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# Charged Point Particles with Magnetic Moment in General Relativity*

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Halbwachs Lagrangean formalism for the theory of charged point, particles with spin ($g$ = 2) is generalized and formulated in General Relativity for particles of arbitrary charge and magnetic moment. Equations are obtained, both corresponding to Frenkel's condition $S_{\mu\nu}\dot{z}^{\nu} = 0$ and to Nakano's condition $S_{\mu\nu}P^{\nu} = 0$. With the later condition the exact equations are highly coupled and non linear. When linearized in the electromagnetic and gravitational fields they coincide with de Groot-Suttorp equations for vanishing gravitational field and with Dixon-Wald equations in the absence of electromagnetic field. The equations corresponding to Frenkel's condition, when linearized in $S_{\mu\nu}$, coincide with Papapetrou's and Frenkel's equations in the corresponding limits.

O formalismo Lagrangeano de Halbwachs para partículas puntiformes carregadas e com spin ($g$ = 2) é generalizado e estendido para Relatividade Geral, para partículas com carga e momento magnético arbitrãrios. Obtêm-se equações de movimento correspondentes ás condições de Frenkel $S_{\mu\nu}\dot{z}^{\nu} = 0$ e de Nakano $S_{\mu\nu}P^{\nu} = 0$. Com a ultima condição, as equações de movimento exatas são fortemente acopladas e não lineares. Quando estas são linearizadas nos campos eletromagnético e gravitacional coincidem com as equações de Suttorp e de Groot para campos gravitacionais nulos, e com as equações de Dixon-Wald na ausência de interação eletromagnética. As

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equações correspondentes à condição de Frenkel coincidem com as equações de Papapetrou e Frenkel nos limites correspondentes.

## 1. INTRODUCTION

In 1926 Thomas[1] successfully made a kinematic analysis of the precession of the electron spin using approximate relativistic equations, to explain the gyromagnetic factor g = 2 of the electron as evidenced by the Hidrogen atom spectroscopy. Since then many consistent equations for classical particles with spin were produced, using several procedures, such as:

a) Use a variational methods for point particles[2-5].

b) Obtention of consistent equations for a fluid of particles with spin and passage to point particles by analogy with the usual correspondence between fluid equations and those for point particles[6,7].

c) Use of the conservation equations for obtainment of the equations of motion for a small spinning body:

$$T^{\mu\nu}{}_{;\nu} = f^{\mu} \tag{1}$$

for electromagnetic (E.M.) or (and) gravitational interaction.[8-13]

d) Use of quantum relativistic equations, such as Dirac's one, to obtain, by correspondence, the classical theory.[12,14]

Thomas[1] and Frenkel[2] realized already that a supplementary condition such as

$$S^{\mu}{}_{\nu}U^{\nu} = S^{\mu}{}_{\nu}\frac{dx^{\nu}}{d\tau} = S^{\mu}{}_{\nu}\dot{x}^{\nu} = 0 \tag{2}$$

should be satisfied by the spin tensor ($S^{\mu\nu} = - S^{\nu\mu}$). Frenkel's condition (2) has been widely used. However Möller[15] showed that it leads to unsatisfactory helicoidal motions.

This is the case of Frenkel's equations:

$$\frac{dP^{\mu}}{d\tau} = qF^{\mu\nu}\dot{x}_{\nu} + g\,\frac{\mu_0}{2}\,S_{\nu\lambda}F^{\nu\lambda,\mu} \tag{3a}$$

$$\frac{dS^{\mu\nu}}{d\tau} = 2P^{[\mu}\dot{x}^{\nu]} + 2g\mu_0\,S^{[\mu}{}_{\rho}F^{\nu]\rho} \tag{3b}$$

$$P^{\mu} = m\dot{x}^{\mu} + S^{\mu\nu}(\ddot{x}_{\nu} - g\mu_0 F_{\nu\rho}\dot{x}^{\rho}) \tag{3c}$$

$$\mu_0 = \frac{q}{2m_0} \tag{3d}$$

that leads to helicoidal motions even in the absence of E.M. fields. This is due to the fact that they are second order equations in $U^{\nu}$. In eqs. (3) a comma means partial derivative and: $2P^{[\mu}\dot{x}^{\nu]} = P^{\mu}\dot{x}^{\nu} - P^{\nu}\dot{x}^{\mu}$.

The same problem appears in some group-c theories, like Weyssenhoff's[8] and Papapetrou-Schild's[13] which lead to eqs. (3) for the free field case.

Nakano[16] proposed a different supplementary condition:

$$S^{\mu}{}_{\nu}P^{\nu} = 0 \tag{4}$$

where $P^{\mu}$ is the gauge-invariant moment. It can be shown that with condition (4) the equations of motion do not allow the helicoidal solutions.

Another kind of problem which appears in group-c theories is related with the passage to the limit of point particles. Indeed Möller[..] proved that if the energy-density is definite positive, there is a minimum size of a spinning particle ($r_0 = S/M$, $S$ being the spin and $M$ the mass).

We might argue, however, that even for a spinless electron consideredas the limit of a distribution of charge and matter, a minimum size occurs. To take the limit of point particle, we have to introduce negative energies. Thus we should also permit this procedure for obtainment of spinning point particles. This problem does not appear, however, in variational formalism which assumes from the beginning point particles.

Using method (c) Suttorp and de Groot[12] obtained, with condition (4), equations of motion of spinning particles which they linearized in $F^{\mu\nu}$. They also assumed, as usual, the magnetic moment to be proportional to the spin:

$$M^{\mu\nu} = \mu S^{\mu\nu} \quad (5a)$$

where $\mu$ is arbitrary, being

$$\mu = g\mu_0 = gq/2m \quad (5b)$$

for charged particles. Actualy $\mu$ can be different from zero even for neutral particles, i.e., when $\mu_0 = 0$.

They obtain

$$\frac{dP^\mu}{d\tau} = qF^{\mu\nu}U_\nu + \frac{\mu}{2} S_{\nu\lambda}F^{\nu\lambda,\mu} \quad (6a)$$

$$P^\mu = mU^\mu + (2\mu_0 - \mu)S^{\mu\nu}F_{\nu\rho}U^\rho - \frac{\mu}{2m} S_{\nu\lambda}F^{\nu\lambda,\rho}S_\rho{}^\mu \quad (6b)$$

$$\frac{dS^{\mu\nu}}{d\tau} = 2P^{[\mu}U^{\nu]} + 2\mu S^{[\mu}{}_\lambda F^{\nu]\lambda}, \quad (6c)$$

where

$$m = -P_\alpha U^\alpha \quad (6d)$$

They also showed that these equations coincide with those obtained, in the same approximation and as a classical limit, from Dirac's equation with anomalous magnetic moment ($g \neq 2$), after they were submitted to a generalized Foldy-Wouthuysen transformation.

In General Relativity (G.R.) Papapetrou[9] was the first to obtain the equations of point spinning particles, by method (c), for pure gravitational interaction:

$$\frac{D}{D\tau}\left(mU^{\mu} - \frac{DS^{\mu\nu}}{D\tau} U_{\nu}\right) = -\frac{1}{2} R^{\mu}{}_{\nu\rho\sigma} U^{\nu} S^{\rho\sigma} \quad ; \tag{7a}$$

$$\frac{DS^{\mu\nu}}{D\tau} = 2U^{[\mu} \frac{DS^{\nu]\lambda}}{D\tau} U_{\lambda} \ . \tag{7b}$$

These equations are not complete in the sense that he did not assume a supplementary condition like (2) or (4). In a latter paper[17], he uses the condition $S^{0i} = 0$ in a priviledged coordinate system, although this is not necessary. Indeed Schild[13] obtained Papapetrou's equations (7) using condition (2).

In the present paper weobtainthe equations of motion of a charged particle with arbitrary magnetic moment in General Relativity.

In section 2 we reformulate Halbwachs Lagrangean formalism using it to extend his equation to arbitrary g-factor.

In section 3 this Lagrangean formalism is extended to G.R. first using local variation in Riemannian coordinates and then extending it to covariant form. This is used to derive the equations of motion inpresence of E.M. and gravitational field with condition (4), which generalizes de Groot-Suttorp equations when linearized in the fields.

For the case of homogeneous E.M. fields, equations with condition (2) or (4) coincide and lead to Bargmann, Michel and Teledgi[18] (BMT) equation for the spin precession.

## 2. VARIATIONAL METHODS IN SPECIAL RELATIVITY (S.R.)

Soon after discovery of the electron spin, Frenkel[2] obtained classical equations of motion of a particle and its spin by a variational Lorentz invariant formalism. However, it was not defined in holonomic form, what is unsatisfactory for many reasons; not only consistency of this equations is not obvious, but even a Hamiltonian formalism is not obtained from it.

Lattes, Schönberg and Schützer[3] reformulated Frenkel's treatment in an holonomic form but, although the results are Lorentz invariant (L. I.), the spin variables are defined in not an obvious L.I. way. We have not found useful to try an extension of this method to curved spaces.

Halbwachs[4] formulated an obviously L.I. holonomic variational method to obtain the equations of motion for a charged particle in the case when the gyromagnetic factor is 2. In all these papers the Frenkel supplementary condition (2) was used.

In this section we shall extend Halbwachs formalism to arbitrary q and μ. Condition (4) can also be used, as will be done in next section in G.R..

The point particle is characterized by its position $x^{\mu} = x^{\mu}(\tau)$, where τ is a parameter, its four-velocity $U^{\mu} = dx^{\mu}/d\tau = \dot{x}^{\mu}$ and its spin $S^{\mu\nu}(\tau)$ which is determined by two Einstein-Kramers (E.K.) variables"

$$b^{(1)\mu}, b^{(2)\mu} \; ; \; b^{(i)}_{\mu} b^{(j)\mu} = \delta^{ij} \quad (i,j = 1,2) \tag{8a}$$

the spin tensor being defined by

$$S^{\mu\nu} = h(b^{(1)\mu} b^{(2)\nu} - b^{(1)\nu} b^{(2)\mu}) \; . \tag{9}$$

If condition (2) is imposed we complete the E.K. tetradic set that includes, $b^{(1)\mu}$ and $b^{(2)\mu}$, with:

$$b^{(0)\mu} = U^{\mu} \; ; \tag{8b}$$

$$b^{(3)\mu} = \varepsilon^{\mu\nu\rho\sigma} b^{(0)}_{\nu} \; b^{(1)}_{\rho} \; b^{(2)}_{\sigma} \tag{8c}$$

with (for α,β = 0,1,2,3):

$$b^{(\alpha)}_{\mu} b^{(\beta)\mu} = \eta^{(\alpha)(\beta)} \; ; \; b^{(\alpha)}_{\mu} b_{(\alpha)\nu} = \eta_{\mu\nu} \tag{8d}$$

Einstein summation convention is used both for tensor as for tetradic indices. $\eta^{\mu\nu}$ is the Minkowski metric (-+++) and $\eta^{(\alpha)(\beta)}$ is formaly the same quantity in the indices (a), (β) of enumeration of the vectors $h_{(\alpha)}{}^{\mu} = \eta_{(\alpha)(\beta)}\, h^{(\beta)\mu}$

The passage to the case $S^{\mu\nu}P_{\nu} = 0$ corresponds to take

$$b^{(0)\mu} = P^{\mu}(-P^{\nu}P_{\nu})^{-1/2} \;. \tag{10}$$

Thus generalizing and modifying Halbwachs formulatíon, we take for the Lagrangean

$$L = m_0 \frac{\lambda}{2}\,(1 + \dot{x}^{\mu}\dot{x}_{\mu}) + qA_{\mu}\dot{x}^{\mu} + \frac{h}{2}\,\varepsilon_{ij}\, b^{(i)\mu}\dot{b}^{(j)}{}_{\mu} +$$

$$+ h\varepsilon_{ij}\, b^{(i)\mu}\, b^{(j)\nu}\,(\frac{\mu}{2}\,F_{\mu\nu} + \lambda_{\mu}\dot{x}_{\nu}) + \lambda_{ij}(b^{(i)\mu}\, b^{(j)}{}_{\mu} - \delta^{ij}) \;. \tag{11}$$

Here $\varepsilon_{ij}$ $(i,j = 1,2)$ is defined by:

$$\varepsilon_{12} = -\varepsilon_{21} \text{ E ]} \quad ; \quad \varepsilon_{11} = \varepsilon_{22} = 0 \;,$$

and μ is arbitrary (equal 2μ for Halbwachs).

In (11) $\lambda_{\mu}$, $\chi_{ij}$ and λ are Lagrangean multipliers used to impose conditions (2), (8a) and (12):

$$\dot{x}^{\mu}\dot{x}_{\mu} = -1 \tag{12}$$

The term

$$\frac{h}{2}\,\varepsilon_{ij}\; b^{(i)\mu}\dot{b}^{(j)}{}_{\mu} \tag{13}$$

represents the energy of proper rotation[4] not explicitly defined by Frenkel. The remaining are obvious terms of interaction.

The equations of motion are obtained by a varíation

$$\delta I = \delta \int_{B}^{A} L(x^{\mu}, \dot{x}^{\mu}, b^{(i)\mu}, \dot{b}^{(i)\mu})\, d\tau = 0 \quad , \tag{14a}$$

when the arbitrary extreme points $A$, $B$ are fixed. The actual trajectory is the one for which $I$ is extreme in relation to other parametrized paths $\bar{x}^\mu(\tau)$, $\dot{\bar{x}}^\mu(\tau)$, $\bar{b}^{(i)\mu}(\tau)$, $\dot{\bar{b}}^{(i)\mu}(\tau)$, with

$$\delta x^\mu = \bar{x}^\mu(\tau) - x^\mu(\tau) \quad ; \quad \delta b^{(i)\mu} = \bar{b}^{(i)\mu}(\tau) - b^{(i)\mu}(\tau) \quad ; \tag{14b}$$

$$\delta \dot{x}^\mu = \frac{d}{d\tau}\delta x^\mu \quad ; \quad \delta \dot{b}^{(i)\mu} = \frac{d}{d\tau} \delta b^{(i)\mu} \ , \tag{14c}$$

$\delta x^\mu$ and $\delta b^{(i)\mu}$ vanishing at points $A$, $B$.

Here all quantities are functions of $\tau$ alone and are independent:

$$\frac{\partial \dot{x}^\mu}{\partial x^\nu} = \frac{\partial b^{(i)\mu}}{\partial x^\nu} = \frac{\partial \dot{b}^{(i)\mu}}{\partial x^\nu} = 0 \ . \tag{14d}$$

$$\frac{\partial b^{(i)\mu}}{\partial \dot{x}^\nu} = \frac{\partial \dot{x}^\mu}{\partial b^{(i)\nu}} = \frac{\partial \dot{x}^\mu}{\partial \dot{b}^{(i)\nu}} = 0 \tag{14e}$$

$$\frac{\partial \dot{b}^{(i)\mu}}{\partial \dot{x}^\nu} = 0 \tag{14f}$$

The well known details given after eq. (14a) are not superfluous since they must be extended to G.R..

Thus we obtain the supplementary conditions (2), (8a) and (12) plus the equations

$$\frac{dP^\mu}{d\tau} = qF^{\mu\nu}\dot{x}_\nu + \frac{\mu}{2} S_{\nu\lambda}F^{\nu\lambda,\mu} \ ; \tag{15a}$$

with

$$P^\mu = \frac{\partial L}{\partial \dot{x}_\mu} - qA^\mu = m\dot{x}^\mu - S^{\mu\nu}\lambda_\nu \ ; \tag{15b}$$

$$m = m_0\lambda = - P_\mu \dot{x}^\mu \tag{15c}$$

and, for the E.K. variables:

$$h\varepsilon_{ij}\dot{b}^{(i)\mu} = -\lambda_{ij}b^{(i)\mu} + \frac{\hbar}{2}\varepsilon_{ij}b^{(i)\mu} + h\varepsilon_{ij}b^{(i)}_{\nu}(\mu F^{\nu\mu} + \lambda\,\dot{x}^{[\nu}\dot{x}^{\mu]}) . \quad (16)$$

In (15c) we used (15b), (2) and (12).

Notice that for our Lagrangean, the canonical angular momentum[4]

$$S_{\mu\nu} = b^{(i)}{}_{\mu}\frac{\partial L}{\partial \dot{b}^{(i)\nu}} - b^{(i)}{}_{\nu}\frac{\partial L}{\partial \dot{b}^{(i)\mu}} \quad (17)$$

coincides with expression (8).

From (8a, (9), (15b) and (16) we get h = 0 and

$$\frac{dS^{\mu\nu}}{d\tau} = 2P^{[\mu}\dot{x}^{\nu]} + 2\mu S^{[\mu}{}_{\lambda}F^{\nu]\lambda} \quad ; \quad (15d)$$

and from (2), (15b) and (15d),

$$S^{\mu\nu}\lambda_{\nu} = S^{\mu\nu}(\mu F_{\mu\rho}\dot{x}^{\rho} - \ddot{x}_{\nu}) \quad . \quad (15e)$$

Equations (15) are the same as Frenkel's eqs. (3), now obtained by an holonomic procedure.

Also, using (12) and (15) we obtain

$$\frac{d\tilde{m}}{d\tau} = 0 \quad ; \quad (18a)$$

$$\tilde{m} = m_0\lambda + \frac{\mu}{2}S^{\mu\nu}F_{\mu\nu} = m_0 \quad (18b)$$

where we took $\tilde{m} = m_0$ as this occurs, asymptotically, for vanishing $F^{\mu\nu}$.

We notice[4] that (13) is not the most general expression for the energy of proper rotation but we use it for simplicity. Regge and Hanson[5] have indeed formulated a very general relativistic theory for spherical top in interaction with E.M. field, using also a variational principle. After this work was accomplished it came to our attention a preprint of R. Hojman and S. Hojman[19] which extends that formulation to G.R., for charged particles without magnetic moment, with some unsatisfactory results such as the 4-velocity becoming a space vector in regionsof strong fields. The possible relation of these papers to our treatment isbeing examined.

## 3. LAGRANGEAN FORMALISM IN A RIEMANNIAN SPACE

A central point in the Lagrangean formalism for spinning particles in S.R. was that the E.K. varlables dependend only on the proper time and not explicitly in the coordinates $x^{\mu}$ of the particles. If this is correct in a particular coordinate system, it would, however, not be so after a general coordinate transformation is performed.

Thus we postulate that in G.R. the previledged coordinate system where the E.K. variables are independent of the coordinates is precisely the largest locally inertial coordinate system associated to an arbitrary point O in particles's trajectory: a normal Riemann coordinate system $\hat{R}$ with origin in O.

In a small region containing O we have

$$g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}} + O^2(x^{\hat{\mu}}) \; ; \tag{19a}$$

$$\{{}_{\hat{\nu}}{}^{\hat{\mu}}{}_{\hat{\lambda}}\} = O(x^{\hat{\mu}}) \tag{19b}$$

However the existence of a curvature implies that

$$\{{}_{\hat{\nu}}{}^{\hat{\mu}}{}_{\hat{\lambda}}\}_{,\hat{\rho}} \neq 0 \qquad \text{(in O)} \; . \tag{19c}$$

In that region we use a local variational principle with constraints:

$$\delta I = \delta \int_A^B L d\tau = 0 \; , \tag{20a}$$

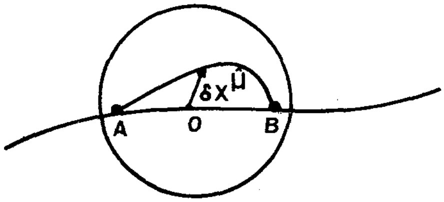


Fig. 1

with fixed points A and $B$ infinitesimaly near, and assume that the path which makes $I$ extreme (trajectory) contains 0 (Fig.1).

Thus we have:

$$\frac{\partial b^{(i)\hat\mu}}{\partial x^{\hat\nu}} = \frac{\partial \dot b^{(i)\hat\mu}}{\partial x^{\hat\nu}} = \frac{\partial \dot x^{\hat\mu}}{\partial x^{\hat\nu}} = O(x^{\hat\mu}) \tag{20b}$$

and, to first order,

$$\delta \dot x^{\hat\mu} = \frac{d}{d\tau}\, \delta x^{\hat\mu} \quad ; \quad \delta \dot b^{(i)\hat\mu} = \frac{d}{d\tau}\, \delta b^{(i)\hat\mu} \; . \tag{20c}$$

Thus, besides the supplementary conditions which take the same form as in S.R. (eqs.(2), (8d) and (12)) we obtain, at the point 0, the Euler-Lagrangean equations:

$$\frac{d}{d\tau}\, \frac{\partial L}{\dot x^{\hat\mu}} - \frac{\partial L}{x^{\hat\mu}} = 0 \quad ; \tag{21a}$$

$$\frac{d}{d\tau}\, \frac{\partial L}{\partial \dot b^{(i)\hat\mu}} - \frac{\partial L}{\partial b^{(i)\hat\mu}} = 0 \; . \tag{21b}$$

In special we take

$$L = m_0\, \frac{\lambda}{2}\, (1 + \dot x^{\hat\mu} \dot x_{\hat\mu}) + qA_{\hat\mu} \dot x^{\hat\mu} + \frac{h}{2}\, \varepsilon_{ij}\, b^{(i)\hat\mu}\, \frac{Db^{(j)}{}_{\hat\mu}}{D\tau}$$

$$+ h\varepsilon_{ij}\, b^{(i)\hat{\mu}}\, b^{(j)\hat{\nu}}\, (\dot{x}_{\hat{\nu}}\lambda_{\hat{\mu}} + \frac{\mu}{2} F_{\hat{\mu}\hat{\nu}})$$

$$+ \lambda_{ij}\, b^{(i)\hat{\mu}} b^{(j)}{}_{\hat{\mu}} - \delta^{ij}) \ , \tag{22a}$$

i.e., the covariant generalization of (11), with

$$\frac{Db^{(j)\hat{\mu}}}{D\tau} = \dot{b}^{(i)\hat{\mu}} + \{{}_{\hat{\rho}}{}^{\hat{\mu}}{}_{\hat{\sigma}}\} b^{(i)\hat{\rho}}\, \dot{x}^{\hat{\sigma}} \tag{22b}$$

(Notice that $\{{}_{\hat{\rho}}{}^{\hat{\mu}}{}_{\hat{\sigma}}\}$ vanishes only at 0). We obtain

$$\frac{dP_{\hat{\mu}}}{d\tau} = qF_{\hat{\mu}\hat{\nu}}\dot{x}^{\hat{\nu}} + \frac{\mu}{2} S^{\hat{\nu}\hat{\lambda}} F_{\hat{\nu}\hat{\lambda},\hat{\mu}} + \frac{1}{2} (\{\hat{\mu}\hat{\rho},\hat{\nu}\},\hat{\tau} - \{\hat{\tau}\hat{\rho},\hat{\nu}\},\hat{\mu})\dot{x}^{\hat{\tau}} S^{\hat{\nu}\hat{\rho}} \ , \tag{23a}$$

where

$$P_{\hat{\mu}} = \frac{\partial L}{\partial \dot{x}^{\hat{\mu}}} - qA_{\hat{\mu}} - \frac{1}{2} \{\hat{\mu}\hat{\rho},\hat{\nu}\} S^{\hat{\nu}\hat{\rho}} = m\dot{x}_{\hat{\mu}} - S_{\hat{\mu}\hat{\nu}}\lambda^{\hat{\nu}} \tag{23b}$$

is the gauge-invariant moment, relative both to E.M. and to gravitational transformations.

The equation for $\dot{b}^{(i)\hat{\mu}}$ are formally identical to (16) in $\hat{R}$, so we obtain in 0:

$$\frac{dS^{\hat{\mu}\hat{\nu}}}{d\tau} = 2P^{[\hat{\mu}}\dot{x}^{\hat{\nu}]} + 2\mu\, S^{[\hat{\mu}}{}_{\hat{\lambda}} F^{\hat{\nu}]\hat{\lambda}} \tag{23c}$$

and the expression (15c) for $S^{\hat{\mu}\hat{\nu}}\lambda_{\hat{\nu}}$ (in R). As in the Riemann frame we have in 0

$$\frac{DP^{\hat{\mu}}}{D\tau} = \frac{dP^{\hat{\mu}}}{d\tau} \ ; \ \frac{DS^{\hat{\mu}\hat{\nu}}}{D\tau} = \frac{dS^{\hat{\mu}\hat{\nu}}}{d\tau} \ ; \ \frac{DU^{\hat{\mu}}}{D\tau} = \frac{dU^{\hat{\mu}}}{d\tau} \ , \tag{24a}$$

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} = \{\hat{\mu}[\hat{\sigma}, \hat{\rho}]\},_{\hat{\nu}} - \{\hat{\nu}[\hat{\sigma},\hat{\rho}]\},_{\hat{\mu}} \tag{24b}$$

eqs. (23) can be written in'a covariant form in the arbitrary point 0 in $\hat{R}$ and thus take the **same** form in any coordinate system *(R)* at any point:

$$\frac{DP^{\mu}}{D\tau} = qF^{\mu\nu}\dot{x}_{\nu} + \frac{1}{2}\,(\mu F_{\rho\sigma}{}^{;\mu} - \frac{1}{2}\,R^{\mu}{}_{\nu\rho\sigma}\dot{x}^{\nu})S^{\rho\sigma} \quad ; \tag{25a}$$

$$P^{\mu} = m\dot{x}^{\mu} - S^{\mu\nu}\lambda_{\nu} \; ; \tag{25b}$$

$$\frac{DS^{\mu\nu}}{D\tau} = 2P^{[\mu}\dot{x}^{\nu]} + 2^{\mu}\, S^{[\mu}{}_{\lambda}F^{\nu]\lambda} \quad ; \tag{25c}$$

$$S^{\mu\nu}\lambda_{\nu} = -P^{\mu} + m\dot{x}^{\mu} = S^{\mu\nu}(\mu F_{\nu\rho}\dot{x}^{\rho} - \frac{D\dot{x}_{\nu}}{D\tau}) \; . \tag{25d}$$

For vanishing gravitational fields these equations reduce to Frenkel's equations (3), and for vanishing E.M. fields they reduce to Papapetrou's equations (7) with condition (2).

We are now prepared to formulate the variational principle in a covariant form.

Thus if in $\hat{R}$ an arbitrary 4-vector $B^{\mu}$ çatisfies $\partial B^{\mu}/\partial x^{\nu} = 0$, in R where $B^{\mu} = B^{\hat{\nu}}\,\partial x^{\mu}/\partial x^{\nu}$, we have $DB^{\mu}/Dx^{\nu} = 0$.

In particular we find that

$$\frac{D\dot{x}^{\nu}}{Dx^{\rho}} = \frac{Db^{(i)\nu}}{Dx^{\rho}} = 0 \; , \tag{26a}$$

but we still have:

$$\frac{D}{Dx^{\rho}}\,\frac{Db^{(i)\mu}}{D\tau} \neq 0 \tag{26b}$$

Thus for a Lagrangean $L = L(x^{\mu}, \dot{x}^{\mu}, b^{(i)\mu}, Db^{(i)\mu}/D\tau)$,

$$\delta L = \delta x^{\mu} \frac{DL}{dx^{\mu}} + \partial \dot{x}^{\mu} \frac{\partial L}{\partial \dot{x}^{\mu}} + \delta b^{(i)\mu} \frac{\partial L}{\partial b^{(i)\mu}} + \delta\left(\frac{Db^{(i)\mu}}{D\tau}\right) \frac{\partial L}{\partial\left(\frac{Db^{(i)\mu}}{D\tau}\right)} , \tag{27a}$$

since $\partial L/\partial x^{\mu} = \&/h^{p}$.

In the same way as we obtained (26), we can prove from eq. (20) that, at any point,

$$\delta \dot{x}^{\mu} = \frac{D}{D\tau} \delta x^{\mu} \; ; \quad \delta\left(\frac{Db^{(i)\mu}}{D\tau}\right) = \frac{D}{D\tau} b^{(i)\mu} \; . \tag{27b}$$

Now we can eliminate the condition that the fixed points A, $B$ are very near each other.

So the condition of extreme action and relations (27) permit us to obtain the generalized Euler-Lagrange equations

$$\frac{D}{D\tau} \frac{\partial L}{\partial \dot{x}^{\mu}} - \frac{DL}{Dx^{\mu}} = 0 \; ; \tag{28a}$$

$$\frac{D}{D\tau} \frac{\partial L}{\partial\left(\frac{Db^{(i)\mu}}{D\tau}\right)} - \frac{\partial L}{\partial b^{(i)\mu}} = 0 \; , \tag{28b}$$

valid on particle's trajectory. We notice that (14e) are still valid, but not (14f), as:

$$\frac{\partial}{\partial \dot{x}^{\mu}} \frac{Db^{(i)\mu}}{D\tau} = \left\{ {\mu \atop \rho \; \nu} \right\} b^{(i)\rho} \; . \tag{14g}$$

With the Lagrangean (22a) in an arbitrary coordinate system, we thus obtain directly equations (25) (besides the supplementary conditions).

Now we will obtain the equations consistent with condition (4). Thus we take again (22a) in R, with the substitution $\dot{x}_\nu \lambda_\mu \to P_\nu \lambda_\mu$, as now $b^{(0)\mu}$ is given by (10).

We assume that $P^\mu = P^\mu(x^\mu, \dot{x}^\mu, S^{\mu\nu})$ is an unknown functional latter to be identified with the gauge-invariant moment.

Proceeding similarly to the previous case we obtain, besides the supplementary conditions (4), (8a) and (12), the equations

$$\frac{DP_\mu}{D\tau} = qF_{\mu\nu}\dot{x}^\nu + \frac{\mu}{2} S^{\nu\rho} F_{\nu\rho;\mu} - \frac{1}{2} R_{\mu\nu\rho\sigma}\dot{x}^\nu S^{\rho\sigma} + \lambda_\nu S^{\nu\rho} P_{\rho;\mu} \; ; \tag{29a}$$

$$P_\mu = \frac{\partial L}{\partial \dot{x}^\mu} - qA_\mu - \left\{ {}^{\nu}_{\rho\;\mu} \right\} b^{(i)\rho} \frac{\partial L}{\partial\left(\frac{Db^{(i)\nu}}{D\tau}\right)} = m\dot{x}_\mu - \frac{\partial P_\nu}{\partial \dot{x}^\mu} S^{\nu\rho}\lambda_\rho \tag{29b}$$

where (14g) was used, and

$$\frac{DS^{\mu\nu}}{D\tau} = 2S^{[\mu}{}_\tau \tilde{F}^{\nu]\tau} + S^{[\mu}{}_\tau K_\gamma{}^{\nu]\tau} S^{\gamma\rho}\lambda_\rho \tag{29c}$$

where

$$\tilde{F}^{\mu\nu} = \frac{\mu}{2} F^{\mu\nu} + 2\lambda^{[\mu} P^{\nu]} \tag{29d}$$

$$K_{\gamma\nu\lambda} = \frac{\partial P_\gamma}{\partial S^{\nu\lambda}} \; . \tag{29e}$$

Condition (4) and eq. (29c) imply that

$$S^{\rho\nu}\left[\frac{DP_\nu}{D\tau} - \mu F_{\nu\mu}P^\mu - \lambda_\nu P^\mu P_\mu - \frac{1}{2}\lambda_\rho S^{\rho\gamma} K_{\gamma\mu\nu} P^\mu\right] = 0 \tag{29f}$$

which is an implicite equation for $S^{\rho\nu}\lambda_\nu$.

In (29a) we used the relation

$$\left[\frac{D}{Dx^\rho}, \frac{D}{Dx^\alpha}\right] b^{(j)\mu} = R^\mu{}_{\lambda\rho\alpha}\, b^{(j)\lambda}$$

Equations (29) constitute a coupled system much more complex than the system (25) for condition (2), being highly non linear. However in weak electromagnetic and gravitational fields, eqs. (29) can be much simplified (linear field approximation):

$$\frac{DP_\mu}{D\tau} = qF_{\mu\nu}\dot{x}^\nu + \frac{\mu}{2} S^{\nu\lambda} F_{\nu\lambda,\mu} - \frac{1}{2} \overset{(o)}{R}_{\mu\nu\rho\sigma}\dot{x}^\nu S^{\rho\sigma} \tag{30a}$$

$$P_\mu = m\dot{x}_\mu - S_{\mu\nu}\left[(\mu - 2\mu_0) F^{\nu\lambda}\dot{x}_\lambda - \frac{\mu}{2m_0} S_{\lambda\tau}F^{\lambda\tau,\nu} + \frac{1}{2m_0} \overset{(o)}{R}{}^\nu{}_{\lambda\rho\sigma}\dot{x}^\lambda S^{\rho\sigma}\right] \tag{30b}$$

$$\frac{DS^{\mu\nu}}{D\tau} = 2P^{[\mu}\dot{x}^{\nu]} + 2\mu\, S^{[\mu}{}_\lambda F^{\nu]\lambda} \; ; \tag{30c}$$

Conditions (4) and (12) imply with (30):

$$m = - P_\mu\dot{x}^\mu = (- P_\mu P^\mu)^{1/2} = m_0 \tag{30d}$$

where we identified the constant of motion $-P_\mu P^\mu$ (in the linear approximation) with it's asymptotical value $m^2$. In (30),

$$\overset{(o)}{R}_{\alpha\mu\nu\rho} = g_{\alpha[\rho,\nu]\mu} - g_{\mu[\rho,\nu]\alpha}$$

is the linearized Riemann tensor.

For vanishing gravitational field eqs. (30) coincide with Suttorp and de Groot equations (6). For pure gravitational field they lead to Dixon-Wald equations[20], which can be written in the same form as eqs. (21).

If we define the spin 4-vector

$$S_\rho = \frac{1}{2m_0}\sqrt{-g}\,\varepsilon_{\sigma\mu\nu\rho}P^\sigma S^{\mu\nu}\ , \qquad (31a)$$

we find from (30):

$$\frac{DS^\mu}{D\tau} = \mu F^{\mu\lambda}S_\lambda + (\mu - 2\mu_0)(\dot{x}_\lambda F^{\lambda\nu}S_\nu)\dot{x}^\mu +$$

$$+ \frac{\dot{x}^\mu}{2m_0\sqrt{-g}}\dot{x}_\alpha S_\beta\,\varepsilon^{\alpha\beta\nu\lambda}(\mu F_{\nu\lambda,\gamma} - R_{\gamma\tau\nu\lambda}\dot{x}^\lambda)S^\gamma \qquad (31b)$$

which generalizes the B.M.T. precession equations[20], valid for constant E.M. fields and no gravitational interaction.

Contrary to eqs. (28) which were linearized in the external fields with $P_\mu$ of the form (30b), this cannot be done to eqs. (25) due to the term $S^{\mu\nu}\frac{D\dot{x}^\nu}{D\tau}$. However, if we linearize it in the spin $S_{\mu\nu}$, we obtain equations which coincide with eqs. (30) when they are also linearízed in the spin. For homogeneous fields they become identical and lead to B.M.T. precession equations (as the terms quadratic in $S_{\mu\nu}$ also disappear in eqs. (30) and (31)).

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