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Neutron Stars with Equation of State Given by Nuclear Thomas-Fermi Model

K.C. CHUNG and T. KODAMA

Centro Brasileiro de Pesquisas Físicas, Rio de Janeiro, RJ*

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A equation of state for neutron gas, based on Thomas-Fermi model, is used to recalculate the maximum mass of neutron stars. The complete equation of state is found to present a first order phase transition between the subnuclear regime without free neutron and the nuclear regime. This suggests that the sudden disintegration of the neutron-rich-nuclei may be very competitive with relation to the continuous neutron drip process. The mass limit for neutron stars was found to be 3.26 $M_\odot$.

Usa-se uma equação de estado para um gás de neutrons, baseada no modelo de Thomas-Fermi, para se calcular a máxima massa de estrelas de neutrons. Verifica-se que a equação de estado completa apresenta uma transição de fase de primeira ordem entre o regime sub-nuclear sem neutrons livres e o regime nuclear. Isto sugere que a desintegração subita dos nucleos ricos em neutrons, possa ser muito competitiva em relação ao processo de gotejamento continuo de neutrons. O limite de massa para estrelas de neutrons encontrado é de 3,26 $M_\odot$.

INTRODUCTION

It is firmly believed today that the pulsars are rotating neutron stars[1]. This fact encourages the study of neutron star structures, not only to investigate the pulsars themselves but also to understand other related astrophysical phenomena, such as supernova explosions, black holes and r--process nucleosynthesis.

* Postal address: Centro Brasileiro de Pesquisas Físicas, Av. Wenceslau Braz, 71 (fundos), Rio de Janeiro, RJ.

Neutron star calculations are very sensitive to the equation of state in nuclear regime. Recently, a equation of state, based on nuclear Thomas--Fermi model, was developped in connection with the study of heavy-ion physics, giving reasonable results[2]. As a valid alternative to the envolved nuclear manybody calculations and in order to examine the plausibility of this equation of state, we apply the Thomas-Fermi method to a new calculation of neutron star configuration.

In this paper, we investigate catalyzed neutron star configurations in static and spherically symmetric case (non-rotating star) with zero temperature. At densities of our interest (about 10^{14} g cm^{-3}), metrics of the spacetime deviate from the flat Euclidean metric[3]. Hence we utilized the hydrostatic equilibrium equations obtained from the General Relativity.

The hydrostatic equilibrium equations are presented in Chap. 1 and the equation of state in Chap. 2. The results and discussions are presented in Chap. 3.

1. HYDROSTATIC EQUILIBRIUM EOUATIONS

Assuming a spherically symmetric distribution of matter, we can write the line element as

$$ds^2 = e^{\nu}(cdt)^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + \mathrm{sen}^2\theta\ d\phi^2) \tag{1}$$

The diagonalyzed energy-momentum tensor, in the case of perfect gas, assumes the following form, obtained from Pascal's law,

$$T^1_1 = T^2_2 = T^3_3 = -p \quad ; \quad T^0_0 = \varepsilon = dc^2 \tag{2}$$

where p, ε and d denote, respectively, pressure, energy density and matter density (including the rest mass of the particles); c is the velocity of light.

In this case, the Einstein's equations reduce to

$$
\begin{aligned}
p &= \kappa^{-1} \left[e^{-\lambda}(r^{-1}\nu' + r^{-2}) - r^{-2}\right] \\
\varepsilon &= \kappa^{-1} \left[e^{-\lambda}(r^{-1}\lambda' + r^{-2}) + r^{-2}\right] \\
p' &= -\frac{p+\varepsilon}{2}\nu'
\end{aligned}
\tag{3}
$$

where the prime denotes derivative with respect to r; $\kappa \equiv 8\pi G/c^4$ and G the gravitational constant.

In the **external** region of the star ($T_{\mu\nu} = 0$),

$$
e^{-\lambda} = 1 - 2GM/rc^2 \tag{4}
$$

where M is the mass of the star.

The mass is given by[4]

$$
M = \frac{4\pi}{c^2}\int_0^R \varepsilon\, r^2 dr \tag{5}
$$

where R is the radius of the star. In eq. (5) the gravitational energies are included. The total energy E of the star is given by

$$
E = M c^2 \tag{6}
$$

and the total neutron (proton) number by[5]

$$
N_{n(p)} = 4\pi \int_0^R \rho_{n(p)}\, e^{\lambda/2}\, r^2 dr \tag{7}
$$

where $\rho_{n(p)}$ is the neutron (proton) number density.

The stellar structure can be inmediately determined by eqs. (3) and (5), since the equation of state

$$
\varepsilon = \varepsilon(\rho) \tag{8}
$$

is explicitly given. In eq. (8), ρ is the total number density of baryons. Note that, in the degenerate case, the equation of state has only one degree of freedom since the energy density does not depend on temperature, and the chemical composition depends only on matter density.

We are mainly interested in the configuration of the star, which is usually specified by the function

$$\rho = \rho(r) \tag{9}$$

so that we use the following equation obtained from eq. (3)[6]

$$\frac{d\rho}{dr} = -\frac{1}{2}\frac{\partial\varepsilon}{\partial\rho}\left(\frac{\partial^2\varepsilon}{\partial\rho^2}\right)^{-1} e^{\lambda}\,\kappa r\left[p + r^{-3}\int_0^{r}\varepsilon\, r'^2 dr'\right] \tag{10}$$

where we utilized the following relation between pressure and energy density,

$$p = \rho\frac{\partial\varepsilon}{\partial\rho} - \varepsilon \tag{11}$$

The configuration of a neutron star is obtained from eq. (10) with boundary conditions:

$$\begin{aligned} &\rho = \rho_c\,, \quad E = O && \text{for } r = 0 \\ &p = 0 && \text{for } r = R \end{aligned} \tag{12}$$

where ρ_c is the central value of number density.

2. EQUATION OF STATE

Due to changes in the chemical composition of matter and possible phase transitions, we have to consider the neutron star matter in the following phases:

a) electron-nucleus, corresponding to a density between 500 g cm^{-3} and d_{nd} (d_{nd} is the neutron drip density $\sim 10^{11}$ g cm^{-3}), with the pressure gradient given mainly by the electron kinetic energy. This phase is also called as subnuclear *regime* without free neutrons.

b) electron-nucleus-neutron, corresponding to a density between d_{nd} and d_n (d_n is the density in which all the nuclei dissolve out $\sim 10^{13}$ g cm^{-3}), with the free gas beginning to contribute to the pressure, but yet the dominant term is due to the relativistic electron gas. This phase is called subnuclear regime with free neutrons.

c) neutron-proton-electron, corresponding to a density between d_n and approximately 10^{15} g cm^{-3}, which is referred to as the nuclear regime. In this phase, almost the entire pressure is provided by the non-relativistic neutron gas.

d) neutron-proton-electron-muon-hyperon, corresponding to a density greater than 10^{15} g cm^{-3}, which is called the supernuclear or hyperonic regime. Here various elementary particles come into play, according to their threshold energies.

Let us discuss the above phases separately.

a) Subnuclear Regime Without Free Neutrons

We assume that, for a given density, there is only one type of nuclide and that such nuclei form a crystal structure, immersed in a degenerate electron gas. For densities $\sim$ 500 g cm^{-3} the electrons may be considered as free particles, and when the density rises up to about 10^6 g cm^{-3}, they should be treated as relativistic particles. If the density is around 10^7 g cm^{-3}, where the electron Fermi energy gas to $\sim$ 1 MeV, the electron capture

$$(Z,A) + e^- \rightarrow (Z-1, A) + \nu \tag{13}$$

can take place and it causes a gradual neutronization of the nuclei, so that they deviate more and more from the beta stability line ($N >> Z$).

The equation of state for this regime can be written as

$$\varepsilon_1 = M(Z,A)\ c^2\ \rho_c + \varepsilon_F + \varepsilon_C \tag{14}$$

The first term at right side of the equation represents the energy of nuclei (Z,A) with mass $M(Z,A)$ and number density $\rho_c = \rho/A$; ε_F is the average Fermi anergy of the gas (including the rest energy), which in the completely degenerate case is given by

$$\varepsilon_F = \frac{m_e^4 c^5}{32\ \pi^2\ \hbar^3}\ (\text{senh}\ \xi - \xi) \tag{15}$$

with

$$\xi = 4\ \ln\left[x + (1+x^2)^{1/2}\right]$$

and the relativity parameter x is related to the electron number density ρ_e by

$$x = \frac{\hbar}{m_e c}\ (3\pi^2\ \rho_e)^{1/3} \tag{16}$$

The last term in eq. (14), ε_c, results from the electron-nucleus interaction, and is calculated by the Wigner - Seitz sphere approximation, where the crystal structure is chosen as being a bcc structure, and given by[7]

$$\varepsilon_c = -0.902\ \left(\frac{4\pi}{3}\right)^{1/3}\ \hbar c\ \alpha Z^{2/3} \rho_e^{4/3} \tag{17}$$

where α is the fine structure constant. The small deviations of the negative charges from a uniform distribution, due to the electron screening, cause another correction term which is proportional to $Z^{4/3}$ per electron. Nevertheless, as this term is of same order of magnitude than the binding energy of atomic electrons but opposite in sign, so they cancel each other.

In order to write down $M(Z,A)$, we chose the mass formula of Garvey et *al.*[8]. The validity of the extrapolation of this mass formula, like any other known, to the region far from the beta stability valley is not certain. However, this serves to estimate the mass of neutron-rich-nuclei, within the uncertainty inherent to this type of calculation.

The species of nuclei existing for each value of ρ is determined by the condition that the nuclei are in equilibrium with the electron gas[9], so that Z and A are function of ρ. The neutron drip density may be determined by the condition

$$M[Z(\rho),A(\rho)-1] - M[Z(\rho),A(\rho)] + m_n \leq 0 \qquad (18)$$

where m_n is the neutron rest mass.

b) Subnuclear Regime With Free Neutrons

Neglecting interactions between the nuclei and free neutrons, aswell as the possible existence of protons outside of nuclei, we write the equation of state as

$$\varepsilon_2 = M(Z,A)c^2\rho_c + \varepsilon_F + \varepsilon_c + m_n c^2\rho_\ell + \varepsilon_{int}(\rho_\ell) \qquad (19)$$

The first three terms in above equation are the same as in eq.(14). The fourth term, with ρ_ℓ equal the number density of free neutrons, gives the contribution from the rest energy of the free neutrons. The last term represents the kinetic and interaction energies of free neutrons. For ε_{int}, we utilized the expression given by Küpper et *al.*[2] (eq.(24) of reference 2), which is based on the Thomas-Fermi model, extended to the pure neutron gas with $T = 0$.

c) Nuclear Regime

As the density increases, the Fermi level of the degenerate neutron gas become higher and higher. This inhibits the neutron ejection from the neutron-rich-nuclei, because it reduces the phase space acessible to emitted neutrons. When the density is $\approx 10^{13}$ g cm^{-3}, the neutron-rich-nuclei become unstable and begin to disintegrate. It is known that in this region the effect of protons is negligible, so that the total energy is practically given by the *same* expression as eq. (19) without the first three terms. We again use the Thomas-Fermi model.

d) Supernuclear Regime

At supernuclear density (greater than the density of ordinary nuclei $\sim 2.4 \times 10^{14}$ g cm^{-3}), other particles appear, the muons first and then the hyperons. Since there are many uncertainties in this regime, we simply extend eq. (24) of reference 2 to the hyperonic regime, at least until 10^{16} g cm^{-3}, which is the upper limit of interest to this work.

3. RESULTS AND DISCUSSION

a) Phase Transition

The equation of state of neutron star matter in equilibrium, $\varepsilon = \varepsilon(\rho)$, has the behavior shown schematically in fig. 1, where the curve I represents the behavior in phases a and b, and the curve II in phases c and d. The discontinuity of derivative $d\varepsilon/d\rho$ is unphysical. Thus we impose the condition of continuity on the pressure and the chemical potential, which leads to a kind of phase transition of matter.

Regarding the various regimes as homogeneous phases, we have, in the surface of contact,

$$\begin{aligned} p^{(1)} &= p^{(2)} \\ \mu_i^{(1)} &= \mu_i^{(2)} \end{aligned} \tag{20}$$

In eq. (20), the figures denote the phases and μ_i denotes chemical potential of the particle i. Hence we can make Maxwell construction, which is equivalent to find a tangent to curves I and II in fig. 1. The points of contact, by our equation of state (eqs. (14), (19)) are: $d_1 = 2.6 \times 10^{11}$ g cm^{-3} and $d_2 = 3.3 \times 10^{13}$ g cm^{-3}.

The fact that $d_1 \neq d_2$ implies the existence of a first order phase transition between phase I and II, with common pressure and temperature, but with different first derivatives of chemical potential. The value of d_2 may be considered as the initial point of nuclear regime.

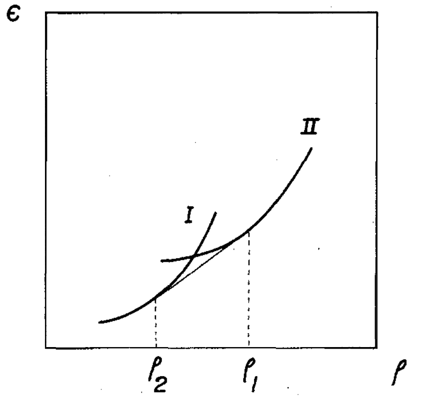

Fig.1 - The points of the phase transition are given by p and p .

The neutron drip point was found to occur at $d_{nd} = 6.0 \times 10^{11}$ g cm^{-3}, which is a little more than the values calculated by Langer et *al.*[10] (3×10^{11} g cm^{-3}) and by Baym et *al.*[11] (4.3×10^{11} g cm^{-3}). As the first point of phase transitions is sligthly below the neutron drip point, this suggests that the sudden nuclear disintegration process is very competitive with relation to the continuous neutron drip process.

A similar phase transition had been also observed for a different equation of state of neutron gas[12], but in this case the first point of transition is found to be slightly above the neutron drip point.

Beside various uncertainties from the equation of state, the small difference between d_{nd} and d_1 is not enough to tell us what process is dominant. Some authors[13,14] claim that there is no distinct phase transition between phase a and phase c. However, in any case, the pressure gradient becomes extremely small in the region, so that we can conclude that the neutron star is formed by two distinct parts: 1) a sharp core, from the center of the star to the core surface where $d = d_1$; 2) and a envelope or mantle, from the core surface, where $d = d_2$, to the star surface.

Even if the phase b exists, its region would be very narrow and probably gives no effect to the neutron star structure.

b) Maximum Mass and Other Results

It is believed that the black holes are indefinitely collapsing objets, whose mass is beyond the upper limit of stable neutron star configuration. In order to determine the limit, we plotted the total mass of the neutron star against the central number density (fig. 2). The condition of stability for neutron star, $dM/d\rho_c > 0$ [4], is used to point out the maximum mass of a neutron star as 3.26 $M_\odot$. This means that, beyond this value, the pressure gradient due to the neutron gas is insufficient to counterbalance the general relativistic effects of gravitation whichthen become dominant. The maximum mass wefound is greater than the values obtained by Cohen et *al.*[15] (1.92 $M_\odot$) and by Canuto and Chitre[16] (1.39 $M_\odot$).

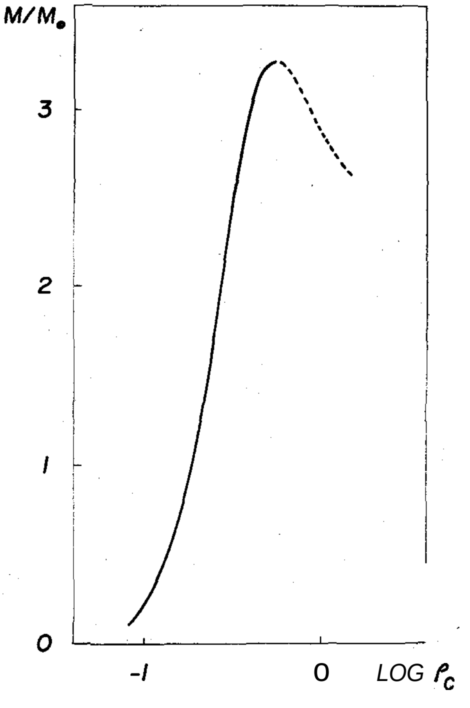

Fig.2 - Total mass of neutron star versus central number density. Stable neutron star configurations are represented by the solid line, with the maximum mass equal to 3.26 $M_\odot$, corresponding to a central density equal to 10^{15} g cm^{-3}. Unstable configurations are represented by the dashed line.

We also plotted the radius of the star versus the total mass (fig.3) and versus the central number density (fig. 4). We find that the radius of neutron stars is not monotonically decreasing function of mass in contrast to the most of neutron star models. In fig. 5, we present the configuration of the core of three neutron stars with central density chosen arbitrarily as 0.08, 0.1 and 0.3 fm^{-3}. For these values of density, the total neutron number and total proton number, as well as the total mass, are shown in Table I. We can see that the neutron star mass is largely concentrated in the core and the proton number is only a few percents of the neutron number.

TABLE I

$\rho_c (fm^{-3})$	$N_n/N_\odot$	$N_p/N_\odot$	$M/M_\odot$
0.08	0.116	8.06×10^{-4}	0.115
0.1	0.214	1.76×10^{-3}	0.212
0.3	2.65	6.77×10^{-2}	2.35

$N_\odot$ = solar baryonic number = 1.2×10^{57} baryons.

As is shown in fig. 6, the thickness of mantle decreases rapidly as the total mass increases, which may give some information to determine the mass of pulsars through the observed sudden change of period due to starquake mechanisms[17].

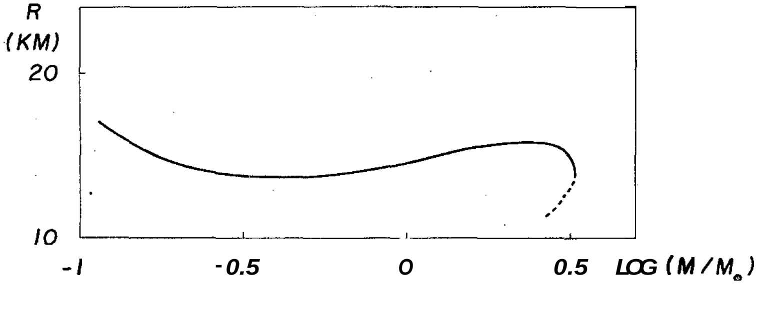

Fig.3 - Radius of the star versus total mass.

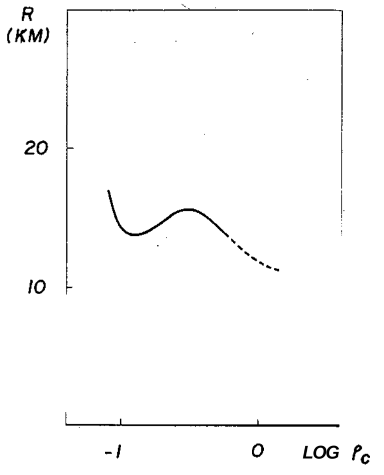

Fig.4 - Radius of the star against the central number density.

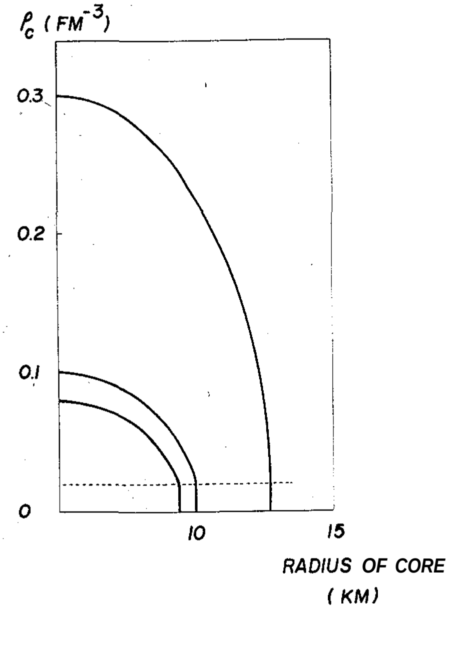

Fig.5 - Configuration of the core of three neutron stars, with central density equal to 0.08, 0.1 and 0.3 fm^{-3}.

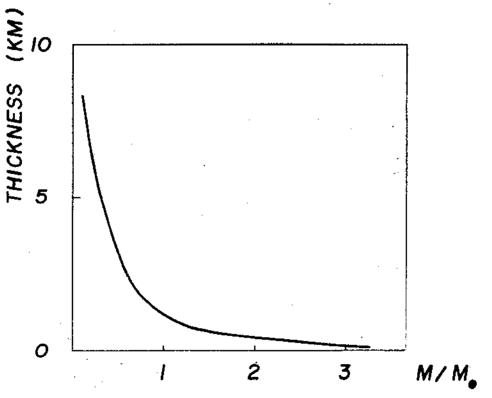

Fig.6 - The thickness of neutron star surface versus total mass.

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