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Geometric Properties of Unified Matrix Formulation of Gravitation, Electromagnetism and the Yang-Mills Field

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The geometric properties of a recent extension of Einstein's nonsymmetric unified field theory which includes isotopic gauge fields are discussed. The affinity associated to the isotopic gauge and the corresponding curvature are introduced in terms of the Yang-Mills potentials and field componen'ts. The Maxwell and Yang-Milis field equations are derived by imposing conditions on the matrix that represents the full covariant derivative of the matrix G_{aB} that replaces the tensor field h_{ab} of the nonsymnetric unified theory.

As propriedades geométricas de uma recente extensão' da teoria não- simétrica do campo unificado de Einstein que inclui campos de gauge isotópico são discutidas. A afinidade associada ao gauge isotópico e a curvatura correspondente são introduzidas em termos dos potenciais e das componentes dos campos de Yang-Mills. As equações de campo de Maxwell e Yang-Mills são deduzidas impondo-se condições sobre a matriz que representa a derivada covariante completa da matriz G_{aB} que substitui o campo tensorial h_{aB} da teoria unificada não-simétrica.

1. INTRODUCTION

The nonsymmetric unified field theory proposed by Einstein's recently reviewed by Moffat, Boal[2] and Borchsenius[3], may be presented in terms of tensor field $h_{\alpha\beta}$ and of a linear connection $\Delta^{\alpha}_{\beta\gamma}$ which are defined by

$$h_{\alpha\beta} = g_{\alpha\beta} + pF_{\alpha\beta} \qquad (1.1)$$

$$\Delta^{\alpha}_{\beta\gamma} = \{^{\alpha}_{\beta\gamma}\} - (2/p)\delta^{\alpha}_{\beta} A_{\gamma} \qquad (1.2)$$

where $g_{\alpha\beta}$ is the symmetric tensor field representing gravitation, and A_γ is defined by

$$A_\gamma = -1/3 \cdot p\, \Delta^{\beta}_{[\beta\gamma]}$$

p is a universal constant, and in the limit $p \to 0$ the Einstein-Maxwell theory is obtained, with $g_{\alpha\beta}$ and $\{^{\alpha}_{\beta\gamma}\}$ as the metric and connection of general relativity (Christoffel symbols). The quantities $F_{\alpha\beta}$ and A_a are the field tensor and potentials of Maxwell's theory.

Under an electromagnetic Abelian gauge transformation $A'_\alpha = A_\alpha - (1/2)p\lambda_{,\alpha}$ the tensor field $h_{\alpha\beta}$ remains invariant and the linear connection transforms as $\Delta'^{\alpha}_{\beta\gamma} = \Delta^{\alpha}_{\beta\gamma} + \delta^{\alpha}_{\beta} A_{,\gamma}$. Einstein has called such transformation the λ-transformation.

From the similarity between general relativity and the Yang-Mills theory which arises from the property that both theories are self-coupled gauge theories, it has been proposed a generalization of the unified Einstein's theory that contains both formulations[4]. Such extended gauge theory is obtained generalizing the objects $h_{\alpha\beta}$ and $\Delta^{\alpha}_{\beta\gamma}$ from ordinary e-numbers to 2×2 matrices which are related to a unitary symmetry ($SU(2)$ symmetry).

In this paper we study the geometric properties of such extended formulation. Explicit definition of the covariant, gauge invariant, derivatives involving objects which are 2 x 2 matrices is given. The concept of gauge transformations is now taken over by 2×2 maírices which are a linear combination of the Paulit matrices and the 2 x 2 identity matrix with coefficients given by arbitrary functions of the coordinates.

The notation used is the following: all relations involving matrices are always written in the matrix notation (that means, without explicit use of matrix indices). Greek letters denote spacetime degree of freedom. Small Latin letters designate degrees of freedom going from 1 to 4 and are used for labelling the basis elements in the space of the 2×2 matrices.

2. MATRIX FORMULATION OF THE NONSYMMETRIC UNIFIED FIELD THEORY

The tensor field $h_{\alpha\beta}$ of (1.1) describes gravitation and electromagnetism. In order to unify this formulation with the Yang-Mills field theory[5] it has been assumed that the components of the tensor field $h_{\alpha\beta}$ and of the linear connection $\Delta^{a}_{\beta\gamma}$ are matrices instead of ordinary functions[4]. This generalized tensor field is denoted by the symbol $G_{\alpha\beta}$ and may be written in the general form

$$G_{\alpha\beta} = q_{\alpha\beta\underline{i}}\,\omega_{\underline{i}} \tag{2.1}$$

where $\omega_{\underline{i}} = (\omega_{i},\omega_{4})$ form a basis in the space of the 2 × 2 matrices. The $\omega_{i}(\underline{i} = 1,2,3)$ are the three Pauli matrices and ω_4 is the 2 × 2 identity matrix. The basis elements $\omega_{\underline{i}}$ satisfy the conditions of commutation and of anticommutation

$$[\omega_{\underline{i}}\ \omega_{\underline{j}}] = l_{\underline{i}\underline{j}k}\,\omega_{k} \tag{2.2}$$

$$\{\omega_{\underline{i}}\ \omega_{\underline{j}}\} = f_{\underline{i}\underline{j}k}\,\omega_{k} \tag{2.3}$$

with the coefficients taking values

$$l_{\underline{i}44} = l_{\underline{i}4\underline{r}} = l_{\underline{ik}4} = 0,\quad l_{\underline{iks}} = 2i\ \varepsilon_{\underline{iks}}$$

$$f_{\underline{iks}} = f_{\underline{i}44} = f_{44\underline{i}} = 0,\ f_{\underline{ik}4} = f_{\underline{i}4\underline{k}} = 2\ \delta_{\underline{ik}},\ f_{444} = 2\ .$$

The generalized field theory which uses $G_{\alpha\beta}$ as field potentials is assumed to be invariant under the group of general coordinate transformations, as in general relativity, and under local transformations with generating matrices that satisfy the conditions $|M| = 1, M^{\dagger} = M^{-1}$ (local SU (2) transformations). Under these transformations we assume that $G_{\alpha\beta}$ transforms as

$$G'_{\alpha\beta}(x') = \frac{\partial x^{\mu}}{\partial x'^{\alpha}}\frac{\partial x^{\nu}}{\partial x'^{\beta}}\,G_{\mu\nu}(x) \tag{2.4}$$

$$\bar{G}_{\alpha\beta}(x) = M(x)\ G_{\alpha\beta}(x)\ M^{-1}(x) \tag{2.5}$$

For a discussion on the choice of the transformation law (2.4) see the appendix. The general form for the matrices M is

$$M(x) = \lambda_i(x)\ \omega_i \tag{2.6}$$

with the conditions

$$|M| = \lambda^2_4 - \vec{\lambda}^2 = 1, \quad i\vec{\lambda}^* \times \vec{\lambda} + 2\ \mathrm{Re}(\vec{\lambda}^* \lambda_4) = 0, \quad |\vec{\lambda}\cdot\vec{\lambda}|^2 + |\lambda_4|^2 = 1$$

Thus only three real parameters are involved in these local transformations, corresponding to local isotopic gauge paramaters. Any matrix in isotopic space transforms as $G_{\alpha\beta}$ of (2.5), and any two-component wave function describing a field with isotopic spin 1/2 transforms as

$$\bar{\psi}(x) = M(x)\ \psi(x) \ .$$

Since the derivatives of ψ do not 'transform as ψ under these transformations, it is necessary to'introduce a gauge covariant derivative, defined by

$$\psi_{||\alpha} = \partial_\alpha \psi + \Gamma_\alpha \psi \ . \tag{2.7}$$

thus

$$\bar{\psi}_{||\alpha} = M\ \psi_{||\alpha} \ . \tag{2.8}$$

Similarly

$$\bar{G}_{\alpha\beta||\gamma} = M\ G_{\alpha\beta||\gamma}\ M^{-1} \tag{2.9}$$

where $G_{\alpha\beta||\gamma}$ is defined by

$$G_{\alpha\beta||\gamma} = \partial_\gamma G_{\alpha\beta} + [\Gamma_\gamma, G_{\alpha\beta}] \tag{2.10}$$

From (2.7) and (2.8) the transformation law for the $SU(2)$ affinity is given by

$$\bar{\Gamma}_\alpha = M \, \Gamma_\alpha \, M^{-1} - M_{,\alpha} \, M^{-1} \qquad (2.11)$$

The four matrices Γ_α (with respect to the index α) form the components of a world four-vector.

3. THE COVARIANT DERIVATIVES

The generalization of the c-number tensor field $h_{\alpha\beta}$ to a 2 × 2 matrix $G_{\alpha\beta}(x)$ implies in a similar extension for the linear connection $\Delta^\alpha_{\beta\gamma}$. Borchsenius in his approach[4] considers this generalized affinity as a 2 × 2 matrix without explicit definition of the full covariant derivative of the $G_{\alpha\beta}$, and takes a decomposition of such matrix in the canonical basis ω_i in such form that the system of field equations derived from the variational principle (with a suitable form for the Lagrangian density) assume the well known form of the equations for gravitation, electromagnetism and for the Yang-Mills theory.

Here we consider an approach which is directly based on geometric concepts, similar to the approach used in general relativity. The isotopic degrees of freedom of $G_{\alpha\beta}$ are treated as internal indices, in a form similar to the theory of spinors in Riemannian spaces. The theory obtained is a theory of iso-spinors in the generalized Riemannian space of Einstein's unitary formulation. This method allows for a separation of the isotopic affinity from the affinity associated to gravitation and electromagnetism, similarly as the internal affinity is separated from the spacetime affinity in the conventional treatment of spinors in Riemannian spaces.

The full covariant derivative of $G_{\alpha\beta}$ is defined as

$$\begin{aligned} G_{\alpha\beta|\gamma} &= \partial_\gamma \, G_{\alpha\beta} + [\Gamma_\gamma \, G_{\alpha\beta}] - G_{\alpha\lambda} \, \Omega^\lambda{}_{\beta\gamma} - G_{\lambda\beta} \, \Omega^\lambda{}_{\alpha\gamma} \\ &= G_{\alpha\beta||\gamma} - G_{\alpha\lambda} \, \Omega^\lambda{}_{\beta\gamma} - G_{\lambda\beta} \, \Omega^\lambda{}_{\alpha\gamma} \qquad (3.1) \end{aligned}$$

where $G_{\alpha\beta||\gamma}$ is given by (2.10). It is also convenient to introduce the derivative

$$G_{\alpha\beta;\gamma} = \partial_\gamma G_{\alpha\beta} - G_{\alpha\lambda}\, \Omega^{\lambda}{}_{\beta\gamma} - G_{\lambda\beta}\, \Omega^{\lambda}{}_{\alpha\gamma} \tag{3.2}$$

The quantity $\Omega^{\lambda}{}_{\alpha\gamma}$ transforms under local rotation of the isotopic axis as $\bar{\Omega}^{\lambda}{}_{\alpha\gamma} = M\, \Omega^{\lambda}{}_{\alpha\gamma}\, M^{-1}$, and transforms as an affinity with respect to coordinate transformations. Thus, we also have

$$G_{\alpha\beta|\gamma} = G_{\alpha\beta;\gamma} + [\Gamma_\gamma, G_{\alpha\beta}] \tag{3.3}$$

The decomposition (3.3) shows that $G_{\alpha\beta|\gamma}$ transform as a third rank tensor under spacetime mappings. From (3.1), (2.5) and (2.9), as well as from the definition of $\Omega^{\lambda}{}_{\alpha\gamma}$, it follows that $G_{\alpha\beta|\gamma}$ transform under rotation of the isotopic axis similarly to $G_{\alpha\beta}$.

The quantities $G_{\alpha\beta|\gamma}$ are in general of the form

$$G_{\alpha\beta|\gamma} = \tau_{\alpha\beta\gamma i}\, \omega_i \tag{3.4}$$

Besides the components of $G_{\alpha\beta}$, namely the quantities $q_{\alpha\beta i}$ of (2.1), the equations (3.1) involve the quantities Γ_a, $\Omega^{\alpha}{}_{\beta\gamma}$ and the $\tau_{\alpha\beta\gamma i}$ of (3.4) as unknows. In order to obtain correct values for these quantities we proceed as follows.: first it should be noted that in absence of electromagnetic field the trace of $G_{\alpha\beta}$ gives the metric of general relativity

$$Tr\, G_{\alpha\beta} = g_{\alpha\beta} \tag{3.5}$$

Along with (3.5) we also impose that the affinity of a Riemannian spacetime is obtained by taking trace in equation (3.1). Then it follows from (3.4) that

$$\tau_{\alpha\beta\gamma 4} = 0 \tag{3.6}$$

and from (3.1) we get

$$\Omega^{\nu}{}_{\lambda\alpha} = \left\{ {\nu \atop \lambda\alpha} \right\} \omega_4 \tag{3.7}$$

where $\left\{ {\nu \atop \lambda\alpha} \right\}$ are the Christoffel symbols. The introduction of electromagnetic **field** is obtained as a correction to equation (3.5), (3.6) and (3.7). The quantity $\tau_{\alpha\beta\gamma 4}$ is chosen in such form that the equations (3. 1) are consistent with the definitions (1.1) and (1.2), which here takethe form

$$Tr\ G_{\alpha\beta} = h_{\alpha\beta} \tag{3.8}$$

$$\Omega^{\alpha}{}_{\beta\gamma} = \Delta^{\alpha}_{\beta\gamma}\ \omega_4 \tag{3.9}$$

We take

$$2\tau_{\alpha\beta\gamma 4} = \frac{4}{p} A_{\gamma}(g_{\alpha\beta} + p\ F_{\alpha\beta}) + p\ E_{\alpha\beta\gamma} \tag{3.10}$$

such that

$$g^{\beta\gamma}\ E_{\alpha\beta\gamma} = j_{\alpha}\ \sqrt{-g} \tag{3.11}$$

A possible expression for $E_{\alpha\beta\gamma}$, function of the electromagnetic variables, satifying (3.11) is

$$E_{\alpha\beta\gamma} = a\ A_{\alpha}\left[\frac{1}{A^2} A_{\beta}\ A_{\gamma} - \frac{1}{4}\ g_{\beta\gamma}\right] + \frac{1}{4}\ g_{\beta\gamma}\ j_{\alpha}\ \sqrt{-g}$$

where a is a constant. From (3.1), (3.8), (3.9) and (3.10) we find

$$F_{\alpha\beta;\gamma} = E_{\alpha\beta\gamma} \tag{3.12}$$

Here $F_{\alpha\beta;\gamma}$ denotes the covariant derivative of the c-number quantity $F_{\alpha\beta}$ with affinity $\{^{\alpha}_{\lambda\nu}\}$. From (3.11) and (3.12) the **Maxwell** equations in presente of gravitation are obtained as

$$g^{\beta\gamma}(Tr\ G_{\alpha\beta|\gamma} - 2\tau_{\alpha\beta\gamma 4}) = 0$$

where $Tr\ G_{\alpha\beta|\gamma}$ is derived from (3.1). Again from (3.1) and from (3.4) we have

$$[\Gamma_{\gamma}, G_{\alpha\beta}]' = \Gamma_{\alpha\beta\gamma i}\ \omega_i - \partial_{\gamma}\ G_{\alpha\beta} + G_{\alpha\lambda}\ \Omega^{\lambda}_{\beta\gamma} + G_{\lambda\beta}\ \Omega^{\lambda}_{\alpha\gamma} \tag{3.13}$$

writing

$$\Gamma_\gamma = p_{\gamma i}\, \omega_i$$

we obtain from (3.13)

$$p_{\gamma i}\, q_{\alpha\beta j}\, l_{ij\underline{k}} = \tau_{\alpha\beta\gamma\underline{k}} - q_{\alpha\beta\underline{k},\gamma} + q_{\alpha\lambda\underline{k}} \left[\{{}^{\lambda}_{\beta\gamma}\} - \frac{2}{p}\, \delta^{\lambda}_{\beta}\, A_\gamma\right]$$

$$+\, q_{\alpha\beta\underline{k}} \left[\{{}^{\lambda}_{\alpha\gamma}\} - \frac{2}{p}\, A_\gamma\, \delta^{\lambda}_{\alpha}\right]$$

After some easy calculations this equation takes the form

$$2i\, \vec{p}_\gamma \times \vec{q}_{\alpha\beta} = \vec{\tau}_{\alpha\beta\gamma} - \vec{q}_{\alpha\beta;\gamma} - \frac{4}{p}\, A_\gamma\, \vec{q}_{\alpha\beta} \qquad (3.14)$$

here again $\vec{q}_{\alpha\beta;\gamma}$ is the covariant derivative with affinity $\{{}^{\alpha}_{\beta\gamma}\}$. We may write this equation in compact form by introducing the covariant derivative with respect to the affinity $\Delta^{\alpha}_{\beta\gamma}$ of (1.2). In this case (3.14) takes the form

$$\begin{aligned} 2i\, \vec{p}_\gamma \times \vec{q}_{\alpha\beta} &= \vec{\tau}_{\alpha\beta\gamma} - D_\gamma\, \vec{q}_{\alpha\beta} \\ D_\gamma\, \vec{q}_{\alpha\beta} &= \partial_\gamma\, \vec{q}_{\alpha\beta} - \vec{q}_{\alpha\lambda}\, \Delta^{\lambda}{}_{\beta\gamma} - \vec{q}_{\lambda\beta}\, \Delta^{\lambda}_{\alpha\gamma} \end{aligned} \qquad (3.15)$$

Equations (3.14) or (3.15), similarly to before, imply in the Yang-Mills equations by contraction on the indices β, γ if

$$\vec{p}_\gamma = \frac{\kappa}{i}\, \vec{b}_\gamma, \quad \vec{q}_{\alpha\beta} = \vec{f}_{\alpha\beta}, \quad g^{\beta\gamma}\, \vec{\tau}_{\alpha\beta\gamma} = \vec{j}_\alpha \sqrt{-g}$$

Here we have the isotopic vector meson interacting with gravitation and electromagnetism. However, the introduction of the electromagnetic interaction of the isotopic meson* generates Abelian gauge dependent terms in the equation of motion (the derivative $D_\gamma\, \vec{q}_{\alpha\beta}$ is dependent on the electromagnetic gauge). It is possible to redefine $\vec{\tau}_{\alpha\beta\gamma}$ in such form that all Abelian gauge dependent terms disappear. Writing

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* the $\vec{b}_\alpha$ and $\vec{f}_{\alpha\beta}$ are real quantities, thus no second kind gauge transformation involving a phase factor is possible.

$$\vec{\tau}_{\alpha\beta\gamma} = \vec{\psi}_{\alpha\beta\gamma} + \frac{4}{p} A_\gamma \vec{q}_{\alpha\beta}$$

$$g^{\beta\gamma} \vec{\psi}_{\alpha\beta\gamma} = \vec{J}_\alpha \sqrt{-g}$$

we obtain the Yang-Mills field equation by contraction on the indices B, γ with covariant derivative $\vec{q}_{\alpha\beta;\gamma}$ and gauge invariance is obtained. A possible choice for $\vec{\psi}_{\alpha\beta\gamma}$ is

$$\vec{\psi}_{\alpha\beta\gamma} = \frac{1}{4} g_{\beta\gamma} \vec{J}_\alpha \sqrt{-g} + c \vec{b}_\alpha \left[\frac{\vec{b}_\beta \cdot \vec{b}_\gamma}{\vec{b}_\lambda \cdot \vec{b}^\lambda} - \frac{1}{4} g_{\beta\gamma} \right]$$

where c is a constant. From the expression of the $SU(2)$ affinity and from the left hand side of (3.13) we see that the coefficient $p_{\gamma 4}$ may be set equal to zero. Thus, the affinity Γ_γ takes the form

$$\Gamma_\gamma = \frac{\kappa}{i} \vec{b}_\gamma \cdot \vec{\omega} \qquad (3.16)$$

4. THE CURVATURES

Formulas like (3.1) define the full covariant derivative of a mixed object with two covariant world indices, and a covariant and another contravariant internal, or isotopic, indices. More general objects may be considered. The process of definition of the full covariant derivative of such objects follows the conventional method used in general relativity.

In this section we determine the curvatures associated to the affinities obtained in the previous section. Curvatures are determined from the commutator of covariant derivatives, in the usual form.

For a field with isotopic spin 1/2 the covariant derivative is given by (2.7). The commutator of this derivative is of the form

$$\psi_{||\alpha\beta} - \psi_{||\beta\alpha} = P_{\alpha\beta} \psi$$

where

$$P_{\alpha\beta} = \Gamma_{\alpha,\beta} - \Gamma_{\beta,\alpha} + [\Gamma_\beta \Gamma_\alpha] \tag{4.1}$$

is the curvature tensor associated to the affinity Γ_α. Under a local rotation in isotopic spin space, which induces on the Γ_α the variation (2.11), the $P_{\alpha\beta}$ transform as a mixed second rank internal tensor

$$\bar{P}_{\alpha\beta} = M\, P_{\alpha\beta}\, M^{-1}$$

Under coordinate transformations the components of each matrix $P_{\alpha\beta}$ (namely the components with respect to the indices α,β) behave as an antisymmetric second rank world tensor.

From (3.16) and (4.1) the explicit value for the internal curvature is

$$P_{\alpha\beta} = \frac{\kappa}{i} \vec{f}_{\alpha\beta} \cdot \vec{\omega} \tag{4.2}$$

where

$$\vec{f}_{\alpha\beta} = \vec{b}_{\alpha,\beta} - \vec{b}_{\beta,\alpha} - 2\kappa\, \vec{b}_\alpha \times \vec{b}_\beta$$

is the Yang-Mills field tensor. Considering a local infinitesimal rotation in the isotopic spin space which induces on the iso-spinors a transformation with matrix $M(x)$ of the form

$$M(x) = \exp(i\vec{\Lambda}(x) \cdot \vec{\omega}) = \omega_4 + i\vec{\Lambda}(x) \cdot \vec{\omega}$$

(here $\vec{\Lambda}(x)$ has the form $\vec{\Lambda}(x) = \varepsilon \vec{f}(x)$ with ε a first order quantity), the variation of the affinity Γ_α given by (2.11) takes the form

$$\delta\Gamma_\alpha = i\vec{\Lambda} \cdot [\vec{\omega}, \Gamma_\alpha] - i\vec{\Lambda}_{,\alpha} \cdot \vec{\omega} \tag{4.3}$$

Substituting the value of Γ_α given by (3.16) in (4.3) we find the Yang-Mills gauge transformation formula for the potentials $\vec{b}_\alpha$. As was mentioned before, presently we consider this unified formulation as a theory in spacetime with internal structure, posessing $SU(2)$ symmetry. The present formalism has some similarities with the theory of two-component spinors in curved spacetime, taking, however, in consideration the two different internal symnetries. At the same time we have different properties for our present formalism, as for instance the fact that the metric tensor (or its nonsymmetric extension) becomes a 2 × 2 matrix. Along with the similarities with the conventional spacetime formulation of theories with internal symmetry structure (in curved or in flat spaces), it should be mentioned that our present formalism of covariant differentiation is directly related to a well known process used in relativity[6], here applied to $SU(2)$ transformations.

Given the object $S_{\mu\nu}$ the same index structure as $G_{\mu\nu}$ we have for the second order full covariant derivative of this quantity

$$S_{\mu\nu|\alpha\beta} - S_{\mu\nu|\beta\alpha} = [P_{\alpha\beta}, S_{\mu\nu}] - S_{\mu\lambda} B^{\lambda}{}_{\nu\alpha\beta} - S_{\lambda\nu} B^{\lambda}{}_{\mu\alpha\beta} + S_{\mu\nu|\lambda} (\Omega^{\lambda}{}_{\beta\alpha} - \Omega^{\lambda}{}_{\alpha\beta}) \tag{4.4}$$

where

$$B^{\lambda}{}_{\mu\alpha\beta} = \Omega^{\lambda}{}_{\mu\alpha,\beta} - \Omega^{\lambda}{}_{\mu\beta,\alpha} - \Omega^{\lambda}{}_{\tau\beta}\, \Omega^{\tau}{}_{\mu\alpha} - \Omega^{\lambda}{}_{\tau\alpha}\, \Omega^{\tau}{}_{\mu\beta} \tag{4.5}$$

from (3.9) we get

$$B^{\lambda}{}_{\mu\alpha\beta} = C^{\lambda}{}_{\mu\alpha\beta}\, \omega_4$$

with

$$C^{\lambda}{}_{\mu\alpha\beta} = \Delta^{\lambda}{}_{\mu\alpha,\beta} - \Delta^{\lambda}{}_{\mu\beta,\alpha} + \Delta^{\lambda}{}_{\tau\beta}\, \Delta^{\tau}{}_{\mu\alpha} - \Delta^{\lambda}{}_{\tau\alpha}\, \Delta^{\tau}{}_{\mu\beta} \tag{4.6}$$

The explicit value for the curvature $C^{X}{}_{\mu\alpha\beta}$ is obtained from the equation (1.2), we find

$$C^{\lambda}{}_{\mu\alpha\beta} = R^{\lambda}{}_{\mu\alpha\beta} + \frac{2}{p}\, \delta^{\lambda}_{\mu}\, F_{\alpha\beta}$$

where $R^{\lambda}{}_{\mu\alpha\beta}$ is the Riemann tensor and $F_{\alpha\beta} = \partial_{\alpha} A_{\beta} - \partial_{\beta} A_{a}$ is the electromagnetic field tensor. Therefore the curvature $C^{\lambda}{}_{\mu\alpha\beta}$ is independent of the choice of the Abelian gauge. It should be noted that the equation (4.

4) contains a term proportional to the torsion $\Omega^{\lambda}{}_{\beta\alpha} - \Omega^{\lambda}{}_{\alpha\beta}$. This terrn depends on the choice of the Abelian gauge. However, this fact does not represent a serious unphysical consequence of the theory, since from the field theoretic point of view the fundamental quantity is the Lagrangian density. This function depends only on the curvatures , and the formalism developed from the Action principle is free of problems of this sort.

Borchsenius[4] in his approach uses the Action principle in the Palatini form and derives the expression for the affinities by variations, and by a convenient decomposition in the canonical basis in such form that the equations resulting from the variational principle (by variations in the affinities) assume the form of the Maxwell and Yang-Mills equations. We have shown that the affinities may be fixed by first order covariant differential conditions on $G_{\mu\nu}$ (similarly to the geometric approach to general relativity based on the Riemannian geometry), and by suitable choice of the right hand side of such conditions it is possible to derive the Maxwell and Yang-Mills field equations. In this method the variational principle has to be taken in the usual form by considering only variations with respect to $G_{\mu\nu}$, and the affinities assume the expression derived in the previous sections.

APPENDIX

Coordinate transformations under matrix form

The expression (2.4) for the transformation law of $G_{\alpha\beta}$ under coordinate transformations may be generalized in order to take into full account the fact that $G_{\alpha\beta}$ is a matrix. We begin by considering four functions of the new coordinates x', denoted by $\xi^{\alpha}_{i}(x') = x^{a}_{i}$, and form the matrices $M^{\alpha} = \xi^{\alpha}_{i}\omega_{i}$.
The generalized transforrnation law is then taken as

$$G'_{\mu\nu}(x') = M^{\alpha}_{,\mu}\, G_{\alpha\beta}\, M^{\beta}_{,\nu} = \xi^{\alpha}_{i,\mu}\, \xi^{\beta}_{j,\nu}\, \omega_{i}\, G_{\alpha\beta}\, \omega_{j} \qquad (A.1)$$

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* here there is no reference to methods involving the torsion as a physical quantity.

In this formula the expression'of $G_{\alpha\beta}$ has to be substituted by (2.1). Now, we recall that $Tr\, G_{\alpha\beta} = g_{\alpha\beta}$ is the Einstein's metric, or in the more general case is equal to the nonsymnetric tensor $h_{\alpha\beta}$. In any of these two cases a usual tensor transformation law has to be obtained by taking trace in both sides of (A.1). Using the formula

$$Tr(\omega_i\, \omega_j\, \omega_k) = 1/2 \cdot f_{ijs}\, f_{sk4} + 1/2 \cdot f_{ijs}\, 1_{sk4} + 1/2 \cdot 1_{ijs}\, f_{sk4} + 1/2 \cdot 1_{ijs}\, 1_{sk4}$$

we obtain

$$Tr\, G'_{\mu\nu} = 2\, q_{\alpha\beta 4}\, \xi^{\alpha}_{4,\mu}\, \xi^{\beta}_{4,\nu} + 2\, \xi^{\alpha}_{4,\mu}\, \vec{q}_{\alpha\beta} \cdot \vec{\xi}^{\beta}_{,\nu} + 2\, \xi^{\beta}_{4,\nu}\, \vec{q}_{\alpha\beta} \cdot \vec{\xi}^{\alpha}_{,\mu} + q_{\alpha\beta 4}\, \vec{\xi}^{\alpha}_{,\mu} \cdot \vec{\xi}^{\beta}_{,\nu} + 2i\, \vec{q}_{\alpha\beta} \cdot (\vec{\xi}^{\alpha}_{,\mu} \times \vec{\xi}^{\beta}_{,\nu})$$

this formula shows that true coordinate transformations are of the form $\vec{\xi}^{\alpha} = 0$, or in other terms, only the first factor on the right hand side of this equation has significance. Similar result holds for other geometrical objects, for instance for a vectór

$$V_{\alpha} = v_{\alpha i}\, \omega_i, \qquad V'_{\mu} = M^{\alpha}_{,\mu}\, V_{\alpha} = \xi^{\alpha}_{i,\mu}\, v_{\alpha j}\, \omega_i\, \omega_j$$

Thus

$$Tr\, V'_{\alpha} = 2\, \xi^{\mu}_{4,\alpha}\, v_{\mu 4} + 2\, \vec{v}_{\mu} \cdot \vec{\xi}^{\mu}_{,\alpha}$$

Therefore the expression (2.4) is the correct form of considering the transformation law of geometrical objects (with, or without, internal indices) under coordinate transformations. The present type of generalization (Eq. (A.1)) may be of interest if we try to formulate a unified law of transformation involving simultaneously the coordinate transformations and the isotopic gauge transformations. However this is not the intention in this paper, since we considered the isotopic gauge transformation as a internal transformation.

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