Revista Brasileira de Física, Vol. 8, N? 2, 1978

Cylindrically Symmetric Stationary Fields in General Relativity

M.M. SOM and N.O. SANTOS

Departamento de Física Matemática, Universidade Federal do Rio de Janeiro, Rio de Janeiro, RJ*

Recebido em 21 de Novembro de 1977

It is demonstrated that cylindrically symmetric stationary electrovac fields are always reducible to static electrovac fields when the metric elements g_{tt}, g_{t4} and g_{44} are linearly connected.

Demonstra-se que campos eletrovac estacionários com simetria cilíndrica são redutiveis a campos eletrovac estáticos sempre que os elementos da métrica g_{tt}, g_{t4} e g_{44} forem relacionados linearmente.

1. INTRODUCTION

In an earlier work, Lewis[1] first examined the stationary cylindrically symmetric vacuum field and obtained a special class of solutions of Einstein's equations, which are linear combinations of the static cylindrically symmetric solutions[2] with constants coefficients. Later Som *et al.*[3] extended his work to a stationary axially symmetric case and presented a class of solutions which are linear combinations of the static axially symmetric Curzon[4] fields. Recently Frehland[5], Som *et.al.*[6] studied the general solution of Einstein's equations for stationary cylindrically symmetric vacuum field and found that these solutions are always reducible to the static field by a suitable coordinate transformation. In fact they obtained the coordinate transformations which diagonalize the stationary cylindrically symmetric line element.

It seems worthwhile to the present author to study whether this property

---

* Postal address: Ilha do Fundão, Cidade Universitária, Rio de Janeiro, RJ, Brasil.

is a general feature of the cylindrical symmetry with respect to other stationary fields. In the present work some special solutions corresponding to the source-free stationary cylindrically symmetric electromagnetic fields are investigated. It is found that if g_{tt}, $g_{\phi\phi}$ and $g_{\phi t}$ are linearly related, the stationary solutions can be reduced to the known solutions corresponding to the static electrovacs.

2. FIELD EQUATIONS

For a stationary cylindrically symmetric metric, the most general line element may be written as:

$$ds^2 = f\, dt^2 - e^{2\psi}(dr^2 + dz^2) - \ell\, d\phi^2 + 2m\, d\phi\, dt \tag{1}$$

where f, ψ, R and m are functions of r alone. We number the co-ordinates t, r, z, ϕ as 0,1,2,3, and take the units in which G = c = 1. The Einstein-Maxwell field equations in an otherwise empty space are

$$R^{\mu}_{\nu} - \frac{1}{2}\, \delta^{\mu}_{\nu}\, R = -\, 8\pi\, T^{\mu}_{\nu} \quad , \tag{2}$$

where

$$T^{\mu}_{\nu} = \frac{1}{4\pi}\, (F^{\mu\alpha} F_{\alpha\nu} - \frac{1}{4}\, \delta^{\mu}_{\nu}\, F^{\alpha\beta}\, F_{\beta\alpha}) \quad , \tag{3}$$

$$F^{\mu\nu}{}_{;\nu} = 0 \quad , \tag{4}$$

$$F_{[\alpha\beta;\gamma]} = 0 \, , \tag{5}$$

$F^{\alpha\beta}$ being the electromagnetic field tensor whose only non-vanishing contravariant components for a stationary cylindrically symmetric field are $F^{13} = -F^{31}$ and $F^{10} = -F^{01}$.

For such a field one has

$$T^{0}_{0} = -\, T^{3}_{3} \quad , \tag{6}$$

so that from Eq. (2) one obtains

$$R^0_0 + R^3_3 = 0 \ . \tag{7}$$

In view of (7), one can introduce Weyl-like canonical co-ordinates such that[3]

$$f\ell + m^2 = r^2 \ . \tag{8}$$

From Eq. (4) one has

$$F^{31} = \frac{A}{\sqrt{-g}} \tag{9}$$

and

$$F^{01} = \frac{B}{\sqrt{-g}} \tag{10}$$

With the help of Eqs. (1), (3), (8), (9) and (10) the field equations can be written from Eq. (2) as

$$r\psi_{11} - \psi_1 - \frac{1}{2r}(\ell_1 f_1 + m_1^2) = \frac{1}{r}(fB^2 - \ell A^2 + 2mAB) \ , \tag{11}$$

$$r\psi_{11} + \psi_1 = -\frac{1}{r}(fB^2 - \ell A^2 + 2mAB) \ , \tag{12}$$

$$\frac{1}{2}\frac{d}{dr}\left[\frac{f\ell_1 \quad mm_1}{r}\right] = -\frac{1}{r}(\ell A^2 + fB^2) \ , \tag{13}$$

$$\frac{1}{2}\frac{d}{dr}\left[\frac{\ell f_1 + mm_1}{r}\right] = -\frac{1}{r}(\ell A^2 + fB^2) \ , \tag{14}$$

$$\frac{1}{2}\frac{d}{dr}\left[\frac{fm_1 - mf_1}{2}\right] = -\frac{2}{r}(mA^2 + fAB) \ , \tag{15}$$

$$\frac{1}{2}\frac{d}{dr}\left[\frac{m\ell_1 - \ell m_1}{r}\right] = -\frac{2}{r}(mB^2 - \ell AB) \quad . \tag{16}$$

If one writes $f/R = u$ and $m/\ell = v$, then Eq. (8) reduces to

$$u + v^2 = \frac{r^2}{\ell^2} \tag{17}$$

and Eqs. (13)-(16) take the simple forms

$$\frac{d}{dr}\left[\frac{u^2 r}{u + v^2}\frac{d}{dr}(v/u)\right] = -\frac{4}{r}(mA^2 + fAB) \quad , \tag{18}$$

$$\frac{d}{dr}\left[\frac{r}{u + v^2}\frac{dv}{dr}\right] = \frac{4}{r}(mB^2 - \ell AB) \quad , \tag{19}$$

$$\frac{d}{dr}\left[\frac{r}{u + v^2}\frac{du}{dr}\right] = \frac{4}{r}(\ell A^2 + fB^2) \quad . \tag{20}$$

If one now assumes that there is a linear relation between u and v

$$v = au + b \quad , \tag{21}$$

then for non-null real fields one must have $1 + 4ab = \mu^2 > 0$. One class of solutions of Eqs. (11)-(12) and (18)-(20) isobtained[7] as

$$e^{\psi} = k\, r^{c^2/4} \cosh \log(r/r_0)^{c/2} \quad , \tag{22}$$

$$f = \frac{1}{2a}\, r\, \frac{1 + \xi^2}{\xi} - \frac{1}{4a}\frac{1 + \mu^2}{\mu}\, r\, \frac{1 - \xi^2}{\xi} \tag{23}$$

$$\ell = \frac{a}{\mu}\, r\, \frac{1 - \xi^2}{\xi} \quad , \tag{24}$$

$$m = \frac{1}{2} r \frac{1 + \xi^2}{\xi} - \frac{1}{2\mu} r \frac{1 - \xi^2}{\xi} , \tag{25}$$

where k, c and r_0 are constants and

$$\xi = \frac{4B^2}{c^2} \frac{\mu}{a} r \cosh^2 \log(r/r_0)^{c/2} . \tag{26}$$

Another class of solutions is[8]

$$e^{2\psi} = r^{c(c-2)} R(r,c) , \tag{27}$$

$$f = \gamma^2 \left[R^2 - \omega^2 r^2 R^{-2}\right], \tag{28}$$

$$R = \gamma^2 \left[r^2 R^{-2} - \omega^2 R^2\right], \tag{29}$$

$$m = \gamma^2 \omega \left[r^2 R^{-2} - R^2\right], \tag{30}$$

where c is a constant of integration and $0 \leq \omega < 1$, $Y = (1-\omega^2)^{-1/2}$

$$R = r^c + br^{2-c} , \quad b = \text{constant of integration} . \tag{31}$$

Eqs. (28)-(30) give

$$v = - \frac{\omega}{1 + \omega^2} u + \frac{\omega}{1 + \omega^2} \tag{21'}$$

3. DIAGONALIZATION OF THE METRIC

Let us now consider the diagonalization of the metric corresponding to these two cases of solutions. The line element (1) can easily be reduced to a fundamental quadratic form by the following transformation:

$$\begin{aligned} t &= \bar{t} - \frac{2a}{1 \pm \mu} \bar{\phi} , \\ \phi &= \bar{\phi} - \frac{1 \pm \mu}{2a} \bar{t} , \end{aligned} \tag{32}$$

where ± signs before μ correspond to positive and negative values of μ.

For positive of μ we have

$$F = \frac{|\mu|}{a} \xi r \quad , \tag{23a}$$

$$L = \frac{4a\,|\mu|}{(1 + |\mu|)^2} \frac{r}{\xi} \tag{24a}$$

and

$$F^{31} = -\frac{1 + |\mu|}{4\,|\mu|} \frac{B}{\sqrt{-g}} , \; F^{01} = 0 \; . \tag{33}$$

The solution thus corresponds to the axial magnetic field. With negative μ we have

$$F = \frac{|\mu|}{a} \frac{r}{\xi} \quad , \tag{23b}$$

$$L = \frac{4a\,|\mu|}{(1 + |\mu|)^2} r\,\xi \tag{24b}$$

and

$$F^{01} = \frac{B}{\sqrt{-g}} \quad , \quad F^{31} = 0 \; . \tag{33'}$$

This solution corresponds to the radial electric field. Both these solutions are equivalent to those already given in the literature[9-11].

The transformations for the second case may be obtained simply by substituting in Eq. (32) the values of a and μ from Eq. (21'). For positive μ we then have

$$\begin{aligned} t &= \bar{t} + \omega\bar{\phi} \; , \\ \phi &= \bar{\phi} + \omega\bar{t} \; . \end{aligned} \tag{22a}$$

The transformed solution corresponds to a purely static radial field[6-8]. For negative μ we have

$$t = \bar{t} + \frac{1}{\omega}\bar{\phi} ,$$
$$\phi = \bar{\phi} + \frac{1}{\omega}\bar{t} . \tag{22b}$$

The transformed solution is equivalent to those given by Bonnor[10] and Chosh and Sengupta[12] for axial magnetic fields.

4. CONCLUSIONS

It can be shown that the linear relation among f, ℓ and m always induces a coordinate transformation which reduces the line element (1) to a quadratic fundamental form. Let us assume that there exists a linear relation among f, R and m of the form:

$$m = af + b\ell .$$

Then 'the line-element (1) takes the form

$$ds^2 = fdt^2 - e^{2\psi}(dr^2 + dz^2) - \ell d\phi^2 - 2(af + b\ell)d\phi dt$$

If one now makes the coordinate transformation

$$t = \alpha\bar{t} + \beta\phi$$
$$\phi = y\bar{\phi} + \delta\bar{t}$$

one gets immediately

$$ds^2 = F\overline{dt}^2 - e^{2\psi}(dr^2 + dz^2) - Ld\bar{\phi}^2 \tag{1'}$$

where

$$F = \left[(1 - \frac{2\delta\beta}{\alpha y + \beta\delta})\alpha^2 f - (1 - \frac{2\alpha y}{\alpha y + \beta\delta})\delta^2\ell\right]$$

$$L = \left[\left(1 - \frac{2\beta\delta}{\alpha y + \beta\delta}\right) y^2 \ell - \left(1 - \frac{2\alpha y}{\alpha y + \beta\delta}\right) \beta^2 f\right]$$

The line element (1') is the well known form of the cylindrically symmetric metric.

One of the authors (M.M.S.) would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste, where a part of the work has been done. They also acknowledge financial support from CNPq and FINEP of Brasil.

REFERENCES

1. T. Lewis, Proc. Roy. Soc. (London) *A136*, 176 (1932).
2. L. Marder, Proc. Roy. Soc. (London) *A244*, 524 (1958).
3. M.M. Som, A.F. da F. Teixeira and I. Wolk, Acta Physica Austríaca *43*, 15 (1975).
4. H.E. Curzon, Proc. London Math. Soc. 23, 477, (1924).
5. E. Frehland, Com. Math. Phys., 26, 307 (1972).
6. M.M. Som, A.F. da F. Teixeiraand I. Wolk, G.R.G. 7, 263 (1976).
7. B.K. Datta and A.K. Raychaudhuri, J. Math. Phys. 9, 1715 (1968).
8. N. Arbex and M.M. Som, Nuovo Cimento *13B*, 49 (1973).
9. B.C. Mukherjee, Bull. Cal. Math. Soc. 30, 95 (1938).
10. W.B. Bonnor, Proc. Roy. Soc. (London) A66, 145 (1953).
11. A.K. Raychaudhuri, Ann. Phys. (NY) 11, 501 (1960).
12. R. Chosh and R. Sengupta, Nuovo Cimento 38, 1579 (1965).