Revista Brasileira de *Física,* Vol. 8, *N? 2,* 1978

Connections and Spinor Connections Associated with Finite Groups of Transformations

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Recebido em **21** de Novembro de 1977

Modified and generalized Spinor Connections are obtained from the non-symmetric connections of Groups of Transformation on the Riemannian spaces which are invariant varieties of these groups. The cases of the Rotation Groups and Lorentz-De Sitter Groups in n dimensions are worked out as an example.

conexões espinoriais modificadas e generalizadas são obtidas das conexões não-simétricas de grupos de transformação sobre os espaços riemannianos que são variedades invariantes desses grupos. Os casos dos grupos de rotação e de Lorentz-De Sitter em n dimensões são resolvidos como exemplos.

1. INTRODUCTION

Dirac's equation in general covariant form[1] exhibits a spinor connection the traceless part of which is the electromagnetic potential. The remaining part is related to the gravitational field and Klein suggested to obtain Einstein's equations from it[2]. Laurent[3] who developed the formalism of the spinor connection further considered already more general connections than that obtained from the requirement that the covariant derivative of the Gamma matrices:

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$$\gamma^{\mu}_{\ \|\gamma} = \gamma^{\mu}_{;\nu} + [\gamma^{\mu}, \Gamma_{\nu}] \tag{1}$$

should vanish. He obtained this way a somewhat modified gravitational theory, Any such connection Γ_{μ} can easily be seen to keep the covariant derivative of the metric tensor

$$g_{\mu\nu} = \frac{1}{2} \{\gamma_{\mu}, \gamma_{\nu}\} \tag{2}$$

vanishing:

$$g_{\mu\nu|'\alpha} = 0 \tag{2a}$$

Novello considered such generalized spinor connections also in relation with the vector meson of weak interactions[4].

The introduction of the modified spinor connection remained however rather arbitrary and also the relation of the conventional connection with the gravitational law appears not straight forward.

Halpern[5] showed that the conventional spinor connection can serve as potential for the gravitational as well as the electromagnetic field in a unified way with a Lagrangian of the form:

$$L = C_1 R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} + \frac{C_2}{\kappa} R + \frac{1}{4} f_{\mu\nu} f^{\mu\nu} \tag{3}$$

(C_1, C_2 constants, $\kappa = \frac{8\pi G}{c^4}$ Einstein's constant).

The theory can be derived by generalizing the electromagnetic gauge transformations to spin transformations. A subsequent work[6] showed that if Poincaré covariance is modified to De Sitter covariance the constant C_1 can be related to that of the De Sitter group.

The present work seeks to make the introduction of modified spinor connections more consistent by relating them to non-Riemannian connections of invariance groups.

2. GENERAL RELATIONS BETWEEN SPINOR CONNECTIONS AND LINEAR CONNECTIONS IN SPACE

A linear connection Λ^i_{jk} of an n-dimensional space may simply be defined by its transformation law w.r.t. coordinate transformations[7]::

$$\Lambda'^h_{jk} \frac{\partial x^i}{\partial x'^h} = \Lambda^i_{\ell m} \frac{\partial x^\ell}{\partial x'^j} \frac{\partial x^m}{\partial x'^k} + \frac{\partial^2 x^i}{\partial x'^j \partial x'^k} \tag{1}$$

It determines a covariant derivative of vectors which are denoted in case of the connection of Riemannian space: $\Lambda^i_{jk} = \{^i_{jk}\}$ by a semicolon and in the general case by a bar. A parallelism among vectors at different points is thus defined;

A spinor connection Γ_i may be defined by its law of transformation w.r.t. spin transformations $\psi' = S\psi$:

$$\Gamma'_k = S\Gamma_k S^{-1} - \frac{\partial S}{\partial x^k} S^{-1} \tag{2}$$

it defines a covariant derivative of spinors denoted by a double bar

$$\psi_{||k} = \frac{\partial \psi}{\partial x^k} + \Gamma_k \psi \tag{2a}$$

Entities which transform w.r.t. coordinate transformations as well as w.r.t. spin transformations as the γ^k have thus a covariant derivative (also denoted by a double bar) in which the space connection as well as the spinor connection occurs:

$$\gamma^i_{||k} = \gamma^i_{|k} + [\gamma^i, \Gamma_k] = \frac{\partial \gamma^i}{\partial x^k} + \gamma^j \Lambda^i_{jk} + [\gamma^i, \Gamma_k] \tag{3}$$

Clearly because of (1.1) $g_{mn|k} = g_{mn||k}$. We shall deal here chiefly with connections for which $g_{mn|k} = 0$ so that there exist spinor connections Γ_k for which also $\gamma^i{}_{||k} = 0$. The integrability conditions for $\gamma^i{}_{||k} = 0$ are:

$$\gamma^n\left(\frac{\partial\Lambda^i_{nk}}{\partial x^\ell} - \frac{\partial\Lambda^i_{n\ell}}{\partial x^k} - \Lambda^m_{n\ell}\Lambda^i_{mk} + \Lambda^m_{nk}\Lambda^i_{n\ell}\right) =$$

$$= \left[\gamma^i, \left(\frac{\partial\Gamma_\ell}{\partial x^k} - \frac{\partial\Gamma_k}{\partial x^\ell} + [\Gamma_k, \Gamma_\ell]\right)\right] \tag{4}$$

The left-hand side equals $\gamma^n\Lambda^i_{n\ell k}$ where $\Lambda_{n\ell k}$ is the curvature tensor of the connection Λ^i_{nk}. One sees that eq. (4) is only consistent if:

$$g_{ni}\Lambda^i_{m\ell k} + g_{mi}\Lambda^i_{n\ell k} = 0 \tag{4a}$$

Then

$$\frac{\partial\Gamma_\ell}{\partial x^k} - \frac{\partial\Gamma_k}{\partial x^\ell} + [\Gamma_k, \Gamma_\ell] = \frac{1}{2}\gamma^r\gamma^s\Lambda_{rs\ell k} + \frac{\partial a_k}{\partial x^\ell} - \frac{\partial a_\ell}{\partial x^k} \tag{5}$$

with an arbitrary vector field $a_k(r)$. Because of $\gamma^k|_\ell = [\Gamma_\ell, \gamma^k]$ one can calculate the commutator of Γ_R with all possible expressions γ_A $(A=1\cdots16)$ formed out of γ^k and obtains finally the general form of Γ_k expressed in terms of the trace of the $\{\gamma_B, \gamma_{A|\ell}\}$ and the γ_A.[3]

3. GEOMETRICAL PROPERTIES OF THE GROUP OF TRANSFORMATIONS

We contemplate a continuous group of transformations of $n \geq 4$ variables x^i with r group parameters a^α. We deal in general with the case where four among the x^i or functional combination there of describe the space and time variables.

The parameter a^α must be essential, this means their role cannot be replaced by a smaller number of parameters.

A transformation $x^i \to x'^i$ with parameters a^α is thus of the form:

$$x'^i = f^i(x^1\ldots x^n;a^1\ldots a^r) \tag{1}$$

and because of the group property two consecutive transformations with parameters a_1^α and a_2^α can be combined to a single transformation with parameters a_3^α by a law:

$$a_3^\alpha = \phi^\alpha(a_1^1,\ldots,a_1^r;\ a_2^1,\ldots,a_2^r) \tag{1a}$$

The functions f^i, ϕ^α must fulfill all the requirements imposed by their role in a group of transformations.[7]

Each set of x^i may be considered as a point of a manifold M_n and each set of a^α as a point of a manifold M_r. The functions $f^i(x,a)$ of Eq. (1) determine thus point transformations $x\to x'$ in M_n. The x and the a's can both be subject to coordinate transformations. In a suitable canonical coordinate system of M_r the symbols of the group: $\xi_\alpha^i(x)$ are related to the generators:[7]

$$X_\alpha = \left(\frac{\partial x^i}{\partial a^\alpha}\right)_{a^\alpha=a_0^\alpha} \frac{\partial}{\partial x^i} = \xi_\alpha^i(x)\,\frac{\partial}{\partial x^i} \tag{2}$$

a_0^α are those values of the set of parameters which effect the identity transformation: $x^i = f^i(x,a_0)$. $\xi_\alpha^i(x)$ transforms as a contravariant vector w.r.t. coordinate transformations of the x^i and as a covariant vector w.r.t. coordinate transformations of the a^α. The ξ_α^i are linear independent for coefficients which depend only on the a^α but they are in general not linear independent for coefficients which depend on the x. Let the rank of the matrix (ξ_α^i) for nonsingular points be q. If $q<r$, choose the coordinates such that $\xi_1^i\ldots\xi_q^i$ are linear independent; then there exist relations:

$$\xi_\mu^i(x) = \phi_\mu^\varepsilon(x)\,\xi_\varepsilon^i(x) \qquad (\varepsilon=1\ldots q,\ \mu = q+1\ldots r) \tag{3}$$

The group is transitive if $q\geq n$ (any regular point can then be transformed into any other point x'). We shall deal here with intransitive groups for which $r>n>q$. Then a well known theorem[7] by G.Fubini states that there exist q-dimensional minimal invariant varieties such that the group induced on any such variety has r parameters. The coordinates x^i can be chosen such that there are q variables in the variety, so that one has to deal there with the equations of a transitive group.

The group of rotations in a flat space of n dimensions has for example $r = n(n-1)/2$ essential parameters. The symbols in Cartesian coordinates are:

$$\xi^i_{[\alpha,\beta]} = \frac{1}{2}\, x^n(\delta^i_\alpha \eta_{n\beta} - \delta^i_\beta \eta_{n\alpha}) \tag{4}$$

The rank $q=n-1$ and the minimal invariant varieties are the $(n-1)$-dimensional spheres around the origin. This example can be extended to the case of n-dimensional Lorentz groups by choosing an indefinite metric or by making coordinates imaginary. $n=5$ then deals with the De Sittergroup. A further theorem[7] says that if suitable conditions (to be stated later) are fulfilled, there exist Riemannian spaces for which the r-parameter group is a group of motions for which the ξ^i_α are the Killing vectors and the components of the metric tensor g_{ij} of such spaces involve at least $1/2(n-q)(n-q+1)$ arbitrary constants.

The n-1-dimensional spheres or hyperboloids of our example are such Riemannian spaces with one arbitrary constant. One can always, under the general condition stated, introduce a new x-coordinate system such that the symbols have the property:

$$\xi^v_\alpha = 0 \qquad \text{for } (\alpha = 1,\ldots,r;\, v = q+1,\ldots n) \tag{5}$$

in this case the equations $x^v = a^v$ = const. define an invariant variety for each $n-q$ set of constants a^v, because $X_a x^v = 0$. The new coordinates in our example are polar coordinates and the invariant varieties are the concentric spheres (or hyperboloids). The symbols which we denoted by $\xi'^i_{[\alpha,\beta]}$ in our example (with two antisynunetric indices running each from one to n instead of one index running from one to $r = (n/2)$) assume the form

$$\xi'^{i}_{[\alpha,\beta]} = \frac{1}{2} y^n \left(\frac{\partial y^n}{\partial x^\beta}\frac{\partial y^i}{\partial x^\alpha} - \frac{\partial y^n}{\partial x^\alpha}\frac{\partial y^i}{\partial x^\beta}\right) \quad .$$

We call the new coordinates $x'^m = y^m$ ($m = 1 \ldots n$) with $y^n = R = (\sum_{K=1}^{n} (x^k)^2)^{1/2}$ (no summation over the elevated index n), We have

$$\begin{aligned}
x^1 &= -R \sin y^{n-1} \sin y^{n-2} \ldots \sin y^1 \\
x^2 &= R \sin y^{n-1} \sin y^{n-2} \ldots \cos y^1 \\
x^3 &= R \sin y^{n-1} \sin y^{n-2} \ldots \sin y^3 \cos y^2 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x^n &= R \cos y^{n-1} \\
y^k &= \arc\cos\left(\frac{x^{k+1}}{r_{k+1}}\right) \quad \text{for } k = 1 \ldots n-1 \quad \text{with} \quad r_m = \left(\sum_{\ell=1}^{m} (x^\ell)^2\right)^{1/2} \\
y^n &= R = r_n
\end{aligned} \tag{6}$$

We introduce now in the new coordinate system, where the ξ^i_a fulfill eq. (5), new functions ξ^β_j for $\beta,\alpha = 1\ldots q$, $j,i = 1\ldots q$ such that:

$$\xi^j_\alpha \xi^\beta_j = \delta^\beta_\alpha \quad \text{and} \quad \xi^i_\beta \xi^\beta_j = \delta^i_j \qquad (i,j,\alpha,\beta = 1\ldots q) \tag{7}$$

here we have assumed that the ξ^i_a with a ranging from 1 to q are linear independent. In the case of our example, we must be careful not to identify the double indices each of which runs from 1 to n with one single index from 1 to $q = n-1$. We may identify this single index α with the double index $[\alpha,n]$ and obtain:

$$\xi^i_\alpha = \frac{R}{2}\left(\frac{\partial y^n}{\partial x^n}\frac{\partial y^i}{\partial x^\alpha} - \frac{\partial y^n}{\partial x^\alpha}\frac{\partial y^i}{\partial x^n}\right) \qquad \xi^\beta_j = \frac{2}{x^n}\frac{\partial x^\beta}{\partial y^i} \tag{7'}$$

(no summation over index n)

With these funstions $\xi^i_\alpha, \xi^j_\beta$ in the new coordinate system with the property of eq. (7) we form now:

$$\Lambda^{i}_{\ell k} = \xi^{j}_{\beta}\ \frac{\partial\, \xi^{\beta}_{k}}{\partial\, x^{\ell}} \qquad \begin{pmatrix} j,k,\beta = 1,\ldots,q \\ \ell = 1\ldots n \end{pmatrix} \tag{8}$$

this entity has the transformation character of a connection (see eq. 2-1); it is in general not symmetric in the two lower indices.

$$\frac{\partial \xi^{i}_{\alpha}}{\partial x^{k}} + \xi^{j}_{\alpha}\ \Lambda^{i}_{kj} = 0 \tag{8a}$$

Using the relations $\xi^{\blacksquare}_{\mu} = \phi^{\alpha}_{\mu}(x)\ \xi^{i}_{\alpha}\left(\begin{matrix}\alpha = 1\ldots q \\ \mu = q+1\ldots r\end{matrix}\right)$ by which all the ξ^{i}_{ρ} can be expressed by q of them and the definition of the structure constants: $[X_{\rho}, X_{\sigma}] = c^{\tau}_{\rho\sigma} X_{\tau}$ one finds. For the antisymmetric part:

$$\Lambda^{i}_{jk} - \Lambda^{i}_{kj} = (c^{\gamma}_{\alpha\beta} + c^{\mu}_{\alpha\beta}\ \phi^{\gamma}_{\mu})\xi^{\alpha}_{j}\, \xi^{\beta}_{k}\, \xi^{i}_{\gamma} \tag{9}$$

where all indices run from $1\ldots q$ except $\mu = q+1\ldots r$.

The entity[7]

$$\Lambda^{i}_{jk\ell} = \frac{\partial \Lambda^{i}_{jk}}{\partial x^{\ell}} - \frac{\partial \Lambda^{i}_{j\ell}}{\partial x^{k}} + \Lambda^{m}_{jk}\ \Lambda^{i}_{m\ell} - \Lambda^{m}_{j\ell}\ \Lambda^{i}_{mk} \tag{10}$$

is equal to

$$\Lambda^{i}_{jk\ell} = c^{\mu}_{\alpha\beta}\ \frac{\partial \phi^{\gamma}_{\mu}}{\partial x^{j}}\ \xi^{\alpha}_{k}\, \xi^{\beta}_{\ell}\, \xi^{i}_{\mu} \qquad \begin{pmatrix} \mu = q+1,\ldots,r \\ j = 1,\ldots,n \\ \text{all others}: 1,\ldots,q \end{pmatrix} \tag{10a}$$

In case of our example

$$\Lambda^{j}_{\ell k} = \frac{\partial^2 x^{\beta}}{\partial y^{k} \partial y^{\ell}}\ \left(\frac{\partial y^{j}}{\partial x^{\beta}} - \frac{\partial y^{j}}{\partial x^{n}}\, \frac{\partial y^{n}}{\partial x^{\beta}}\, \frac{R}{x^{n}}\right) - \delta^{i}_{k}\ \frac{1}{x^{n}}\, \frac{\partial x^{n}}{\partial y^{\ell}} \tag{8'}$$

(no summation over n)

The structure constants are:

$$C^{[g,h]}_{[b,c]\,[\ell,f]} = \frac{1}{4}\{\delta^h_c(\delta^g_\ell\eta_{bf} - \delta^g_f\eta_{b\ell}) - \delta^h_b(\delta^g_\ell\eta_{cf} - \delta^g_f\eta_{c\ell}) - (g\leftrightarrow h)\} \qquad (11)$$

for c = $f = n$ this becomes:

$$C^{[\gamma,\delta]}_{[\alpha,n]\,[\beta,n]} = \frac{1}{4}\eta_{nn}(\delta^\gamma_\alpha\,\delta^\delta_\beta - \delta^\gamma_\beta\,\delta^\delta_\alpha) \qquad (\alpha,\beta,\gamma = 1,\ldots,q{=}n{-}1) \qquad (11')$$

eepressed in the original x-coordinates:

$$\xi^i_{[\alpha,\beta]} = \phi^{[\gamma,n]}_{[\alpha,\beta]}\;\xi^i_{[\gamma,n]} \qquad\qquad \phi^{[\gamma,n]}_{[\alpha,\beta]} = \frac{x^\rho}{x^n}(\eta_{\rho\beta}\delta^\gamma_\alpha - \eta_{\rho\alpha}\delta^\gamma_\beta) \qquad (3')$$

with which one can verify eq.(9) and finds from (10a):

$$\Lambda^i_{jk\ell} = \frac{1}{2(x^n)^2}\,\eta_{rs}\;\frac{\partial x^r}{\partial y^j}\,\frac{\partial x^s}{\partial y^p}\,(\delta^i_k\;\delta^p_\ell - \delta^i_\ell\;\delta^p_k) \qquad \left\{\begin{matrix} j,r,s = 1,\ldots,n \\ i,p,\ell,h=1,\ldots,q=n-1\end{matrix}\right\} \qquad (10'a)$$

We are searching now for a metric for which our group is a group of motions so that the metric properties remain unchanged for an infinitesimal point transformation. Thus:

$$\xi^k_\alpha\,\frac{\partial g_{ij}}{\partial x^k} + g_{ik}\;\frac{\partial \xi^k_\alpha}{\partial x^j} + g_{kj}\;\frac{\partial \xi^k_\alpha}{\partial x^i} = 0 \qquad (12)$$

We can consider either the n-dimensional space where the group is intransitive or the q-dimensional space of the invariant variety where the induced group is transitive. We have in any case $r>q\geq n$. Multiplying eq. (12) with ξ^a_R and summing over $(a \equiv 1\ldots q)$ gives:

$$\frac{\partial g_{ij}}{\partial x^k} - g_{i\ell}\,\Lambda^{\ell}_{jk} - g_{\ell j}\,\Lambda^{\ell}_{ik} = 0 \qquad \begin{Bmatrix} \ell,k = 1\ldots q \\ i,j = 1\ldots n \end{Bmatrix} \qquad (12a)$$

There remain however, additional equations for $\alpha > q$ in eq.(12):

$$\xi^{\ell}_{\beta}\,\left(g_{i\ell}\,\frac{\partial \phi^{\beta}_{\alpha}}{\partial x^j} + g_{\ell j}\,\frac{\partial \phi^{\beta}_{\alpha}}{\partial x^i}\right) = 0 \qquad \begin{Bmatrix} \ell,\beta = 1,\ldots,q \\ i,j = 1,\ldots,n \\ \alpha = q+1,\ldots,r \end{Bmatrix} \qquad (12b)$$

Eqs. (12) have as conditions of integrability:

$$g_{i\ell}\,\Lambda^{\ell}_{jmk} + g_{\ell j}\,\Lambda^{\ell}_{imk} = 0 \qquad \begin{Bmatrix} \ell,m,k = 1,\ldots,q \\ i,j = 1,\ldots n \end{Bmatrix} \qquad (12c)$$

which, according to eq.(10), is satisfied if eq.(12b) is satisfied. Eqs. (12a,b) are a system of differential equations with the independent variables $y^1 \ldots y^q$. It is shown in ref. 7 that if eqs. (12b) can be fulfilled for certain values of the independent variables by a set of g_{ij} with det $g_{ij} \neq 0$ the solution with such values determines a Riemannian space with metric $g_{ij}(x)$. One finds by inspection that

$$g_{ij} = \frac{\partial x^r}{\partial y^i}\,\frac{\partial x^s}{\partial y^j}\,\eta_{rs}$$

is a solution to eqs. (12) or (12a,b). The n-dimensional space is thus flat and the $q=n-1$-dimensional spaces have the metric of an imbedded hypersurface.

4. RELATION TO THE SPINOR CONNECTIONS

We have seen that a nontransitive group of transformations can define a Riemannian space on which the induced group is transitive, and a connection (3.8) on this space which has a skew symmetric part (3.9). This connection fulfills relations (3.12, 2.4a) and therefore according to the results of Section 2, one can construct a spinor connection on the Riemannian space from it.

We are mainly interested in physical space time where $q=4$ and themetric hes the appropriate signature. The group in our example is then the De Sitter or anti-De Sitter group.[6] We have the four-dimensional γ^μ which fulfill relation (1.2) and want to find the spinor connection for which $\gamma_{\mu||\nu} = \gamma_{\mu|\nu}$ $[\gamma_\mu, \Gamma_\nu] = 0$. One can thus express Γ_ν in terms of the $\gamma_{\mu|\nu}$:

$$\Gamma_\nu = \frac{1}{16n} \sum_{r=1}^{n} \frac{1}{r!} T_r\{\gamma_{\mu|\nu}, \gamma^\mu \gamma_{\beta_s \beta_{s-1} \ldots \beta_1}\} \gamma^{\beta_1 \ldots \beta_s} \qquad (1)$$

with the tensor-spinor operator:

$$\gamma^{\beta_1 \ldots \beta_s} = \frac{1}{n!} \binom{n}{s} \xi^{\beta_1 \ldots \beta_s \beta_{s+1} \ldots \beta_n} \xi_{\beta_{s+1} \ldots \beta_n \alpha_1 \ldots \alpha_s} \gamma^{\alpha_1} \gamma^{\alpha_2} \ldots \gamma^{\alpha_s} \qquad (1a)$$

γ^μ transforrn here as the component of a vector w.r.t. coordinate transformations.

We have still to calculate the tensor $A^j_{Rk} = \{^j_{\ell k}\} - A$ in tha case of our example. The physical case is $q = n-1 = 4$ where the invariant varieties are De Sitter or anti-De Sitter spaces. We start out witha set of basic Clifford numbers $\overset{0}{\gamma}{}^1, \ldots, \overset{0}{\gamma}{}^4$ and remaining in a four-dimensional representation we choose. $\overset{0}{\gamma}{}^5 = \overset{0}{\gamma}{}^1 \overset{0}{\gamma}{}^2 \overset{0}{\gamma}{}^3 \overset{0}{\gamma}{}^4$ for the Cartesian x^k-coordinates. A well-known formulation, valid in any coordinate system, is:

$$\overset{0}{\gamma}{}^5 = \gamma_{(5)} = \frac{1}{24\sqrt{g}} \xi_{\alpha\beta\gamma\delta} \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta$$

One can now form the tensor $A^j_{\ell k} = \{^j_{\ell k}\} - \Lambda^j_{\ell k}$ from the two coonections. If one considers these connections and $A^j_{\ell k}$ as four-dimensional entities, one obtains

$$A^j_{\ell k} = \frac{1}{x^5} \{ -\frac{\partial y^j}{\partial x^n} g_{\ell k} + \delta^j_k \frac{\partial x^n}{\partial y^\ell} \} \qquad (2)$$

(rio summation overn) $(j, k, \ell = 1 \ldots 4)$

$g_{\ell k}$ is the metric tensor on the invariant variety $y^n=R$=const. in y-coordinates.

$A^j_{\ell k}$ vanishes at the point: $x^5=R$ $x^c=0$ $(c=1,2,3,4)$ and consequently also any contribution of it to Γ_k. This is due to our choice of the symbols $\xi_{[\alpha,5]}$ $(\alpha=1\ldots4)$ as independent entities. $\Delta\Gamma_k$ depends on the choice of the independent symbols and can thus be hardly itself a physical entity. (Contributions of $\Delta\Gamma_k$ to the curvature tensor may however be physical).

One can also consider $\{^j_{\ell k}\}$, Λ^j_{0k} as connections in five-dimensional space and form the five-dimensional $A^j_{\ell k}$ (with $A^5_{\ell k}=0$). Such a five-dimensional entity can also be employed in a theory of four dimensions by assuming suitable homogeneity conditions [6].

One obtains then with the γ^ℓ of the y-coordinates:

$$\gamma^\ell A^j_{\ell k} = \frac{1}{x_5} (\gamma_{(5)} \delta^j_k - \frac{\partial y^j}{\partial x^5} \gamma_k) \tag{3}$$

This gives a contribution to Γ_k:

$$\Delta\Gamma_k = \frac{1}{96} T_r\{(\gamma_{j;k} - \gamma_{j|k}), \gamma^j\gamma_{\beta_3}\gamma_{\beta_2}\gamma_{\beta_1}\}\gamma^{\beta_1\beta_2\beta_3} \tag{3a}$$

with $\sigma_{\alpha\beta} = \frac{1}{2} [\gamma_\alpha, \gamma_\beta]$.

One can introduce a natural coordinate system in the neighbourhood of the point $\left(\begin{smallmatrix} x^5=R \\ x^\alpha=0 \end{smallmatrix}\right.$, where R is the constant of the De-Sitter space, and obtain then:

$$\Delta\Gamma_n = -\frac{1}{R}\sigma_{k5} \tag{2a}$$

The physical interpretation of this additional term to the spinor connection, which is related to the connection of the invariance group, is however not straight forward because in the Dirac equations the Clifford numbers should be chosen as generators of the invariance group. Details about these features are found in ref. 6.*

The work of L. Halpern was supported D.O.E. grant no. EY-76-S-05-3509. He would like to thank Prof. P.A.M. Dirac and J. Lannutti for the possibility to work in their Institute.

This work was initiated during a visit at the Centro Brasileiro de Pesquisas Físicas. Most of it was written during a visit at the Instituut van Theoretische Physika of the University of Amsterdam when the author was confined to a hospital in Amsterdam. He would like to thank particularly Professor S. Wouthuysen for his extremely kind and everpresent assistance with discussions and scientific material and to M. J. Hoek, A. Perk for their help.

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* (The difference of a factor 2 is due to the choice of the generators of the group).