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Short-Range Scalar Field in General Relativity

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The role of a short-range repulsive scalar field in preventing the grávitational of a cylindrical distribution of incoherent dust is studied.

E estudado o papel de um campo escalar repulsivo no impedimento do colapso gravitacional de uma distribuição cilíndrica de poeira incoerente.

1. INTRODUCTION

The study of relativistic field equations in the presence of a scalar field initiated by Bramhachary in the case of static spherically symmetric fields, has attracted considerable attention of many workers. However the studies were confined mainly to long-rangefield. Duan-I-Shi first obtained a class of solutions to equations in which a reals scalar field of short-range is coupled with gravitational field, containing some functions whose explicit forms are not known. Later Stephenson presented an approximate solution to static spherically symmetric Einstein - Maxwell-Yukawa fieid equations.

Recently Teixeira, Wolk and Som extended the idea of the real scalar field of short-range to investigate elementary structures. They considered a static spherically symnetric distribution of incoherent dust. The gravitational collapse of the dust distribution is supposed to be prevented by a stronger short-range repulsion. They obtained some interesting re-

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sults which are structurally important in srnall as well as in large dimensions. As cylindrical systems played an important role in nvestigating highly non-spherical collapse, it seems worthwhile to direct attention to the effect of a real short-range scalar field in the case of cylindrical systems.

In this paper we propose to extend the idea of repulsive Yukawa type field introduced by Teixeira et *aZ.*, to the case of cylindrically symmetric system. We consider here a static cylindrical distribution of matter in the form of incoherent dust. The constituents of the dust are supposed to be simultaneously sources of gravitational as well as repulsive short-range scalar fields. A class of approximate solutions are presented. The interesting feature of the solutions is that within the validity of our approximation, the radius of the cylindrical distribution lies within a definite range. Moreover, at large distances from the source, the only interaction that can exist, is a gravitational one and of Levi--Civita type.

2. BASIC EQUATIONS

In a region containing matter and source of short-range scalar field, the Einstein field equations are

$$R^{i}_{\ j} - \frac{1}{2}\delta^{i}_{\ j}R = -\frac{2\varepsilon}{c^2}T^{i}_{\ j} \tag{2.1}$$

$$S^{;i}_{\ \ ;i} + \mu^2 S = \varepsilon\sigma \tag{2.2}$$

$$T^{i}_{\ j} = \rho c^2 u^i u_j - \frac{c^2}{2\varepsilon}\left|2S^{,i}S_{,j} - \delta^{i}_{\ j}(S^{,k}S_{,k} - \mu^2 S^2)\right| \tag{2.3}$$

where

$$\varepsilon = \frac{4\pi G}{c^2} \tag{2.4}$$

p is the matter density, s is a short-range $(1/\mu)$ repulsive scalar field, σ is the density of the source of s, and u^i is the 4-velocity field of p and a.

For a static cylindrical distribution of incoherent dust, we consider the general line element with cylindrical symmetry

$$ds^2 = e^{2\eta}dx^{02} - e^{2\lambda}dr^2 - r^2e^{2\beta}d\phi^2 - e^{2y}dz^2 \qquad (2.5)$$

where all the metric elements are functions of r only. The coordinates (x^0,r,ϕ,z) are numbered as $(0,1,2,3)$. The static condition implies that $u^i = \delta^i_0\ e^{-\eta}$.

From the equations (2.1) - (2.4) we have

$$R^0_{\ 0} + R^2_2 = -2\mu^2S \qquad (2.6)$$

The four unknown functions in the line element (2.5) can be reduced at most to three unknown functions (Synge). So we take the line element in the form considered by Teixeira, Wolk and Som where

$$\lambda = \eta + \beta + y \qquad (2.7)$$

The field equãtions can now be expressed with the help of the equations (2.1) - (2.3) and (2.5) - (2.7) as

$$R^0_{\ 0} = -e^{-2\lambda}[\eta_{11} + \eta_1/r] = -\varepsilon\rho - \mu^2S^2 \qquad (2.8)$$

$$R^1_{\ 1} = -e^{-2\lambda}[\eta_{11} + \beta_{11} + \gamma_{11} + (\beta_1 - \eta_1 - \gamma_1)/r -$$

$$-2(\eta_1\beta_1 + \eta_1\gamma_1 + \beta_1\gamma_1)] = \varepsilon\rho + 2S^1S_1 - \mu^2S \qquad (2.9)$$

$$R^2_{\ 2} = -e^{2\lambda}(\beta_{11} + \beta_1/r) = \varepsilon\rho - \mu^2S \qquad (2.10)$$

$$R^3_{\ 3} = -e^{-2\lambda}(\gamma_{11} + \gamma_1/r) = \varepsilon\rho - \mu^2S^2 \qquad (2.11)$$

$$r^{-1}e^{-2\lambda}(rS_1)_1 + \mu^2S = \varepsilon\sigma \qquad (2.12)$$

where the subscript $_1$ denotes differentiation with respect to coordinate r. The conservation laws $T^i_{\ j;i} = 0$ yield:

$$\rho\eta_1 + \sigma S_1 = 0 \qquad (2.13)$$

3. SOLUTIONS OF THE EOUATIONS

The total number of field equations is five. The number of unknown functions $(\eta,\beta,\gamma,\rho,\sigma,S)$ is six. So we add one more equation by assuming that the source densities bear a constant ratio such as

$$\sigma = f\rho \qquad (3.1)$$

As we have failed to obtain the exact solutions of the field equations, we proceed to obtain the approximate solutions following the method used by Teixeira, Wolk and Som. We expand the four potentials (η,β,γ,S) and the two densities (ρ,σ) in integral powers of a small dimensionless constant k to be identified later. In the lowest approximation, we take the densities (ρ,σ) independent of k and the potentials (η,β,λ,S) proportional to k. Then the equations (2.8) - (2.13) reduce to

$$\eta_{11} + \eta_1/r = \varepsilon\rho \qquad (3.2)$$

$$\eta_{11} + \beta_{11} + \lambda_{11} + (\beta_1 - \eta_1 - \lambda_1)/r = -\varepsilon\rho \qquad (3.3)$$

$$\beta_{11} = \beta_1/r = -\varepsilon\rho \qquad (3.4)$$

$$\lambda_{11} = \lambda'_1/r = -\varepsilon\rho \qquad (3.5)$$

$$\gamma^{-1}(\gamma S_1)_1 = +\mu^2 S - \varepsilon\sigma \qquad (3.6)$$

$$\rho(\eta_1 + fS_1) = 0 \qquad (3.7)$$

In this order of approximation the field equations are decoupled into gravitational and scalar equations. However in view of the equation (3.7) which comes from the conservation laws, the gravitational and scalar fields are, in fact not independent.

When $\rho \neq 0$, we have from the equations (3.2), (3.6) and (3.7)

$$S_i = \varepsilon \frac{f^2-1}{\mu^2} \rho \tag{3.8}$$

where the subscript i means interna1. Now on plugging the equation (3.8) into the equation (3.6), one gets

$$\rho_{11} + \frac{\rho_1}{r} + \frac{\mu^2}{f^2-1} \rho = 0 \tag{3.9}$$

The equation (3.9) is the Bessel's equation. Two possible cases arise: I) $f^2 < 1$ and II) $f^2>1$.

Case I) when $f^2 < 1$, the equation (3.9) takes the form of Bessel's modified differential equation, The solutions of this equation are modified Bessel functions of order zero. However the solutions diverge at the axis of symnetry which implies infinite density of matter at the axis. Therefore the case $f^2<1$ is of less physical interest.

Case II) when $f^2>1$, the general solution of the equation (3.9) is given by

$$\rho(r) = \alpha J_0(\lambda r) + \beta Y_0(\lambda r)$$

where J_0 and Y_0 are Bessel's functions of the first and second kindof order zero. a and β are constants of integration and $\lambda^2 = \mu^2/f^2-1$. We put $\beta=0$ to ensure finite density at the axis of symmetry. The density of matter is then given by

$$\rho(r) = A\lambda^2 J_0(\lambda r) \tag{3.10}$$

From the equations (3.8) and (3.10) we have

$$S_i = \frac{\varepsilon'}{f} J_0(\lambda r) \tag{3.11}$$

where $\varepsilon' = \varepsilon A$ a dimensioniess constant. When $\rho \neq 0$ we obtain from (3.7)

$$\eta_i = - K - fS_i \tag{3.12}$$

where K is constant integration. Now on plugging (3.11) into (3.12) weget

$$\eta_i = \varepsilon' |a + J_0(\lambda r)| \tag{3.13}$$

On subtracting the equation resulting from the sum of (3.2), (3.4) and (3.5) from the equation (3.3) one obtains

$$(\eta+\lambda)_1/r = 0 \tag{3.14}$$

the integration of (3.14) yields

$$\gamma_i = -\eta_i + b \tag{3.15}$$

where b constants of integration. Since in this approximation $R^0_0 + R^2_2 = 0$, we choose $b = 0$ so that $\eta_i + \gamma_i = 0$.

On integrating (3.4) one gets

$$\beta_i = \varepsilon' J_0(\lambda r) + g \ln r + h \tag{3.16}$$

where g and h constants of integration. To avoid singularities at $r = 0$, we choose $g = 0$.

4. EXTERIOR SOLUTION

The exterior solution corresponds to the case $\rho = \sigma = 0$. Then the equations (3.2) - (3.6) can be easily integrated as

$$\eta_e = -\gamma_e = \alpha \ln r/r_0$$

$$\beta_e = \delta \ln r/r_0$$

and

$$S_e = \theta K_0(r\mu) \tag{4.2}$$

where $K_0(r\mu)$ is modified Bessel function of second kind and of order zero. The subscript e means external and a, δ and θ are constants of integration.

We now impose that at the boundary of the cylinder $(r=R)$ the potentials (η, β, γ and S) be continuous as well as the radial first derivatives of η and S. The continuity of η and η_1 fixes the constants a and α, so that we have

$$\eta_i = +\varepsilon' J_0(\lambda R)\left|1 - \frac{J_0(\lambda r)}{J_0(\lambda R)}\right. + \varepsilon' \lambda R J_1(\lambda R) \ln R/r_0 \tag{4.3}$$

$$\eta_e = \varepsilon' \lambda R J_1(AR) \ln r/r_0 \tag{4.4}$$

The continuity of γ is identically satisfied. The continuity of β gives δ in terms of h and $\eta_i(R)$.

The continuity of S and S_1 not only fixes the constants δ but also prescribes to the radius R an upper bound:

$$\lambda R > 2.4 \tag{4.5}$$

given by the relation

$$\frac{J_0(\lambda R)}{J_1(\lambda R)} = \frac{\lambda}{\mu}\frac{K_0(\mu R)}{K_1(\mu R)} \tag{4.6}$$

and the positive definiteness of the density of matter.

From (3.1) and (4.2) we get

$$S_i = \frac{E'}{f} J_0(Ar) \tag{4.7}$$

(4.7)

$$S_e = \frac{\varepsilon'}{f} \frac{J_0(\lambda R)}{K_0(\lambda R)} K_0(\lambda r)$$

Now in the Levi-Civita type systems the mass parameter is defined by

$$\eta_e = \frac{2GM}{C^2} \ln(r/r_0) \tag{4.8}$$

Where M represents mass per unit length of the cylinder. Comparing (4.4) and (4.8) we get

$$\varepsilon' = \frac{2GM}{c^2\lambda R}\,\frac{1}{J_1(\lambda R)} \tag{4.9}$$

Since $1/J_1(\lambda R) > 1$ for $\lambda R > 2.8$, our approximation is valid if $\frac{2GM}{c^2\lambda\ \ln R} < 1$.

It is then natural to identify the constant ε' with the small constant K in terms of which the series expansion of the functions were made.

5. DISCUSSION

The approximate solution represents a static distribution of incoherent dust whose gravitational collapse is prevented by a stronger repulsion from Yukawa type source. An important feature of the solution is that no singularity occurs either in the source density or in the potentials within the prescribed condition (4.5). The mass density $\rho(r)$ has a maximum finite value at the axis of symmetry and decreases monotonically to a finite value at the boundary. The scalar field falls off rapidly at large distances. At large distances the only interaction one has is a gravitational one and of Levi-Civita type.

The case where $\lambda R << 2$ is particularly interesting. For $f^2 >> 1$, $R >> 1/\mu$, so that $S_e \to 0$. The potentials η_i, β_i can be obtained expanding $J_0(\lambda r)$ in series and neglecting terms higher than $\lambda^2 r^2$

$$\eta_i = -Y_i = -\frac{\varepsilon'(\lambda R)^2}{4}\,|1 - r^2/R^2| + A \ln r/r_0$$

$$\eta_e = A \ln r/r_0$$

where $A = \varepsilon' \lambda R J_1(\lambda R)$.

For β, we may similarly obtain a simple expression containing terms up to $\lambda^4 r^4$.

The solution is then equivalent to that given by Raychaudhuri and Som for a stationary cylindrically symmetric clusters of particles moving in circles perpendicular to the axis of symmetry in such a way that the total angular momentum of the system vanishes. The effect of the repulsive scalar field in the source distribution resembles to that produced by the circular motions of the particles around the axis of symmetry in counter directions.

The difficulty with cylindrical analyses is that space-time is not asymptotically Minkowskiian far outside the system. In the next paper we propose to analyse the bounded locally cylindrical systems.

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