Revista Brasileira de Física, Vol. 8, Nº 2, 1978

Syrnmetry Properties of Ernbedded Space-Times

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Recebido em 21 de Novembro de **1977**

The properties of the groups $\sim 0(r,s)$ in the local embeddíng formalism applied to general relativity are investigated. It is shown that these groups are the common generators of the Poincaré group in general relativity. Furthermore it a sufficiently small neighborhood of the embedding point is chosen those groups can also generate isometries of space-time in that neighborhood.

As propriedades dos grupos $\sim 0(r,s)$ no formalismo de imersão local aplicado 5 Relatividade Geral são investigadas. Mostra-se que esses grupos são os geradores comuns do grupo de Poincaré na Relatividade Geral. Se, além disso, escolhe-se uma vizinhança suficientemente pequena do ponto de imersão, aqueles grupos podem gerar também isometrias do espaço-tempo nessa vizinhança.

1. INTRODUCTION

The local isometric embedding of the de Sitter space-time in a five dimensional pseudo Euclidean manifold has proven to be a useful too fordealing with the representations of the 10 parameters group of isometries of that space-time. The identification between the group of isometriesof the de Sitter space-time and the group of isometries of the embedding space is a consequence of the constant curvature of that space-time[1].

It would be desirable to obtain similar results for an arbitrary space-time of general relativity. However it is almost an intuitive fact that

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the isometries of an arbitrary space-time, if they exist, can not be obtained from the homogeneous groups of isometries of the embedding spaces, unless additional conditions on these groups are imposed. The purpose of this note is to determine these conditions. Two results emerge: One of them says that the Poincaré-group of the tangent spaces is a common limit of all homogeneous groups of isometries of the embedding spaces. The second result says that in a sufficiently small neighborhood of the embedding point it is possible to derive local isometries from the homogeneous groups of isometries of the embedding space.

These two results seem to indicate that the various groups of isometries of the minimal embedding spaces are endowed with a certain physical meaning which has not been exploited except in the particular cases of constant curvature space-times.

2. LOCAL ISOMETRIC EMBEDDINGS

Let V be any space-time of general relativity embedded minimally, isometrically and locally in a flat p-dimensional pseudo Euclidean space with metric signature $r(+)$, $s(-)$, denoted by $M(r,s)$. There are 22 such spaces sufficient to embedd all possible space-times of general relativity[2,3].

Since $M(r,s)$ are flat spaces, use of Cartesían coordinates denoted by X^μ, $\mu = 1...p$ is possible throughout (in the following all Greek indices run from 1 to p. The metric of $M(r,s)$ in a Cartesian frame is denoted by $\eta_{\mu\nu}$),

Space-time functions and tensor fields in $M(r,s)$ are more conveniently written in a Gadssian coordinate system based on V, formed by space-time coordinates: x^i, $i = 1...4$ and coordinates x^A, A = 5.. .p measured along straight lines orthogonal to V and to themselves (small case Latin indices run from 1 to 4 while capital Latin indices run from 5 to p)[3]. The Gaussian coordinate system thus formed has coordinates denoted by x^α = (x^k, x^A), a = $1...p$. In this system the space-time hypersurface is defined simply by the equation

$$x^A = 0. \qquad (1)$$

Let $f(x^\alpha)$ be any function or tensor field defined in $M(r,s)$. The restriction of $f(x^\alpha)$ to space-time is a space-time function or tensor field defined by

$$f(x^\alpha)\big|_V = \lim_{x^A \to 0} (f(x^\alpha)) .$$

A space-time defined function is denoted by $f(V)$.

The embedding of V in $M(r,s)$ is realized when a coordinate transformation like

$$X^\mu = X^\mu(x^i) . \tag{2}$$

is given, so that

$$g_{ij}(V)\, dx^i dx^j = \eta_{\mu\nu} dX^\mu dX^\nu ,$$

where $g_{ij}(V)$ denotes the space-time metric components in the coordinates x^i. Notice that *(2)* is a coordinate transformation for space-time points only. If points outside space-time are considered the transformation should also include the coordinates x^A:

$$X^\mu = X^\mu(x^\alpha) .$$

with inverse transformation given by

$$x^a = x^\alpha(X^\mu) .$$

The components of the Jacobian matrices of the above transformation are denoted by

$$X^\mu{}_\alpha = \frac{\partial X^\mu}{\partial x^\alpha} , \quad x^\alpha{}_\mu = \frac{\partial x^\alpha}{\partial X^\mu} ,$$

so that

$$X^\mu{}_a\, x^\beta{}_\mu = \delta^\beta_\alpha \quad \text{and} \quad X^\mu{}_\alpha\, x^a{}_V = \delta^\mu_\nu .$$

The Gaussian components of the metric tensor of $M(r,s)$ are given by

$$g_{\alpha\beta} = X^{\mu}{}_{\alpha} X^{\nu}{}_{\beta} \eta_{\mu\nu}, \quad g^{\alpha\beta} = x^{\alpha}{}_{\mu} x^{\beta}{}_{\nu} \eta^{\mu\nu} \tag{3}$$

It is interesting to notice that the embedding formalism canbe regarded as completion of the tetrad formalism in the sense that in the embedding formalism, besides the tangent plane T_0 to V at a point 0, an orthogonal space N_0 is also considered, the dimension of N_0 depending on the curvature of V.

The components of the metric tensor of $M(r,s)$ which are tangent to space-time are $g^{ij}(x^{\alpha})$, $g_{ij}(x^{\alpha})$ and these are not necessarily equal to $g^{ij}(V)$, $g_{ij}(V)$. From simple geodesic calculations in $M(r,s)$, it is found that[3]

$$(g_{\alpha\beta}(x^{\alpha})) = \begin{pmatrix} g_{ij}(x^{\alpha}) & g_{iA}(x^{\alpha}) \\ g_{iA}(x^{\alpha}) & g_{AB} \end{pmatrix} \tag{4}$$

where

$$g_{iA} = P_{iAB} x^{B}, \quad P_{iAB}(x^{i}) + P_{iBA}(x^{i}) = 0, \quad g_{AB} = \varepsilon_{A} \delta_{AB}, \quad \varepsilon_{A} = \pm 1 .$$

It follows that

$$g_{ij}(x^{\alpha})|_{V} = g_{ij}(V), \quad g_{iA}(x^{\alpha})|_{V} = 0, \quad g_{AB}(x^{\alpha})|_{V} = g_{AB}(x^{\alpha}) = \varepsilon_{A}\delta_{AB}, \ \varepsilon_{A} = \pm 1 .$$

The Christoffel symbols of the metric affine connection ∇M of $M(r,s)$ in the Gaussian frame are given by

$$\Gamma^{\alpha}_{\beta\gamma}(\nabla M) = \frac{1}{2} g^{\alpha\delta}(g_{\beta\delta,\gamma} + g_{\gamma\delta,\beta} - g_{\beta\gamma,\delta}) . \tag{5}$$

On the other hand the metric affine connection of V has Christoffel symbols given by

$$\Gamma^{i}_{jk}(V) = \frac{1}{2} g^{im}(V)(g_{jm,k}(V) + g_{km,i}(V) - g_{jk,m}(V)) .$$

It is not difficult to see that (tangent components)

$$\Gamma^{i}_{jk}(\nabla M)\,|_V = \Gamma^{i}_{jk}(V) \quad . \tag{6}$$

The remaining components of ∇M are not necessarily null. Their space-time projections are:

$$\Gamma^{A}_{mk}(\nabla M)\,|_V = \frac{1}{2}\, g^{\alpha\delta}(g_{m\alpha,k} + g_{k\alpha,m} - g_{mk,\alpha})\,|_V = -\frac{1}{2}\,(g^{AB}\, g_{mk,B})\,|_V \,,$$

$$\Gamma^{i}_{mB}(\nabla M)\,|_V = \frac{1}{2}\, g^{i\alpha}(g_{m\alpha,B} + g_{B\alpha,m} - g_{mB,\alpha})\,|_V = \frac{1}{2}\,(g^{ik}\, g_{mk,B})\,|_V.$$

Therefore the following identity holds:

$$(g^{ik}\,\Gamma^{A}_{mk} + g^{AB}\,\Gamma^{i}_{mB})\,|_V = 0. \tag{7}$$

On the other hand

$$\Gamma^{A}_{mC}|_V = \frac{1}{2}\, g^{A\delta}(g_{m\delta,C} + g_{C\delta,m} - g_{mC,\delta})\,|_V = g^{AD}(g_{m[D,C]})\,|_V \,,$$

which imply in the identity:

$$g^{C(B}\Gamma^{A)}_{mC}\,|_V = \frac{1}{2}\,(g^{CB}\Gamma^{A}_{mC} + g^{CA}\Gamma^{B}_{mC})\,|_V = g^{B(C}g^{D)A}g_{m[D,C]}\,|_V = 0. \tag{8}$$

The Gauss-Codazzi equations which define the embedding are obtained from the vanishing of the Gaussian components of the Riemannian tensorof $M(r, s)$ restricted to V:

$$R^{i}_{jk\ell}|_V = 0\,, \quad R^{A}_{ijk}|_V = 0\,, \quad R^{A}_{Bij}|_V = 0\,, \quad R^{A}_{iBj}|_V = 0.$$

The first of these equations gives the metric signature of $M(r,s)$:

$$R^{i}_{jke}(V) = 2\sum_{A=5}^{p} g^{im}(V)\,\varepsilon_A K_{Am[k} K_{|Aj|\ell]} \tag{9}$$

where

$$K_{Aij} = -\frac{1}{2}\, g_{ij\,A}|_V \,.$$

Theorem 1.

In a minimal embedding the signature numbers ε_A are uniquely defined.

Syppose there are two sets of signature numbers ε_A, ε'_A satisfying (9). If $\varepsilon_A \neq \varepsilon'_A$ for all values of A then (9) gives

$$\sum_{A=5}^{p} (\varepsilon_A - \varepsilon'_A) g^{im}(V) K_{Am} [k^K |Aj| \ell] . \qquad (10)$$

Since the values of ε_A, ε'_A are 1 and they are distinct it follows that $\varepsilon''_A = \frac{1}{2}(\varepsilon_A - \varepsilon'_A)$ can be set (numerically) equal to ε_A (or ε'_A). However in this case equation (10) would imply that the curvature of space-time is zero. Now suppose that $\varepsilon_A = \varepsilon'_A$ for A = 5...r but $\varepsilon_A \neq \varepsilon'_A$ for A = r + 1...p then equation (10) holds for A = r+1...p , giving

$$1 \quad (\varepsilon_p - \varepsilon'_p) g^{im}(V) K_{pm} [k^K |pj| \ell] = - \sum_{A=r+1}^{p-1} \frac{1}{2} (\varepsilon_A - \varepsilon'_A) g^{im}(V) K_{Am} [k^K |Aj| \ell] .$$

Again $\varepsilon''_A = \frac{1}{2}(\varepsilon_A - \varepsilon'_A)$ can be set numerically equal to ε_A. Replacing in (9) it follows that

$$R^i_{jke}(V) = 2 \sum_{A=5}^{r} \varepsilon_A g^{im}(V) K_{Am} [k^K |Aj| \ell] .$$

However this equation implies a r- dimensional embedding which is impossible if $r < p$.

3. ISOMETRIES OF $M(r,s)$ AS SYMMETRY GROUPS

The minimal local embedding is invariant under isometric transformation in $M(r,s)$ with fixed origin. In this and the next section the relationships between $SO(r,s)$ and physically interesting groups such as space--time isometries and the Poincaré group are studied.

An infinitesimal transformation of $SO(r,s)$ relating two arbitrary Gaussiam systems may by written as

$$x'^{\alpha} = x^{\alpha} + \xi^{\alpha}(x^{\beta}), \quad \xi^{(\alpha;\beta)}(\nabla M) = 0, \quad \text{(fixed origin)}, \tag{11}$$

where the covariant derivative, as indicated, is calculated with respect to the affine connection of $M(r,s)$. In particular if the two Gaussian systems are based on the same hypersurface V and the definition of V in these systerns is to be invariant, then it is required the additional condition

$$\xi^{A}|_{V} = 0. \tag{12}$$

Since Killing's equation in (11) are defined for any point in $M(r,s)$ it would be desirable to evaluate this equation in V:

$$\xi^{(\alpha:\beta)}(\nabla M)|_{V} = 0. \tag{13}$$

Assuming (11), the set of equations (12) and (13) together with the homogeneity condition (fixed origin) define a subgroup of $SO(r,s)$ denoted $SO(r,s)|_{V}$ and refered to as the space-time subgroup of $SO(r,s)$.

Theorem 2.

If V admits isometries then $\zeta^{i} = \xi^{i}|_{V}$ is a killing vector field in V where $\xi^{\alpha}(x^{\beta})$ satisfy (12) and (13).

In fact, from (5), (6), (12) and (13) it follows that

$$\xi^{(i;j)}(\nabla M)|_{V} = (g^{k(i}(V)\xi^{j)}|_{V})_{,k} + g^{k(i}(V)\Gamma^{j)}_{mk}(V)\xi^{m}|_{V} = \phi^{(i;j)}(V) = 0 .$$

It is clear that if ξ^{α} is a descriptor of $SO(r,s)$ then ξ^{i} generate only local rotations in V.

Theorem 3.

Given a Killing vector field in V, there is a vector field $\xi^{\alpha}(x^{\beta})$ in $M(r,s)$ such thaf $\xi^{(\alpha;\beta)}(\nabla M)|_{V} = 0$.

Let ϕ^i be the given Killing vector field in V:

$$\phi^{(i;j)}(V) = 0.$$

Now define the vector field $\xi^\alpha(x^\beta)$ in $M(r,s)$ by

$$\xi^i = \phi^i(x^k), \quad \xi^A = 0.$$

This vector field is tangent to V and is such that

$$\xi^{(\alpha;\beta)}(\nabla M) = g^{\gamma(\alpha}\xi^{\beta)}{}_{,\gamma} + g^{\gamma(\alpha}\Gamma^{\beta)}_{\delta\gamma}\xi^\delta ,$$

or equivalently

$$\xi^{(i;j)}(\nabla M) = g^{k(i}\phi^{j)}{}_{,k} + g^{k(i}\Gamma^{j)}_{mk}\phi^m + g^{A(i}\Gamma^{j)}_{mA}\phi^m , \tag{14}$$

$$\xi^{(i;A)}(\nabla M) = \frac{1}{2} g^{kA}\phi^i{}_{,k} + g^{k(i}\Gamma^{A)}_{mk}\phi^m + g^{B(i}\Gamma^{A)}_{mB}\phi^m , \tag{15}$$

$$\xi^{(A;B)}(\nabla M) = g^{\gamma(A}\Gamma^{B)}_{m\gamma}\,\phi^m. \tag{16}$$

The fact that ϕ^i is a Killing vector field in V has not yet been used. For that purpose the above equations are evaluated at V. Using $g^{iA}|_V = 0$ it foilows from (14) that

$$\xi^{(i;j)}(\nabla M)|_V = g^{k(i}\phi^{j)}{}_{,k} + g^{k(i}\Gamma^{j)}_{mk}\phi^m|_V = \phi^{(i;j)}(V) = 0.$$

Furthermore from the identities (7), (8), the equations (15), (16) in V are identically null.

$$\xi^{(i;A)}(\nabla M)|_V = (g^{ki}\Gamma^A_{mk} + g^{AB}\Gamma^i_{mB})|_V\phi^m \equiv 0$$

$$\xi^{(A;B)}(\nabla M)|_V = g^{C(A}\Gamma^{B)}_{mC}|_V\,\phi^m \equiv 0.$$

Notice that if V does not admit Killing vector fields then $\phi^i = \xi^i|_V = 0$, but $\xi^i \neq 0$ so that even in this case $SO(r,s)|_V$ is not the identity group in V.

Theorem 4.

The group $SO(r,s)|_V$ has as many parameters as the group of rotations in V plus a number $n \leq (p-4)(p-3)/2$.

Let $\varepsilon^{\mu\nu}$ be the $p(p-1)/2$ parameters of $SO(r,s)$ in Cartesian coordinates. These parameters can be translated to the Gaussian system by

$$\varepsilon^{\alpha\beta} = x^\alpha_\mu x^\beta_\nu \varepsilon^{\mu\nu} \quad .$$

The parameter of $SO(r,s)|_V$ are given by $\varepsilon^{\alpha\beta}|_V$. The number of independent parameters will depend on the values of x^α_μ and of the condition (12). Let $U^\mu = \varepsilon^\mu{}_\nu X^\mu = \varepsilon^{\mu\nu} X_\nu$ and

$$\xi^\alpha = x^\alpha_\mu U^\mu = x^\alpha_\mu \varepsilon^{\mu\nu} X_\nu = x^\alpha_\mu \varepsilon^{\mu\rho} X^\nu{}_\beta x^\beta_\rho X_\nu = \frac{1}{2}\, \varepsilon^{\alpha\beta}(X^\nu X_\nu)_{,\beta} .$$

Assuming that $X^\mu = X^\mu(x^k)$ then $X^\nu X_\nu = f(x^k)$ is a space-time function, $f_{,A} = 0$ and (12) is equivalent to

$$f(x^k)_{,j}\, \varepsilon^{Aj}|_V = 0 \quad .$$

Depending on the value of $f(x^k)$ this last equation may impose a relationship between the parameters ε^{Ai}. The maximum number of parameters ε^{Ai} results when this equations is identically satisfied and in thiscase there are $4(p-4)$ parameters $\varepsilon^{Ai}|_V$. On the other hand there are at most $(p-4)(p-5)/2$ parameters $\varepsilon^{AB}|_V$. Finally from theorems 2 and 3 the parameters $\varepsilon^{ij}|_V$ correspond to the group of local rotations in V.

In the case of the de Sitter space-time the condition (12) is identically satisfied and $SO(r,s)|_V$ has 10 parameters. In the case of the Schwarzschild space-time the function $f(x^k)$ depends only on the radial coordinate r and the final number of parameters is 5, with 3 space-time rotations plus two rotations $\varepsilon^{45}|_V, \varepsilon^{46}|_V$ which are associated to the time translation.

4. THE POINCARÉ GROUP IN GENERAL RELATIVITY

The Poincaré group in general relativity is not well defined because its translational part would not be a space-time transformation and it has to be postulated.[4] However in an embedded space-time the translations in a tangent plane can be derived from $SO(r,s)|_V$.

Theorem 5.

The Poincaré-group in the tangent plane at a point of a space-time is a flat contraction of $SO(r,s)|_V$.

By a flat contraction of a group defined in a curved space-time it is meant a group contraction[5] where the contracting factor depends on the surface curvature and it tends to zero in the flat limit of the surface. Consider that the surface in question is the embedded space-time V and that the group to be contracted is $SO(r,s)|_V$. Denoting by $L_{\alpha\beta}$ the Lie algebra generators of $SO(r,s)$ in the Gaussian system based on V. The commutators between such generators are given by:

$$[L_{\alpha\beta}, L_{\gamma\delta}] = g_{\alpha\gamma}L_{\beta\delta} + g_{\beta\delta}L_{\alpha\gamma} - g_{\alpha\delta}L_{\beta\gamma} - g_{\beta\gamma}L_{\alpha\delta} \;.$$

Separating the indices:

$$[L_{ij}, L_{k\ell}] = g_{ik}L_{j\ell} + g_{j\ell}L_{ik} - g_{i\ell}L_{jk} - g_{kj}L_{i\ell} \;,$$

$$[L_{ij}, L_{kA}] = g_{ik}L_{jA} + g_{jA}L_{ik} - g_{iA}L_{jk} - g_{jk}L_{iA} \;,$$

$$[L_{ij}, L_{AB}] = g_{iA}L_{jB} + g_{jB}L_{iA} - g_{iB}L_{jA} - g_{jA}L_{iB} \;,$$

$$[L_{iA},L_{jB}] = g_{ij}L_{AB} + g_{AB}L_{ij} - g_{iB}L_{Aj} - g_{Aj}L_{iB} ,$$

$$[L_{iA},L_{BC}] = g_{iB}L_{AC} + g_{AC}L_{iB} - g_{iC}L_{AB} - g_{AB}L_{iC} ,$$

$$[L_{AB},L_{CD}] = g_{AC}L_{BD} + g_{BD}L_{AC} - g_{AD}L_{BC} - g_{BC}L_{AD} .$$

Taking these equations in Vand using $g_{iA}|_V = 0$ the Lie algebra of $SO(r,s)|_V$ is obtained:

$$[L_{ij},L_{k\ell}]|_V = g_{ik}(V)L_{j\ell}(V) + g_{j\ell}(V)L_{ik}(V) - g_{i\ell}(V)L_{jk}(V) - g_{jk}(V)L_{i\ell}(V)$$

$$= [L_{ij}(V),L_{k\ell}(V)],$$

$$[L_{ij},L_{kA}]|_V = g_{ik}(V)(L_{jA})|_V - g_{jk}(V)(L_{iA})|_V ,$$

$$[L_{ij},L_{AB}]|_V = 0,$$

$$[L_{iA},L_{jB}]|_V = g_{ij}(V)(L_{AB})|_V + g_{AB}(V)(L_{ij})|_V ,$$

$$[L_{iA},L_{BC}]|_V = g_{AC}(V)(L_{iB})|_V - g_{AB}(V)(L_{iC})|_V ,$$

$$[L_{AB},L_{CD}]|_V = g_{AC}(V)(L_{BD})|_V + g_{BD}(L_{AC})|_V - g_{AD}(V)(L_{BC})|_V - g_{BC}(V)(L_{AD})|_V ,$$

$$= [L_{AB}|_V,L_{CD}|_V] .$$

Consider now that $a^A(V)$ are $p-4$ space-time functions which tend to zero in the flat limit of V. Defining the operators

$$\pi^i(V) = a^A L_{iA}|_V .$$

Then in terms of $\pi^i(V)$ the Lie algebra of $SO(r,s)|_V$ is given by

$$[L_{ij},L_{k\ell}]|_V = [L_{ij}(V),L_{k\ell}(V)] ,$$

$$[L_{ij}(V),\pi_k(V)] = g_{ik}(V)\pi_j(V) - g_{jk}(V)\pi_i(V) ,$$

$$[L_{ij}(V),(L_{AB})|_V] = 0 \ ,$$

$$[\pi_i(V),\pi_j(V)] = g_{ij}(V)\alpha^A(V)\alpha^B(V)(L_{AB})|_V + g_{AB}(V)\alpha^A(V)\alpha^B(V)L_{ij}(V) \ ,$$

$$[\pi_i(V),(L_{BC})|_V] = g_{AC}(V)\alpha^A(V)(L_{iB})|_V + g_{AB}(V)\alpha^A(V)L_{ij}(V) \ ,$$

$$[L_{AB},L_{CD}]|_V = [L_{AB}|_V,L_{CD}|_V] .$$

Now taking the flat limit of V and considering that in this flat limit the minimal embedding space tends to the Minkowski space-time $M(4,1)$ the following Lie algebra is obtained

$$[L_{ij}\ V\ ,L_{k\ell}\ V]|_{\text{flat}} = \eta_{ik}\bar{L}_{j\ell} + \eta_{j\ell}\bar{L}_{ik} - \eta_{i\ell}\bar{L}_{jk} - \eta_{jk}\bar{L}_{i\ell} \ ,$$

$$[L_{ij},\pi_k]|_{\text{flat}} = \eta_{ik}\pi_j - \eta_{jk}\pi_i \ ,$$

$$[\pi_i,\pi_j]|_{\text{flat}} = 0 \ ,$$

where $\bar{L}_{ij}$ denotes the operators of the Lie algebra of the proper Lorentz group. The resulting Lie algebra is isomorphic to the Lie algebra of the Poincaré-group.

This theorem has two main consequences; The first consequence already mentioned, is that the Poincaré-group in general relativity is now properly defined. The second consequence, perhaps with deeper physical implications is that the 22 groups $SO(r,s)$ may be regarded as the common generators of the Poincaré-group. Furthermore is a sufficiently small neigborhood of the embedding point is chosen then $SO(r,s)|_V$ can also generate the isometries of space-time (if they exist) in that neighborhood.

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