Revista Brasileira de Física, Vol. 8, Nº 3, 1978

Soluble Semiclassical Model for a One-Dimensional $\Delta\ell = 1$, $\Delta m = 0$, Decay

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Recebido em 30 de Janeiro de 1978

The radiation emitted by a plane of excited two-level atoms, in $\Delta\ell = 1$, $\Delta m = 0$ transition, is exactly calculated using a semiclassical model in which the electromagnetic field is treated classically (Maxwell's equations) and the coupling between matter and field is described in the electric dipole approximation. The influence of the plane density in the radiation rate is investigated in both limits of weak and strong coupling (defined in the text). It is shown that, in the second case, we can observe infinitely many solutions of the problem, depending on the initial value of the phase difference appearing in the definition of the excited state. Cases of phase choices leading to enhanced or attenuated emission rates are also discussed.

Calcula-se exatamente a radiação emitida por um plano de átomos excitados, a dois níveis, em uma transição com $\Delta\ell = 1$ e $\Delta m = 0$, fazendo-se uso de um modelo semi-clássico no qual o campo eletromagnético é tratado classicamente (via as equações de Maxwell), a interação entre matéria e campo sendo descrita na aproximação de dipolo elétrico. Investiga-se a influência da densidade planar dos átomos, na taxa de radiação, em ambos os limites de acoplamentos fraco e forte, definidos no texto. Constata-se, no segundo caso, a ocorrência de uma infinidade de soluções, dependendo do valor inicial da diferença de fase que aparece na definição do estado excitado. Discutem-se também escolhas de fase que conduzem à intensificação ou atenuação das taxas de emissão.

1. INTRODUCTION

A soluble model, for the spontaneous decay of a plane of two-level atoms, was presented in a previous paper[1] for the case of a $\Delta\ell = \Delta m = 1$ transition. Here, we shall employ the same model to solve the $\Delta R = 1$, $\Delta m = 0$, case, since it presents some features which we deem worthwhile of a separate study. In order to render this paper more readable to those noninitiated in the field, we shall present, in the first two Sections of this work, a brief review of the subject, including a detailed description of the model discussed in Ref.1.

The interaction, between a system of N two-level atoms and the electromagnetic field, was discussed by Dicke[2] who showed how the emission rate could be expressed in terms of the atomic energy anda newquantum number (the cooperation number) which represents the quantum analog of the dipoles phase relationships of the classical description. Theusual treatment today[3-9] cosists in starting with a multimode cavity field, in a volume V, and obtaining Heisenberg's equations of motion for the time-dependend atomic operators and field amplitudes. Here, instead of using perturbation theory (as in Dicke's treatment), we can resort (as in more recent works) to a non-perturbative approach which consists in an approximate integration of the field equations (linear) and the substitution of the results in the atomic equations (non-linear). The approximations more commonly used in these calculations are the rotating wave approximation, and the assumption that the spontaneous radiative decay is a slow process, requiring many cicles of atomic oscillations to be completed. The resulting equations yield a semiclassical limit (also discussed by Dicke) in which the operators are treated as c-numbers and their eigenvalues taken in the limit of large quantum numbers.

The treatment presented in Ref.1 follows a different line, in the sense that it is a thorougly semiclassical (or classical) approach; that is, one in which the electromagnetic field is treated classically from the beginning (through Maxwell's equations). Although some important quantum features will be lost (e.g: the spontaneous decay of *one* atom) the equations of motion for a many-atom system will (as shown below)

be integrated *exactly* (independent of the coupling strength between matter and field, providing an worthwhile comparison with previous results some insight on the behaviour of matter at high densities.

2. MATTER AND FIELD EQUATIONS

Let us consider a medium of two-level atoms, energies equal to $\pm (1/2)\hbar\omega_0$, whose interaction with the transverse electromagnetic field is taken in the electric dipole approximation (the field is assumed to be space-independent in the whole atomic volume). Since there are only two normalized wave functions, ϕ_a and ϕ_b, corresponding respectively to the ground and excited states, we can write the general time-dependent wave function of a single atom as

$$\psi(t) = b(t)\,\phi_b + a(t)\phi_a = \begin{bmatrix} b(t) \\ a(t) \end{bmatrix}, \tag{1}$$

where we have used the vector notation $\phi_b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\phi_a = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. On the other hand, the one-atom Hamiltonian is given by

$$H = H_0 + H_1 \approx H_0 - \vec{p}\cdot\vec{E}(t), \tag{2}$$

where H_0 is the unperturbed Hamiltonian, $H_0 = (\hbar\omega_0/2)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\vec{p}$ is the electric dipole operator, and $\vec{E}$ the *external* electric field produced by all other atoms (no reaction field here). Substituting the wavefunction (1) into the Schrödinger equation corresponding to (2), i.e.

$$(H_0 - \vec{p}\cdot\vec{E}(t))\psi = i\hbar\,\frac{\partial\psi}{\partial t}, \tag{3}$$

we get

$$\begin{aligned} \frac{da}{dt} &= i\,\frac{\omega_0}{2}\,a - \frac{i}{\hbar}\,(\phi_a,\, H_1(t)\phi_b)b\,, \\ \frac{db}{dt} &= i\,\frac{\omega_0}{2}\,b - \frac{i}{\hbar}\,(\phi_b,\, H_1(t)\phi_a)a\,, \end{aligned} \tag{4}$$

where, assuming a spherically symmetric potential for our model atomwe have taken $(\phi_a, \vec{p}\,\phi_a) = (\phi_b, \vec{p}\,\phi_b) = 0$. Instead of using the complex numbers $\underline{a}$ and $\underline{b}$ we introduce the real variables[10]

$$r_x = \langle\sigma_x\rangle = \psi^\dagger\sigma_x\psi = ab^* + a^*b\ ,$$

$$r_y = \langle\sigma_y\rangle = \psi^\dagger\sigma_y\psi = i(ba^* - b^*a)\ , \tag{5}$$

$$r_z = \langle\sigma_z\rangle = \psi^\dagger\sigma_z\psi = bb^* - aa^*\ ,$$

where σ_x, σ_y and σ_z are the usual Pauli spin matrices. $\begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&-i\\i&0\end{pmatrix}$ and $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. In terms of r_x, r_y and r_z, equations (4) can be written as (*)

$$\frac{d\vec{r}}{dt} = \vec{\omega} \wedge \vec{r}\ , \tag{6}$$

where $\vec{r} = r_x\hat{x} + r_y\hat{y} + r_z\hat{z}$ and $\vec{\omega} = \omega_x\hat{x} + \omega_y\hat{y} + \omega_z\hat{z}$, with

$$\omega_x = \frac{2}{\hbar}\,\mathrm{Re}(\phi_a, H_1\,\phi_b)\ ,$$

$$\omega_y = \frac{2}{\hbar}\,\mathrm{Im}(\phi_a, H_1\,\phi_b)\ , \tag{7}$$

$$\omega_z = \omega_0\ .$$

From (6), we have $\vec{r}\cdot(d\vec{r}/dt) = 0$, or $\vec{r}^2 = \text{constant}$, which expresses the conservation of probability, since $r_x^2 + r_y^2 + r_z^2 = (|a|^2 + |b|^2)^2$ (considering only normalized wave functions we shall always take $\vec{r}^2 = 1$). Now we have to calculate the matrix element appearing in the right hand si-

(*) Since the real and imaginary parts of $\underline{a}$ and $\underline{b}$ make four real numbers, we cannot construct $\psi(t)$ from the values of r_x, r_y and r_z alone. However, it follows from (5) that, given r_x, r_y and r_z, we can determine $|\underline{a}|$, $|\underline{b}|$ and the phase difference $\phi_a - \phi_b$, leaving out only an irrelevant multiplicative phase factor.

des of (7): Since in a central potential the ground state has zero angular momentum, ϕ_b must be a p-state, otherwise the interaction Hamiltonian will be identically zero (this is an obvious consequence of the selection rules for the dipole coupling assumed in (2)). There are then two possible choices for the angular momentum of ϕ_b : R = 1, m = ± 1 or ℓ = 1, m = 0, for which, using hydrogen-like wave functions, we get:

$$p_x = p\sigma_x , p_y = \pm p\sigma_y , p_z = 0, \quad m = \pm 1, \tag{8}$$

$$p_x = 0 , p_y = 0, p_z = \sqrt{2}\, p\sigma_x , m = 0 , \tag{9}$$

where p is an overlap integral containing the radial parts of ϕ_a and ϕ_b. From (7), (8) and (9) we obtain

$$\omega_x = -\frac{2p}{\hbar} E_x, \omega_y = \pm \frac{2p}{\hbar} E_y, \quad m = \pm 1, \tag{10}$$

and

$$\omega_x = -\frac{2p}{\hbar}\sqrt{2}\, E_z , \quad \omega_Y = 0, \quad m = 0. \tag{11}$$

The equations of motion can now be obtained from (6):

$$\frac{\partial r_x}{\partial t} = -\omega_0 r_y \pm \frac{2p}{\hbar} E_y r_z, \quad \frac{\partial r_y}{\partial t} = \omega_0 r_x + \frac{2p}{\hbar} E_x r_z , \quad \frac{\partial r_z}{\partial t} = \frac{2p}{\hbar}(\pm r_x E_y - r_y E_x), \quad m = \pm 1, \tag{12}$$

$$\frac{\partial r_x}{\partial t} = -\omega_0 r_y, \quad \frac{\partial r_y}{\partial t} = \omega_0 r_x + \frac{2p}{\hbar}\sqrt{2}\, E_z r_z, \quad \frac{\partial r_z}{\partial t} = -\frac{2p}{\hbar}\sqrt{2}\, E_z r_y, \quad m = 0. \tag{13}$$

Let us now turn our attention to the electromagrietic field considering a medium of n two level atoms per unit volume whose contribution to the sources of Maxwell's equations is entirely described by a macroscopic polarization $\vec{P}(\vec{r},t)$. The detailed expression for $\vec{P}(\vec{r},t)$ will naturally depend on the spatial distribution of the atomic. dipoles at theposition r and time t. Assuming the maximum possible "cooperation" (i.e: all atoms at the same point with the same wave functions) we shall write.

$$\vec{P} = n \langle \vec{p} \rangle \tag{14}$$

From (8) and (9) we then get

$$\vec{P} = np(r_x \hat{x} \pm r_y \hat{y}), \quad m = \pm 1, \tag{15}$$

and

$$\vec{P} = \sqrt{2}\, np\, r_x \hat{z}, \quad m = 0. \tag{16}$$

On the other hand, it follows from Maxwell's equations that

$$\Delta \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = \frac{4\pi}{c^2} \frac{\partial^2 \vec{P}}{\partial t^2}, \tag{17}$$

which, with the help of (15) and (16), gives at last

$$\begin{aligned} (\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) E_x &= \frac{4\pi}{c^2} np \frac{\partial^2 r_x}{\partial t^2}, \\ (\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) E_y &= \pm \frac{4\pi}{c^2} np \frac{\partial^2 r_y}{\partial t^2}, \quad m = \pm 1, \end{aligned} \tag{18}$$

and

$$(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) E_z = \frac{4\pi}{c^2} \sqrt{2}\, np \frac{\partial^2 r_x}{\partial t^2}, \quad m = 0. \tag{19}$$

Equations (12), (18) or (13), (19) constitute a basic self-consistent system for the study of non-linear light propagation in resonant media. Our interest here, however, is directed to emission problems, in which only a limited portion of the space is occupied by the material medium. As the simplest example of this situation we shall consider the spontaneous one-dimension emission produced by an infinitely thin plane of excited atoms:

$$n(x,y,z) = \sigma \delta(z), \tag{20}$$

whereois the number of atoms per unit surface area. Equations (17) is now written as

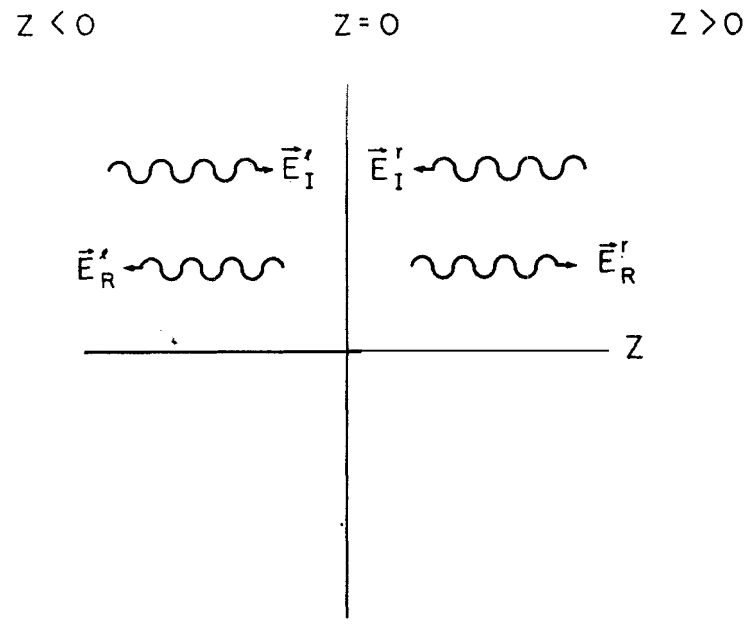

Fig.1 - Emission and absorption of radiation by a material plane, showing the incident and reflected fields (Eq. (22)).

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}(z,t) = \frac{4\pi}{c^2}\sigma\delta(z)\frac{d^2}{dt^2}\langle\vec{p}(0,t)\rangle, \qquad (21)$$

with $\vec{E} = E_x\hat{x} + E_y\hat{y}$. Since all matter is concentrated at $z = 0$ we have for both $z < 0$ and $z > 0$ free-field solutions of (21) (See Fig.1):

$$\begin{aligned} \vec{E}^r(z,t) &= \vec{E}^r_I(z+ct) + \vec{E}^r_R(z-ct), \qquad z > 0, \\ \vec{E}^\ell(z,t) &= \vec{E}^\ell_I(z-ct) + \vec{E}^\ell_R(z+ct), \qquad z < 0, \end{aligned} \qquad (22)$$

(r, ℓ, I and R standing respectively for right, left, incident and reflected) which must obey the boundary condition

$$\vec{E}^r(0,t) = \vec{E}^\ell(0,t) = \vec{E}(t), \qquad (23)$$

where $\vec{E}(t)$ is the electric field actually "seen" by the atoms in the plane (Eqs (12) and (13)). Integrating (21) from $z = -\varepsilon$ to $z = +\varepsilon$ and taking

the limit of the result when $\varepsilon \to 0$ we get the equation

$$\left(\frac{\partial \vec{E}^r}{\partial z} - \frac{\partial \vec{E}^\ell}{\partial z}\right)_{z=0} = \frac{4\pi\sigma}{c^2} \frac{d^2}{dt^2} \langle \vec{p}(0,t) \rangle \ , \tag{24}$$

which with the help of (22) transforms into

$$\frac{1}{c} \frac{\partial}{\partial t} \left(\vec{E}^r_I - \vec{E}^r_R - (\vec{E}^\ell_R - \vec{E}^\ell_I)\right)_{z=0} = \frac{4\pi\sigma}{c^2} \frac{d^2}{dt^2} \langle \vec{p}(0,t) \rangle \ . \tag{25}$$

Integrating (25) with respect to the time, taking into a account the fact that the ground state (where $\langle \vec{p} \rangle = 0$) must be compatible with the absence of external fields, we obtain

$$\vec{E}^r_I(0,t) - \vec{E}^r_R(0,t) + \vec{E}^\ell_I(0,t) - \vec{E}^\ell_R(0,t) = \frac{4\pi\sigma}{c} \frac{d}{dt} \langle \vec{p}(0,t) \rangle \ . \tag{26}$$

This result can be written in a more useful form with the help of boundary condition (23). In fact, using the relations

$$- \vec{E}^r_R(0,t) = \vec{E}^r_I(0,t) - \vec{E}(t) \ ,$$

and (27)

$$- \vec{E}^\ell_R(0,t) = \vec{E}^\ell_I(0,t) - \vec{E}(t) \ ,$$

we can eliminate the reflected field from (26) and obtain

$$0 \quad , + 0 \quad , - \vec{E}(t) = \frac{2\pi\sigma}{c} \frac{d}{dt} \langle \vec{p}(0,t) \rangle \tag{28}$$

Equation (28) together with equations (12), (15) or (13), (16), constitute a system of coupled *ordinary* differential equations which gives the behaviour of matter and field, at $z=0$, under the effectsofa *known* external excitation $\vec{E}^\ell_I + \vec{E}^r_I$. Once this systern solved, thebehaviour of the electric field in the whole space will be readily obtained from (22) and (23). Since we are interested in the *spontaneous* emission we shall, from now on, assume $\vec{E}^\ell_I + \vec{E}^r_I \equiv 0$.

3. SPONTANEOUS EMISSION BY A PLANE OF TWO-LEVEL ATOMS

It is now convenient to treat separately the solutions according to whether m = ± 1 or m = 0.

Case I: m = ±1 Equation (28), taking into account (15), is now written as

$$\vec{E}(t) = -\frac{4\pi p\sigma}{c}\frac{d}{dt}\left(r_x\hat{x} \pm r_y\hat{y}\right). \tag{29}$$

In Eqs (29) and (12) we have a non-linear system of five differential equations for the five unknown r_x, r_y, r_z, E_x and E_y. Its solution is more readily obtained in terms of the complex variables $r^+ = r_x + i\,r_Y$ and $E^\pm = E_x \pm iE_Y$:

$$\begin{aligned} \frac{d}{dt}\,r^+ &= i\omega_0 r^+ + i\,\frac{2p}{\hbar}\,r_z\,E^\pm, \\ E^\pm &= -\frac{2\pi p\sigma}{c}\,\frac{d}{dt}\,r^+ , \end{aligned} \tag{30}$$

with r_z given by the constant of motion $r_x^2 + r_Y^2 + r_z^2 = 1$. Eliminating $E^\pm$ in (30) we obtain, for both values of m:

$$\frac{d}{dt}\,r^+ = \omega_0\,\frac{i + \alpha r_z}{1 + \alpha^2 r_z^2}\,r \tag{31}$$

where $\alpha = 4\pi p^2\sigma/\hbar c$ is a dimensionless constant. Using spherical coordinates $r^+ = \sin\theta e^{i\phi}$, $r_z = \cos\theta$ (it follows from (5) that $\cos\theta$ is proportional to the mean unperturbed energy, whereas ϕ is the phase difference between a and b) we get from (31):

$$\frac{d\theta}{dt} = \alpha\omega_0\,\frac{\sin\theta}{1 + \alpha^2\cos^2\theta} \tag{32}$$

and

$$\frac{d\phi}{dt} = \frac{\omega_0}{1 + \alpha^2\cos^2\theta} \tag{33}$$

The electric field can be obtained eliminating dr^+/dt between the two equations (30). The result is

$$|\vec{E}| = \frac{\hbar\omega_0 \alpha}{2p} \frac{\sin\theta}{(1 + \alpha^2\cos^2\theta)^{1/2}} . \qquad (34)$$

From (34) we have $\alpha = 2pE_m/\hbar\omega_0$, where E_m is the maximum value of the electric field, corresponding to a situation ($\theta = \pi/2$) in which one-half of the atoms are still excited. The constant a measures then the strength of coupling between matter an field. For $a << 1$ the dipole interaction, being much smaller than the binding forces of the atom, can be treated as a perturbation[2]. Numerical approximations for a can be readily established in the following way: writing $p = ed$, where e is the electronic charge and d is of the order of one atomic diameter, we obtain $a = (4\pi e^2/\hbar c)\sigma d^2 = 0.091\sigma d^2$. Here σd^2 plays the role of a "screening constant" giving the fraction (which actually can be larger than one, depending on the number of atomic "layers" in the plane) of the plane area occupied by the atoms. In the case of a solid plane with all atoms participating in the emission process we can have, for a typical optical transition, $a \lesssim 1.6$ (Ref.11).

Let us now turn our attention to the initial conditions involved inthe solution of (32) and (33) (the behaviour of θ and ϕ as functions of time was already discussed in reference 1). The general solution of (32) has the form $\theta(t+c)$, where c is an arbitrary constant (this is an obvious consequence of the invariance of the emission process under time translations). A particular value for c can be obtained in principle, giving the instant of time when the emission begins. Here, however, we run into a tipical difficulty of the semiclassical treatments: -besides the exponential decay for $t \to m$ (ground state) we have an exponential increase in the radiation rate *starting* at $t \to -\infty$ (excited state). This is due to the lack of a "quantum field" term,[2,12] without analogue in the classical limit, which is responsible for the beginning of the radiation process. The effect of this absence can be seen in equation (32) which admits $\theta \equiv 0$ (corresponding to the excitedstate) as an unstable solution. What we shall do here is to choose $t = 0$ as the instant of maximum radiation rate which happens just after the

emission of one-half of the total excitation energy (see Eq. (34)). The solution $\theta(t)$ is then inserted into (33) giving, after integration, the azimuthal angle $\phi(t)$. Here we have andther constant of integration, the general solution being now of the type $\phi(t) + c$ (invariance under rotations). Since $ determines the direction of the polarization, it follows from the above reasoning that we are free to choose the direction of the space in which the bulk of the radiation will be emitted. This may seem, at first sight, paradoxical- as we are used to consider the radiation as being cornpletely determined by the initial state. *This is indeed what happens:* The point is that, strictly speaking, theemission beginning *exactly* in the excited state gives rise to an *undetermined* problem, in the sense that small changes in the initial state may produce sensible changes in the electromagnetic field emitted in the future. If we select, as initial states of the atoms, $\theta = \varepsilon$ ($\varepsilon << 1$) with arbitrary values of ϕ (the same mean energy with all possible phases), the systems will, for all subsequent values of 0, "remember" the initial phase differences producing, accordingly, rotated field patterns. For a<<] those rotated solutions will produce the same physical effects assuming that our time resolution is not too small compared with the life expectation of the system. This is due to the fact that $\phi(t)$, for $\alpha << 1$, changes with time much faster than $\theta(t)$. The polarization vector then passes many times through all directions of the space with approximately the same magnitude, rendering immaterial the presence of an additive phase in $\phi(t)$. For a $\gtrsim$ 1, however, the decay is so fast that the lifetime of the system is reduced to the order of one atomic oscillation ($d\theta/dt \approx \omega_0$) giving rise to a strongly anysotropic and phase-dependent radiation.

Case II: $m = 0$ Let us consider here a plane of excited atoms which *contains* the *z-axis*. Our self-consistent set of equations is obtained from (13), (16) and (28), remembering that now both $\langle \vec{p} \rangle$ and $\vec{E}$ point in the $\hat{z}$ direction.

$$\frac{\partial r_x}{\partial t} = -\omega_0 r_y, \quad \frac{\partial r_y}{\partial t} = \omega_0 r_x + \frac{2p}{\hbar}\sqrt{2}\, E_z r_z, \quad \frac{\partial r_z}{\partial t} = -\frac{2p}{\hbar}\sqrt{2}\, E_z r_y,$$

$$E_z = -\frac{2\pi p\sigma}{c}\sqrt{2}\,\frac{\partial r_x}{\partial t}. \qquad (35)$$

This system is readily transformed into

$$\frac{dr_x}{d\tau} = -r_y, \quad \frac{dr_z}{d\tau} = -2\alpha r_y^2, \quad r_x^2 + r_y^2 + r_z^2 = 1, \tag{36}$$

where $r_x = \langle p \rangle / (\sqrt{2}p)$, $r_y = \sqrt{2}\, pE_z/(\alpha\hbar\omega_0)$ and $\tau = \omega_0 t$.

Using splierical coordinates $r_x = \sin\theta\cos\psi$, $r_Y = \sin\theta\sin\psi$, $r_z = \cos\theta$, we obtain the final form of (36).

$$\frac{d\theta}{d\tau} = 2\alpha \sin\theta\sin^2\psi, \tag{37}$$

$$\frac{d\psi}{d\tau} = 1 + \alpha\cos\theta\sin(2\psi). \tag{38}$$

Before discussing the integration of the system (37), (38) for all values of α, we shall first consider the weak coupling limit. Takíng $\alpha \ll 1$ in (38) we have $d\psi/\delta\tau \approx 1$ and the solution corresponding to $\theta(0) = \pi/2$, $\psi(0) = \psi_0$ is readily obtained

$$\psi = \tau + \psi_0, \tag{39}$$

$$\theta = 2\mathrm{tg}^{-1}\{\exp[\alpha(\tau - \cos(\tau + 2\psi_0)\sin\tau)]\}. \tag{40}$$

Expression (40) can be further simplified: for $\alpha \ll 1$ we have

$$\exp[\alpha(\tau - \cos(\tau + 2\psi_0)\sin\tau)] = \exp(\alpha\tau)\exp[-\alpha\cos(r + 2\psi_0)\sin\tau] \approx \exp(-ar), \tag{41}$$

which makes $\theta(\tau)$ independent of ψ_0. From (39), (40) and (41) we have the approximate solutions

$$r_x = \mathrm{sech}(\alpha\tau)\cos(\tau + \psi_0), \quad r_y = \mathrm{sech}(\alpha\tau)\sin(\tau + \psi_0), \quad r_z = -\mathrm{tgh}(\alpha\tau), \tag{42}$$

where thei amplitude $\mathrm{sech}(\alpha\tau)$ is a slowly changing modulation (lifetime $\approx 1/(\alpha\omega_0)$) compared with the fast oscillations ($1/\omega_0$) of $\cos(\tau+\psi_0)$ and

$\sin(\tau+\psi_0)$. The influence of the phase ψ_0 appears in (42) as a time displacernent of the fast terms with respect to the modulation, giving negligible results for $\alpha<<1$. This fact can also be seen in the radiation rate (proportional to $-dr_z/dt$):

$$-\frac{dr_z}{dt} = \sin\theta\,\frac{d\theta}{dt} = 2\alpha\omega_0 \sin^2\theta \operatorname{sen}^2\psi \simeq 2\alpha\omega_0 \operatorname{sech}^2(\alpha\omega_0 t)\sin^2(\omega_0 t + \psi_0). \quad (43)$$

If we assume a time resolution At, $1/\omega_0 << \Delta t << 1/\alpha\omega_0$ and take the mean of (43) in this time interval we get

$$-\overline{\frac{dr_z}{dt}} = \alpha\omega_0 \operatorname{sech}^2(\alpha\omega_0 t) \quad (44)$$

which is independent of ψ_0.

The solution'for large values of a can be best understood considering the symmetry properties of Eqs.(37) and (38). Let $\theta_1(\tau)$, $\psi_1(\tau)$ bea solution of this system satisfying the initial conditions $\theta_1(0) = \pi/2$, $\psi_1(0) = \psi_0$. Taking $\tau \to -\tau$, $\theta \to \pi - \theta$ and $\psi \to -\psi$ in (37) and (38) it is easily shown that

$$\theta_2(\tau) = \pi - \theta_1(-\tau), \qquad \psi_2(\tau) = -\psi_1(-\tau) \quad (45)$$

is another solution of the same system, satisfying now $\theta_2(0) = \pi/2$, $\psi_2(0) = -\psi_0$ of ψ_0 m Eq. (45) providing the rest of the solutions. In particular, the solutian derived from $\psi_0 = 0$ has the special symmetry $\theta(\tau) = \pi - \theta(-\tau)$, $\psi(\tau) = -\psi(-\tau)$. It is also easy to see that

$$\theta_3(\tau) = \theta_1(\tau), \qquad \psi_3(\tau) = \psi_1(\tau) + \pi \quad (46)$$

is also a solution of (37), (38), corresponding to the initial conditions $\theta_3(0) = \pi/2$, $\psi_3(0) = \psi_0 + \pi$ (reversed field and electric dipole). We can then restrict ourselves to the values of ψ_0 contained in an arbitrary interval of length π. However, in order to better display the symmetry (45) we shall usually consider tha range $[-\pi/2, \pi/2]$. Our next step in the integration of (37), (38) is to note that there are

two different kinds of solutions: for $\alpha<1$ and $\alpha>1$. For $a < 1$ we have always $d\psi/d\tau > 0$, so that $\psi(\tau)$ is an unbounded increasing furiction of the time. For $\alpha>1$ the behaviour of $\psi(\tau)$ can be best described considering the asymptotic form of (37) and (38) for $t \to \pm \infty$. Taking $\cos \to \mp 1$ in (38) (corresponding to $\tau \to \pm\infty$) we get $d\psi/d\tau \approx a \sin 2\psi$, which has the constant solution $\sin(2\psi') = \pm 1/\alpha$. Defining $\psi_\infty = \sin^{-1}/2)\,(1/\alpha)$, $0<\psi_\infty<\pi/2$, we have

$$\text{a) } \psi' = \psi_\infty + n\pi, \quad \text{b) } \psi' = \frac{2n+1}{2}\pi - \psi_\infty, \ \tau \to \infty, \ n = 0, \pm 1 \ldots$$
$$\text{c) } \psi' = n\pi - \psi_\infty, \quad \text{d) } \psi' = \frac{2+1}{2}\pi + \psi_\infty, \ \tau \to \infty, \ n = 0, \pm 1 \ldots \tag{47}$$

These values of ψ' can be thought of as dividing the (τ,ψ) plane in infinitely many sheets parallel to the τ axis. The character of the solution asymptotically contained in each of those sheets depend on the stability of the constant solutions (47). Let us then consider $\theta = \varepsilon$ for $\tau \to -\infty$, $\theta = \pi - \varepsilon$ for $\tau \to +\infty$; $0 < \varepsilon << 1$; and $\psi = \psi' + \mu$, $|\mu| << 1$, $\sin 2\psi' = \pm 1/\alpha$ for $\tau \to \pm \infty$.

Substituting these equations into (37) and (38) we obtain, after straightforward approximations

$$\varepsilon = A \exp\{-2\alpha|\tau|\sin^2\psi'\}, \quad \tau \to \pm\infty, \tag{48}$$

and

$$\mu = B \exp\{-2\alpha|\tau|\cos(2\psi')\}, \quad \tau \to \pm\infty, \tag{49}$$

where A and B are arbitrary constants. It follows from (49) that solutions a) and c) in (47) are stable $(\cos(2\psi') > 0)$ whereas b) and d) are unstable $(\cos(2\psi') < 0)$. From the above discussions it is easy to construct a rough picture of the behaviour of $\psi(\tau)$ $(\alpha>1)$: For $\tau \to \pm\infty$ the curves tend, respectively, to the lines a) and c), and diverge from b) and d).

The two types of solutions are illustrated in Figs. 2 and 3 where we have the result of numerical computations, for both $a < 1$ and $a > 1$, using the initial conditions $\theta(0) = \pi/2$, $\psi(0) = \psi_0$, $-\pi/2 < \psi_0 < \pi/2$. As

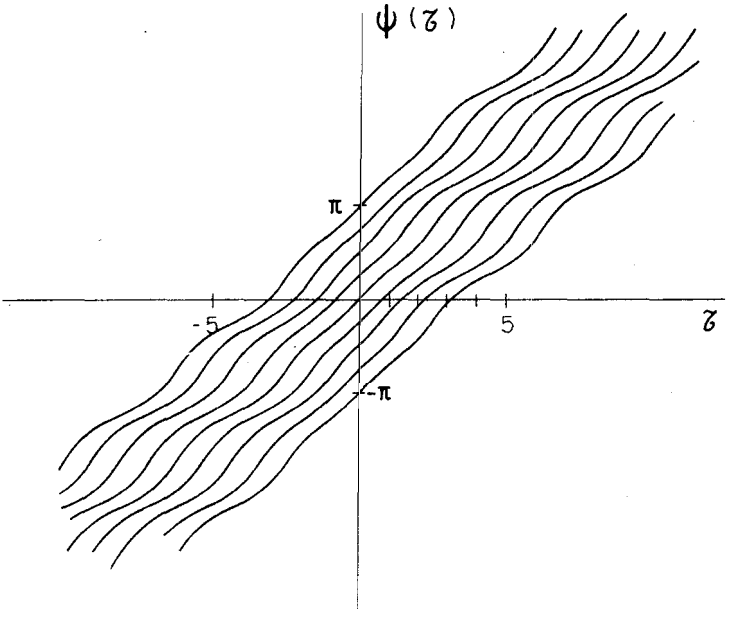

Fig.2 - Graphs of $\psi(\tau)$ for $\alpha = 0.5$. The different values of ψ_0: $0, \pm\pi/4, \pm\pi/2, \pm 3\pi/4$ and $\pm\pi$ are given by the intersections with the vertical axis.

a last remark on $\psi(\tau)$ we note that, for $a > 1$, the electric field and dipole (proportional respectively to $\sin\psi$ and $\cos\psi$) do not oscillate in the limit $\tau \to \pm\infty$ (as in the case when $a < 1$). This situation bears a close analogy with a damped harmonic oscillator. For $a < 1$ we have a sort of *weak damping*, where a small loss of energy per cicle (of the free motion) produces an infinite series of damped oscillations. For $a > 1$ we are in the *overdamped* region where this same loss is of the order of the whole stored energy, giving rise to a situation in which the amplitude decreases without changing its sign. We can now concentrate our attention in the emission rate of the radiation: Starting from (36) we get

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** A more general discussion on the conservatian of the energy can be seen in Ref. 11.

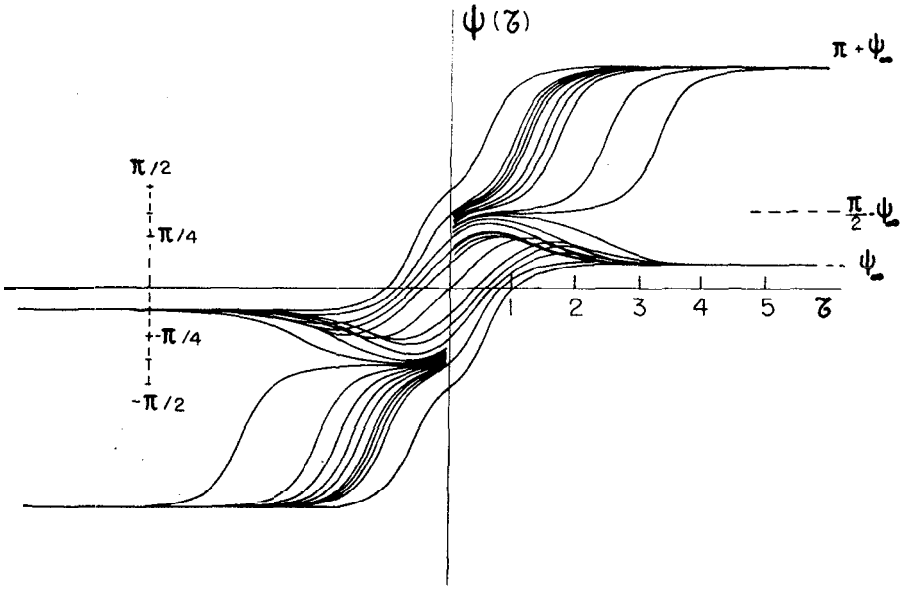

Fig.3 - Graphs of $\psi(\tau)$ for a = 1.5. The values of ψ_0, from $-\pi/2$ to $\pi/2$ are given by the intersections with the vertical axis. Also displayed are the stable ($\psi_\infty = 0.365$ rd) and unstable ($\pi/2 - \psi_\infty = 1.206$ rd) asymptotic solutions.

$$2|\vec{S}| = \frac{c}{2\pi} E_z^2 = -\frac{d}{dt}\left(\frac{1}{2}\hbar\omega_0 \sigma r_z\right), \tag{50}$$

where $\vec{S}$ is the value of Poynting vector on both sides of the planes. Eq. (50) gives the conversion rate of the energy (from the atoms to the field). Using (37), it can also be written as:

$$2|\vec{S}| = \hbar\omega_0^2 \sigma\alpha \sin^2\theta \sin^2\psi \ . \tag{51}$$

From (51) it is easy to see that the emission of the energy may depend critically on the initial conditions: choice of ψ at $\tau = 0$ (we are assuming $\theta(0) = \pi/2$ in all cases). If $\psi(0) = 0$, for example, we have $\vec{S}(0) = 0$, whereas for $\psi(0) = \pi/2$, $\vec{S}(0)$ has the maximum value $\hbar\omega_0^2\sigma\alpha$. It is then easy to predict that, for relatively large values of a, we will have in the second case a strong peak at $r = 0$, whereas in the first case the emission curve will present *two peaks* symmetrically disposed with respect to the origin.

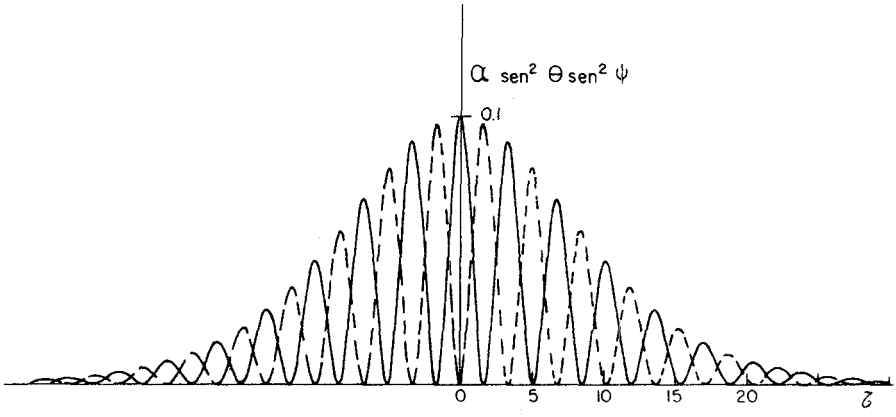

Fig.4 - Emission rates for $\alpha = 0.1$; $\psi = \pi/2$ (solid line) and $\psi_0 = 0$ (dashed line). The envelope of these curves is approximately given by $\alpha \operatorname{sech}^2(\alpha\tau)$.

it is alsa interesting to remark that the exponeritial decay of the radiation for large values of the time presents different features according to whether a < 1 or a > 1. For a << 1, as shown in (44), the attenuation increases with a. For a > 1, however, the exponent in (48), $a \sin^2 \psi' \simeq (a - \sqrt{\alpha^2-1})/2$ actually decreases with a (see also fig. 6). This effect can also be observed in the case $\Delta m = \pm 1$, where, from Eq. (32), $\sin\theta \simeq \exp\{-(\alpha/\alpha^2+1)\omega_0 t\}$. Examples of the radiation rate for different values of a and ψ_0 are given in Figs. 4, 5 and 6. The beha-

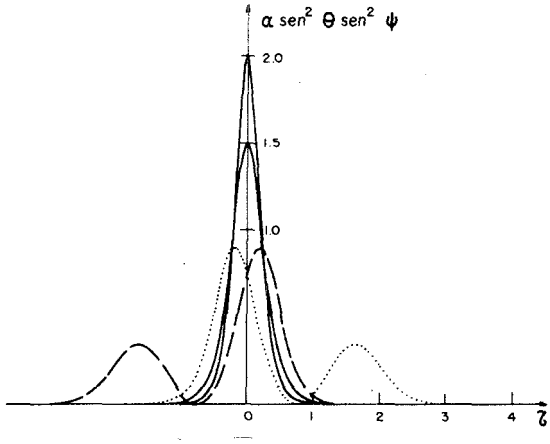

Fig.5 - Emission rates for $\alpha = 1.5$ ($\psi_0 = \pi/2, \pm\pi/4$) and $\alpha = 2$ ($\psi_0 = \pi/2$). The lincs are solid ($\psi_0 = \pi/2$), dashed ($\psi_0 = \pi/4$) or dotted ($\psi_0 = -\pi/4$).

vior of the electric field and dipole is illustrated in Fig. 7 and 8 for both oscillating and non-oscillating decays.

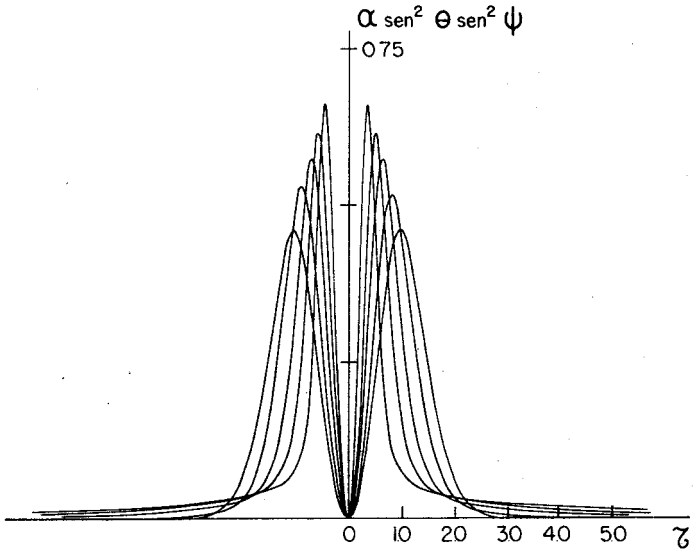

Fig.6 - Emission retes for $\psi_0 = 0$ and a = 1,1.5, 2.5, 4 and 7.5 (in the same order as the height of the peaks).

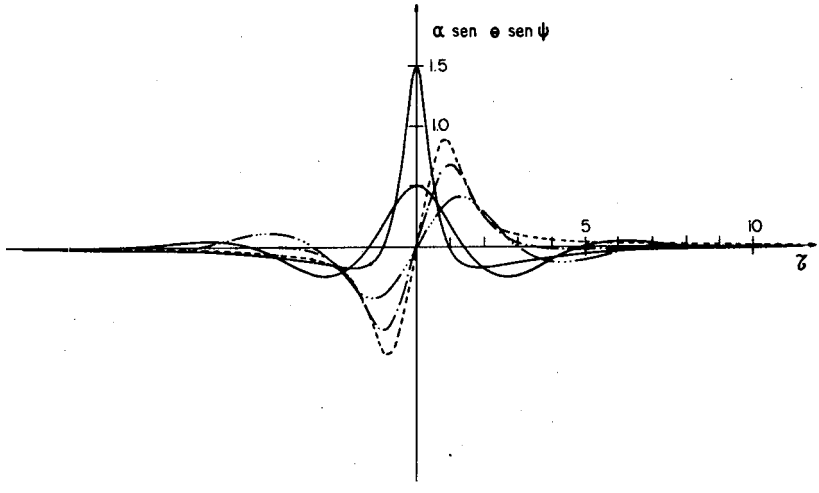

Fig.7 - Graphs of $\alpha \sin\theta \sin\psi$ (proportional to E_z) for $\psi_0 = 0$ (α = 0.5, 1 and 1.5; dashed lines) and $\psi = \pi/2$ (α = 0,5, 1.5 solid lines). The values of a are In the same order as the height of the peaks.

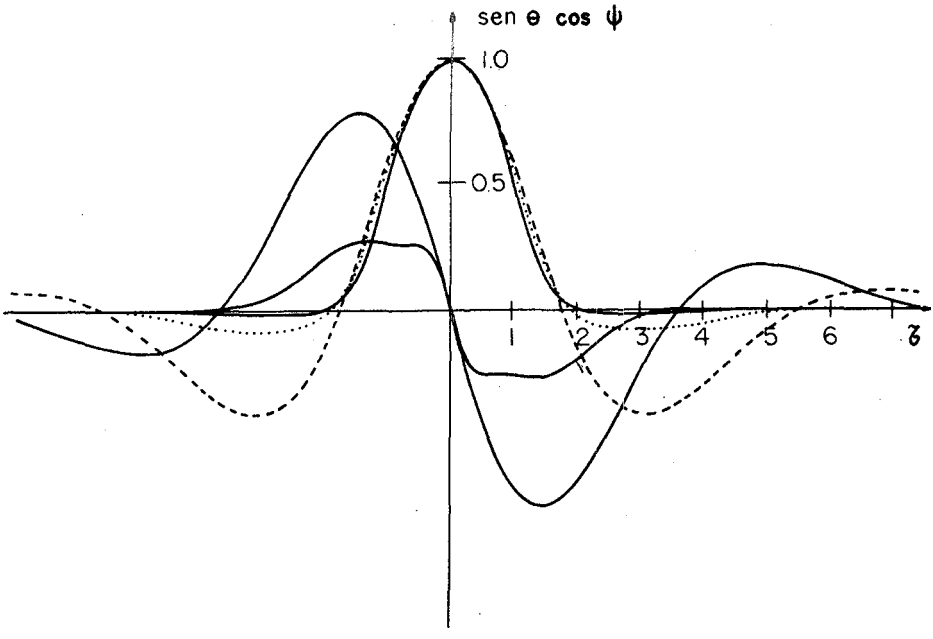

Fig.8 - Graphs of $r_x = \sin\theta\cos\psi$ for $\psi_0 = 0$, a = 0.5 (dashed), a = 1 (dotted), a = 1.5 (solid); and $\psi_0 = \pi/2$, a = 0.5 and 1.5 (solid lines through the origin).

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