part of the Schrödinger equation in the CM system is ($c = \hbar = 1$)

$$\frac{d^2}{dr^2} u(r) + \{m_c(E - V(r)) - \frac{\ell(\ell+1)}{r^2}\} u(r) = 0 \ , \qquad (1)$$

where the radial wave function is $R(r) = u(r)/r$, and m_c denotes the mass of the charmed quark. For potentials of the type $V(r) = Kr^n + C (n > 0)$, C is a constant which is included to take into account the fact that because of confinement one cannot compute absolute energies but only energy differences. The S-wave part of $u(r)$ must vanish at the origin.

Since we can not in general solve Eq.(1) analytically, we may use the WKB-approximation or a variational calculation. The WKB method is a good approximation for energy eigenvalues for stationary states; also, it improves with increasing radial excitation index N. For low excitations, the validity of the method depends on the particular problern under consideration. For example, this approximation gives good energy eigenvalues for the linear potential and reproduces exact results for the harmonic oscillator potential. The validity of this approximation is tested by comparing the results with numerical solution or variational calculations. On rare occasions, one can even check it against analytical solutions.

The N^{th} energy eigenvalue for $\ell = 0$ is given by[6]

$$\int_0^{r_0} \sqrt{m_c(E'_N - Kr^n)}\, dr = (N + \tfrac{3}{4})\pi \ , \qquad (2)$$

where r_0 is the clasrical turning point such that $E' - Kr^n = 0$ and $E'_N = E_N - C$. Making a change of variables $Y = (K/E'_N)r^n$, Eq.2 leads to

$$(E'_N)^{(2+n)/2n} = \frac{2\sqrt{\pi}\, nK^{1/n}\, (N + 3/4)\Gamma((2 + 3n)/2n)}{m_c^{1/2}\, \Gamma(1/n)} \ . \qquad (3)$$

We have used[7]

$$\int_0^1 (1-Y)^{1/2}\, Y^{1/n-1}\, dY = \beta(1/n,\ 3/2) = \frac{\sqrt{\pi}\ \Gamma(1/n)}{2\ \Gamma((2+3n)/2n)}\ , \tag{4}$$

where $\beta(r,s)$ and $\Gamma(z)$ are the usual Beta and Gamma functions.

The mass of the N^{th} bound state is determined by adding the N^{th} energy eigenvalue to the rest mass of the constituents:

$$M_N = 2m_c + E'_N + C \ . \tag{5}$$

The two parameters, K and C, may be fixed by assuming that the observed $\psi(3095)$ and $\psi'(3684)$ are the ground state $(N = 0)$ and the first radial excitation $(N = 1)$ of the $c\bar{c}$ system, respectively. This assumption, together with Eqs. (3) and (5), gives us K and C as a function of the potential exponent n, and m_c. Results are displayed for several values of n and m_c in Table 1.

n		1	2	3	4	6
m_c=1.0 GeV	C in GeV	0.319	0.653	0.761	0.814	0.865
	K	0.193 GeV^2	0.217×10^{-1} GeV^3	0.294×10^{-2} GeV^4	0.042×10^{-2} GeV^5	0.935×10^{-5} GeV^7
m_c=1.5 GeV	C in GeV	-0.681	-0.347	-0.239	-0.186	-0.135
	K	0.237 GeV^2	0.325×10^{-1} GeV^3	0.540×10^{-2} GeV^4	0.095×10^{-2} GeV^5	3.153×10^{-5} GeV^7
mc=2.0 GeV	C in GeV	-1.681	-1.347	-1.239	-1.186	-1.135
	K	0.274 GeV^2	0.434×10^{-1} GeV^3	0.831×10^{-2} GeV^4	0.168×10^{-2} GeV^5	7.449×10^{-5} GeV^7

Table 1. Constant C and potential parameter K for several values of the potential exponent n and charmed quark mass m_c using the standard assumption that ψ and ψ' are the $\ell = 0$ ground state and first radial excitation of the $c\bar{c}$ system.

From Eq.3 and Table 1, we can compute the predicted masses of the higher S-Wave $c\bar{c}$ ressonances.

Above the threshold of charmed meson pair production, the phenomenological Zweig rule preventing strong decay becomes inoperative, ressonances are expected to be broad. The computed masses for the state N = 2, 3, 4 are given in Table 2.

n \ N	2	3	4
1	4.164 GeV	4.519 GeV	4.895 GeV
2	4.272 GeV	4.861 GeV	5.155 GeV
4	4.401 GeV	5.214 GeV	6.103 GeV
6	4.470 GeV	5.421 GeV	6.509 GeV

Table 2. Masses of the first few radial excited states above $\psi'(3684)$, for various values of the potential exponent n.

For linear (n = 1, R = 0) and harmonic-oscillator (n = 2) potentials, we have exact solutions. For all other cases, we need approximate or numerical solutions. For n even (n = 4 in particular), the variational method gives good results by using an oscillator wave function as testfunction, when compared with numerical estimates[8]. Solutions for the ground state, $\Psi(r)$, are given by

$$\Psi_0(r) = (\gamma_0/\pi)^{3/4} \exp(-\frac{\gamma_0}{2} r^2) \qquad (6)$$

with

$$\gamma_0 = \{\frac{1.3.5\ldots(n+1).n.m_c.K}{3.2^{n/2}}\}^{2/(n+2)} , \qquad (7)$$

a parameter which minimizes the energy expectation values. Using the test function given in Eq.(6), energies of the ground states are given by

$$E_0' = (1/2 + 1/n)\ (3/m_c)\gamma_0 \qquad (8)$$

The valueri of E'_0 obtained by the WKB and variational methods for $n=4$ and 6 are displayed in Table 3.

	$n = 4$	$n = 6$
WKB	$E'_0 = 3.752(\frac{K}{m_c})^{1/3}$	$E' = 4.154(\frac{K}{m_c^3})^{1/4}$
Variational	$E'_0 = 3.847(\frac{K}{m_c^2})^{1/3}$	$E'_0 = 4.527(\frac{K}{m_c^3})^{1/4}$

Table 3. **Values** of E'_0, for $n = 4$ and 6, as a function of the parameters K and m_c, cbtained by the WKB and variational methods.

Table 3 shows that the results of the two methods are in good agreement with each other. For $n = 4$, E' differs by only 2%, and for $n = 6$ the difference is 8%. Thus, we may use the values of K and C obtained in Table 1. As' expected, the WKB appr'oximation for E'_N improves as N increases.

3. HYPERFINE SPLITTING

The mass clifference between the triplet and singlet configurations can be computed by noting that the hyperfine interaction behaves as[4]

$$\frac{1}{6m_c^2} \vec{\sigma}_1 \cdot \vec{\sigma}_2 \nabla^2 V(r) \,, \tag{9}$$

where $\vec{\sigma}_1$ and $\vec{\sigma}_2$ are the **Pauli** spin operators. For our case of a power-law confinemerit potential $V(r) = Kr^n + C$, this reduces to

$$\frac{n(n+1)}{6m_c^2} K r^{n-2} \vec{\sigma}_1 \cdot \vec{\sigma}_2 \,. \tag{10}$$

Using the appropriate eigenvalues of the operator $\vec{\sigma}_1.\vec{a}_2$ for the triplet and singlet S-wave states, the mass difference reads

$$\Delta M = \frac{2n(n+1)K}{3m_c^2} \int \psi_0^* \, r^{n-2} \, \psi_0 \, d^3r \quad , \tag{11}$$

where ψ_0 ís the normalized ground-state wave function.

The normalized $\psi_0(r)$ can be approximated by either the WKB method or a variational calculation. The WKB solution for R = 0 is given by

$$\psi_0(r) = \frac{A}{\sqrt{4\pi}} \{\sin\int_0^r (m_c(E' - Kr'^n))^{1/2} \, dr'/r(m_c(E' - Kr^n))^{1/4} \} \quad . \tag{12}$$

For normalization, we can use the approximate relation

$$\int_0^{r_0} \psi_0^* \, (r) \; \psi_0(r) d^3r = 1 \quad , \tag{13}$$

where r_0 is the classical turning point, i.e., the solution of $E' - kr^n =$ 0. In this approximation, one ignores the possibility of tunnelling.

The quantity $|\psi_0(0)|^2$ is exact by construction for the variational method $(n = 2)$, whereas the WKB result is within a 2% accuracy. For the quartic potential $(n = 4)$, the ground state energy eigenvalues given by the variational method agrees better with the numerical results than those given by the WKB approximation[8]. This gives us confidence in the $\psi(r)$ given by the variational method. In view of this comparison, we shall use a wave function determined by the variational method which is more convenient and avoids numerical estimates involved in the WKB method.

The square modulus of ground state function ψ_0, at the origin, is given in Table 4.

Using Eq.(11) and Table 1, we write down the splitting AM in Table 5.

$\|\psi_0(0)\|^2$	$n=2$	$n=4$	$n=6$
	$\frac{(Km_c)^{3/4}}{\pi^{3/2}}$	$\frac{(5Km_c)}{\pi^{3/2}}$	$\frac{(105Km_c/4)^{3/8}}{\pi^{3/2}}$

Table 4. Values of $|\psi_0(0)|^2$ for potential exponents $n=2$, 4 and 6.

		$n=2$	$n=4$	$n=6$
ΔM in MeV	$m_c = 1.0$ GeV	87	66	63
	$m_c = 1.5$ GeV	58	44	42
	$m_c = 2.0$ GeV	43	33	31

Table 5. Mass differences between $R=0$ singlet and triplet states as a function of the potential exponent n (even), for various values of m .

4. DECAY WIDTHS

Knowledge of the wave functíons at the origin allows us to calculate, in the one photon exchange approximation, the leptonic decay widths

$$\Gamma(\psi \to \ell\bar{\ell}) \simeq \frac{16\pi\alpha^2}{M_\psi^2} e_Q^2 \, |\psi(0)|^2 \; ; \qquad (14)$$

e_Q is the charge of the charmed quark which in the usual GIM model is 2/3.

The orthocharmonium interpretation of ψ allows one to compute the hadronic decay width of ψ via the conversion into ordinary hadrons of the three gluons which a $^3S_1$ $c\bar{c}$ system decays into[3]:

$$\Gamma_h(\psi) = \frac{5}{18}\{\frac{16}{9}(\pi^2-9)\frac{\alpha_s^3}{m_c^2}\}\,|\psi(0)|^2 \qquad (15)$$

		$n=2$	$n=4$	$n=6$
$\Gamma(\psi\to\ell\ell)$ in keV	m_c = 1.0 GeV	1.26	1.02	0.99
	m_c = 1.5 GeV	2.32	1.89	1.81
	m_c = 2.0 GeV	3.57	2.90	2.79

Table 6. Leptonic decay widths of ψ as a function of even n, the potential exponent, for various values of m_c.

The factor 5/18 comes from the SU(3) color group: α_s denotes the effective gluon-parton coupling constant. $\psi(0)$ depends on the confining potential. Taking $0.2 < \alpha_s < 0.3$, we compute the upper and lower limits on $\Gamma_h(\psi)$:

		$n=2$	$n=4$	$n=6$
$\Gamma_h(\psi)$ in keV	m_c = 1.0 GeV	> 35 < 118	> 28 < 95	> 27 < 92
	m_c = 1.5 GeV	> 28 < 96	> 23 < 78	> 22 < 75
	m_c = 2.0 GeV	> 25 < 83	> 20 < 68	> 19 < 65

Table 7. Limits on the hadronic decay widths of ψ as a function of (even) n, the potential exponent, taking the gauge coupling α_s in the range $0.2 < \alpha < 0.3$, for various values of m_c.

Experimentally[5,9], $\Delta M \simeq 300$ MeV, $\Gamma(\psi\to\ell\bar{\ell}) \simeq 4.8$ keV, and $\Gamma_h(\psi) \simeq 59$ keV. Tables 5, 6 and 7 show that the computed values of Δ and $\Gamma(\psi\to\ell\bar{\ell})$, assuming ψ and ψ' to be the ground state and the first radial excitation respectively, disagree with observation. $\Gamma_h(\psi)$ is however, compatible.

5. CONSEQUENCES OF CHANGING INPUTS

Let us now abandon[10] the assumption that the first radial excitation is indeed the ψ', and instead use as input the observed mass of ψ, $m(\psi)$, and $\Gamma(\psi\to\ell\bar{\ell})$. This predicts the first radial excitation to be around 4 GeV, above the supposed charmed pair production threshold, and although it increases ΔM considerably, it is still a factor of two too small compared with experiments. Our results are given in Table 8.

		$n=2$	$n=4$	$n=6$
$m_c = 1.0$ GeV	C in GeV	0.018	0.287	0.377
	K	0.1297 GeV^3	0.00925 GeV^5	0.000634 GeV^7
	ΔM in MeV	515	515	515
	$m_{\psi'}$ in GeV	4.530	4.716	4.725
	$\Gamma_h(\psi)$ in keV	> 132 < 447	> 132 < 447	> 132 < 447
$m_c = 1.5$ GeV	C in GeV	-0.623	-0.443	-0.384
	K	0.0859 GeV^3	0.00617 GeV^5	0.000423 GeV^7
	ΔM in MeV	153	153	153
	$m_{\psi'}$ in GeV	4.052	4.176	4.182
	$\Gamma_h(\psi)$ in keV	> 58 < 199	> 58 < 199	> 58 < 199
$m_c = 2.0$ GeV	C in GeV	-1.443	-1.309	-1.264
	K	0.0643 GeV^3	0.00463 GeV^5	0.000317 GeV^7
	ΔM in MeV	64	64	64
	$m_{\psi'}$ in GeV	3.812	3.906	3.910
	$\Gamma_h(\psi)$ in keV	> 33 < 112	> 33 < 112	> 33 < 112

Table 8. Values of C, ΔM, $m_{\psi'}$, and $\Gamma_h(\psi)$ using m = 3.095 GeV and $\Gamma(\psi\to\ell\bar{\ell}) \simeq$ 4.8 keV as input, for various values of m_c. For fixed $\Gamma(\psi\to\ell\bar{\ell})$ and m_c, the values of ΔM are independent of n (Ref. 11).

6. CONCLUSION

Our computations show that the bound state spectra of a heavy quark-antiquark hyperfine splitting between the triplet and singlet S-waves (ΔM), and the lepton decay width of the triplet ground state, $\Gamma(\psi\to\ell\bar{\ell})$, are not very sensitive to the exponent of the confininig potential. The standard assurnption that $\psi'(3684)$ is the first radial excitation results in a serious disagreement between the computed and observed values of AM and $\Gamma(\psi\to\ell\bar{\ell})$. We studied consequences of dropping this assumption and use $\Gamma(\psi\to\ell\bar{\ell})$ as input. This leads to a position of the first radial excitation above 4 GeV, above the supposed charm pair production threshold, where definite structures are seen in the $e^+ e^-$ annihilation cross section. However, the discussion preceding the results given in Table 8 show that one does not achieve agreements with all the observables. Also, there is the problem of reinterpreting $\psi'(3684)$ together with its established properties such as the large branching'ratio for the cascade decay $\psi'\to\psi\pi\pi$. On the other hand, taking m_ψ, $m_{\psi'}$ and $\Gamma(\psi\to\ell\bar{\ell})$ as input, we find for n = 2 case m_c = 2.438 GeV, C = -2.223 GeV, K = 0.0528 GeV, AM= 35 MeV and 22 keV < $\Gamma_h(\psi)$ < 75 keV; i.e., the hyperfine splitting istoo small and the large value of m_c casts doubt on the validity of the non relativistic treatment; n = 4 and 6 give even worse results. In view of those continuing difficulties, there does not seem to be a net gain is changing the standard inputs.

Analysing the results of our computation and considering for definiteness m_c = 1.5 GeV, where relativistic effects are expected to be small, we conclude that potentials with exponents $n > 2$ do not provide satisfactory models for the charmonium. A preliminary study indicates that potentials $n = 1/2$ may describe the charmonium better than the $n \geq 1$ cases. The radial excitation spectrum for this potential, with the standard inputs, using the WKB approximation, given by $m_{\psi''}$ = 4.089 GeV, $m_{\psi'''}$ = 4.415 GeV and $m_{\psi^{iv}}$ = 4.687 GeV, are in good agreement with experimental" indications; also AM and $\Gamma(\psi\to\ell\bar{\ell})$ are estimated to be larger than those of the $n = 1$ case. In view of this, the case $0 < n < 1$ which also satisfies the Martin[13] condition is under numerical study. This, togetherwith the results of spin orbit forces $R' \neq 0$, and radiative decays, will be published elsewhere.

REFERENCES

1. See, for example, H.D. Politzer, Phys. Reports *14C*, 130 (1974).
2. A. De Rújula, S.L. Glashow, Phys. Rev. Letters 34, 46 (1975); T. Appelquist, *Proceedings of the* 1975 *Cargèse Summer Institute on Weak and Electromagnetic Interactions at High Energies*, Cargèse, France, and references therein.
3. See for example, B.J. Harrington, S.Y. Park and A. Yildiz, Phys. Rev. Letts. 34. 168 (1975).
4. E. Eichten, K. Gottfried, T. Kinoshita, J. Kogut, K.D. Lane, and T.M. Yan, Phys. Rev. Lett. *34*, 369 (1975); J.S. Kang and H.J. Schnitzer, Phys. Rev. *D-12*, 841 (1975); J.F. Gunion and R.S. Willey, Phys. Rev. *D-12*, 174 (1975).
5. R. Schwitters, Rapporteur's Talk at the Stanford Conference of August, 1975.
6. See any text on Quantum Mechanics.
7. *Handbook of Mathematical Functions*, edited by M. Abrarnowitz and I. A. Stegun, Dover Publications, Inc. New York; *Table of Integrals, Series and Products*, I.S. Gradshteyn, I.M. Ryzhik (1965) (Academic Press, New York and London).
8. *Introduction to the Quantum Theory*, D. Park (McGraw Hill Book Co., 1964, pp. 99 and 234.
9. G.J. Feldman and M.L. Perl, Phys. Reports 19, 5 (1975).
10. H.J. Schnitzer, Phys. Rev. *D-13*, 74 (1976).
11. T. Appelquist, *et al.*, Phys. Rev. Letters 34, 365 (1975).
12. R.F. Schwitters, SLAC - PUB - 1871 (Jan.1977).
13. A. Martin, Phys. Lett. *67B*, 330 (1977).