Revista Brasileira de Física, Vol. 8, Nº 2, 1978

A Cartesian Operator Algebra for Expansion of Tensor Quantities and Equations in a Spherically Symmetric Background

J. TIOMNO and A.L.L. VIDEIRA

Departamento de Física, Pontifícia Universidade Católica, Rio de Janeiro, RJ*

Recebido em 21 de Novembro de 1977

Tensor (cartesian) operators P_μ^c, $P_{\mu\nu}^{cd} = P_{\nu\mu}^{cd}$, and $Q_{\mu\nu}^{cd} = -Q_{\nu\mu}^{cd}$ are introduced which are useful for expansion of vector, symmetric tensor and anti-symmetric tensor quantities in a curved spherically symmetric space, in terms of rotationally invariant quantities. The algebra of these operators is studied, developped and used to separate Maxwell equations in a time dependent spherically simmetric curved space, without previous expansion in spherical harmonics. Expansion in spherical harmonics and comparison with previous (less general) results is also made.

Introduzem-se operadores tensoriais (Cartesianos) P_μ^c, $P_{\mu\nu}^{cd} = P_{\nu\mu}^{cd}$ e $Q_{\mu\nu}^{cd} = -Q_{\nu\mu}^{cd}$, os quais são úteis para a expansão de quantidades vetoriais, tensoriais simétricas e anti-simétricas em um espaço curvo esfericamente simétrico, em termos de quantidades rotacionalmente invariantes. A álgebra destes operadores é estudada, desenvolvida e utilizada na separação das equações de Maxwell em um espaço esfericamente simétrico e variando no tempo, sem uma prévia expansão em harmônicos esféricos. Faz-se, também, expansão em harmônicos esféricos e compara-se com resultados anteriores (menos gerais).

1. INTRODUCTION

The space-time description of the electromagnetic potential A_μ (or field $F_{\mu\nu}$), and of the gravitational perturbation field $h_{\mu\nu}(=g_{\mu\nu}-\overset{o}{g}_{\mu\nu})$, in the presence of a spherically symmetric static background metric $\overset{o}{g}_{\mu\nu}$ is usually done by expanding them in terms of vector or tensor spherical har-

* Postal address: Cx.P. 38071, Rio de Janeiro, RJ

rnonics. After substitution in the differential field equations (eventually also with a Fourier time-integral expansion), one proceeds to separate the field equations thus obtained.

Separation of Maxwell equations and of Einstein equations without sources were made by Wheeler[1] and by Regge and Wheeler[2], respectively. The separation of Einstein equations with sources was obtained in a paper by Zerilli[3], followed by others on the separation of Maxwell's equations.[4-6] These results have been extensively used in the analysis of electromagnetic and gravitational radiation of particles moving in a Schwarzschild background.[6-7] Finally, the separation of the coupled Maxwell-Einstein equations in a Reissner-Nordström geometry was done by Zerilli[8] (in the presence of sources), by Chitre, Price and Sandberg[9] and by Moncrief[10] (in the absence of sources).

The use of Fourier and tensor harmonics expansions in all those treatments makes the computations somewhat cumbersome. The purpose of the present work is to introduce an alternative method in which the vector and tensor fields depending on $(\vec{x},t)$ are expressed by operators acting on quantities which are scalar under rotations ("scalars"). The differential operators appearing in the field equations (Maxwell or Einstein equations) are also expressed in terms of those new operators.

The advantage of using such aperators, which are defined in *euclidean* four-dimensional space, is that their (cartesian) algebra is rather simple, avoiding, thus, many of the sources of mistakes which arise when dealing with more combersome curvilinear coordinates and with tensor harmonics.

The method, being purely algebraic, is also suitable for the obtention of the field equations by computer programming[11].

In the present paper, this method, besides being described, is also compared to the above mentioned ones, and applied to the separation of Maxwell equations in a spherically symmetric background, which may be time dependent and without the restriction $g = g_0 g_1 = -1$ of the Schwarzschild-Reissner-Nordström external solutions.

In section 2, the vector and tensor operators P^c_μ, $Q^{cd}_{\mu\nu}$, $\tilde{Q}^{cd\underline{\mu\nu}}$ and $P^{cd}_{\mu\nu}$ are introduced, their (cartesian) algebra properties studied, and they are employed for the expansion of vector and tensor quantities in terms of "scalar" ones.

In section 3, expressing these operators in spherical coordinates and expanding the "scalars" in spherical harmonics, we obtain the usual tensor harmonics[3,12-14], and compare our notation with previous ones.

In section 4, completing the expansions of covariant tensors made in section 1, we develop the corresponding expansions of contravariant tensors (in a curved background).

In section 5, as an application of the method, we separate Maxwell equations with sources in the presence of the most general time dependent spherically symmetric geometry.

Finally, in section 6, comments on the results and the advantages of the method are presented.

2. VECTOR AND TENSOR OPERATORS

2.1. Vector Operators

The electromagnetic potential A_μ is represented as

$$A_\mu = P^0_\mu a_0 + P^1_1 a + P^2_\mu a_2 + P^3_\mu a_3 = \sum_{c=0}^{3} P^c_\mu a_c(\vec{x},t) \tag{1}$$

where P^0_μ is the time unit-vector $\tau_\mu = (1,0,0,0)$:

$$P^0_\mu \equiv \tau_\mu = \delta^0_\mu \tag{2a}$$

and $P^1_\mu \equiv \chi_\mu$ is the radial unit-vector

$$P^1_0 = \chi_0 = 0 \quad , \quad P^1_i = \chi_i = \frac{x^i}{r} \quad , \quad r = |\vec{x}| \tag{2b}$$

$i = 1,2,3$ (cartesian coordinates).

These operators P^0_μ and P^1_μ are odd (under space-time reflections).

The tangential operators $P^2_\mu \equiv N_\mu$ (odd) and $P^3_\mu \equiv L_\mu$ (even) have neither time nor radial components, being defined by:

$$P^2_0 = N_0 = 0 \ , \ P^2_i = N_i = -(\vec{\chi} \wedge \vec{x} \wedge \vec{\nabla})_i \tag{2c}$$

$$P^3_0 = L_0 = 0 \ , \qquad P^3_i = L_i = (\vec{x} \wedge \vec{\nabla})_i \ . \tag{2d}$$

Thus:

$$\vec{L} = \vec{\chi} \wedge \vec{N} \quad , \quad \vec{N} = -\vec{\chi} \wedge \vec{L} = r(\vec{\nabla} - \vec{\chi}\partial_r) \ . \tag{2e}$$

We notice that in eq. (1), a_2 and a do not contain a spherically symmetric term ($R = 0$), which would be killed by N_μ or L_μ.

Next we define an *euclidean* scalar product for these vector operators:

$$\alpha \cdot \beta = \alpha^{\underline{\mu}}\beta_\mu = \alpha_0\beta_0 + \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = \alpha_0\beta_0 + \vec{\alpha}\cdot\vec{\beta} \tag{3a}$$

where to indicate summation we raised an index, underlining it in order to avoid confusion with contravariant vectors in curved space. Thus:

$$\alpha^{\underline{\mu}} = \alpha_\mu \neq \alpha^\mu = g^{\mu\nu}\alpha_\nu \ . \tag{3b}$$

The purely time operator τ_μ, the purely space operators χ_μ, N_μ, L_μ and the purely tangential operator $\theta_{\mu\nu}$, defined by

$$\theta_{\mu\nu} \equiv \delta_{\mu\nu} - \tau_\mu\tau_\nu - \chi_\mu\chi_\nu = \theta_{\nu\mu} \tag{4}$$

satisfy the contraction properties given in Table I.

Table I. Contraction properties of P^c_μ and $\theta_{\mu\nu}$

$$\tau\cdot P^c = P^c\cdot\tau = \delta^c_0 \ , \qquad c = 0,1,2,3$$

$$\chi\cdot N = \chi\cdot L = L\cdot\chi = N\cdot L = L\cdot N = 0$$

$$N\cdot\chi = 2 \ , \quad N\cdot N = L\cdot L = L^2$$

$$\tau^{\underline{\mu}}\theta_{\mu\nu} = \theta_{\nu\mu}\tau^{\underline{\mu}} = 0 \ , \quad \chi^{\underline{\mu}}\theta_{\mu\nu} = \theta_{\nu\mu}\chi^{\underline{\mu}} = 0$$

$$\theta_{\mu\nu}N^{\underline{\nu}} = N_\mu \ , \ \theta_{\mu\nu}L^{\underline{\nu}} = L^{\underline{\nu}}\theta_{\nu\mu} = L_\mu \ , \ \theta_{\mu\nu}\theta^{\nu}{}_{\rho} = \theta_{\mu\rho}$$

$$N^{\underline{\mu}}\theta_{\mu\nu} = N_\nu - 2\chi_\nu = -N^\dagger_\nu \ , \quad N^{\underline{\mu}\dagger}\theta_{\mu\nu} = N^\dagger_\nu$$

$$N^\dagger\cdot\chi = N^\dagger\cdot L = 0$$

The operators P^0_μ and P^1_μ are hermitiãn and P^3_μ is antihermitian (actually, $(\hbar/i)\vec{L}$ is the quantum-mechanicai angular momentum operator):

$$P^{0\dagger}_\mu = P^0_\mu \ , \ P^{1\dagger}_\mu = P^1_\mu \ , \ P^{3\dagger}_\mu = -P^3_\mu \ . \tag{5a}$$

However, for P^2_μ we find

$$P^{2\dagger}_\mu = -N_\mu + 2\chi_\mu = N^\dagger_\mu \ . \tag{5b}$$

We have the following orthoyonality relations (cf. Table I):

$$P^{c\dagger}\cdot P^d = P^{c\underline{\mu}\dagger}P^d{}_\mu = \delta_{cd}K_c \tag{6a}$$

with

$$K_0 = K_1 = 1 \ , \ K_2 = K_3 = -L^2 . \tag{6b}$$

We observe that

$$L^2 Y_{\ell m} = -\ell(\ell+1) Y_{\ell m} \quad \text{(not plus!)} \tag{7}$$

For any two operators A and B, we have

$$(AB)^\dagger = B^\dagger A^\dagger \tag{8}$$

from which several relations in Table I result from others.

We can now write the "scalar" components a_μ of the vector potential A_μ, given by eq. (1), as:

$$a_c = K_c^{-1} P^{c\dagger} \cdot A \tag{1a}$$

This equation has a meaning even for c = 2 or 3 because, as already stated, a_2 and a_3 do not contain an R = 0 term.

Table II. Algebraic properties of ∂_μ

$$[\partial_\mu, \tau_\nu] = 0 \ , \ [\partial_\mu, r] = \chi_\mu \ , \ [\partial_\mu, \partial_r] = \frac{1}{r^2} N_\mu$$

$$[\partial_\mu, \chi_\nu] = \frac{1}{r}\theta_{\mu\nu} \ , \ [\partial_\mu, N_\nu] = \frac{1}{r}(\chi_\mu N_\nu - \chi_\nu N_\mu) - \theta_{\mu\nu}\partial_r$$

$$[\partial_\mu, L_\mu] = -\varepsilon_{o\mu\nu\rho}(\chi^\rho \partial_r + \frac{1}{r} N^\rho)$$

$$\theta_\mu{}^\nu \partial_\nu = \frac{1}{r} N_\mu \ , \ \partial_\mu \theta^\mu{}_\nu = \frac{1}{r}(N_\nu - 2\chi_\nu) = -\frac{1}{r} N_\nu^\dagger$$

The operator $\partial_\mu \equiv \partial/\partial x^\mu$ satisfies the relations given in Table II, which are useful to obtain the algebra of commutators of P_μ^c, $\theta_{\mu\nu}$ and L^2, given in Table III. The quantity $\varepsilon_{\lambda\mu\nu\rho}$ appearing in Tables II and III is the totally antisymmetric Levi-Civita symbol $\varepsilon_{0123} = +1$.

From Table III we observe that all the operators P^c_μ and $P^{c\dagger}_\mu$ commute with ∂_0, ∂_r and r.

Obviously, we may also expand ∂_μ as:

$$\partial_\mu = \sum_{c=0}^{3} \delta_c P^c_\mu = \tau_\mu \partial_0 + \chi_\mu \partial_r + \frac{1}{r} N_\mu \tag{9a}$$

Table III. Commutator algebra

$[\tau_\mu, P^c_\nu] = [r, P^c_\nu] = [\partial_0, P^c_\nu] = [\partial_r, P^c_\nu] = 0$
$[N_\mu, \chi_\nu] = [N_\nu, \chi_\mu] = \theta_{\mu\nu}$
$[N_\mu, N_\nu] = \varepsilon_{o\mu\nu\rho} L^\rho$, $[N^\dagger_\mu, N_\nu] = -\varepsilon_{o\mu\nu\rho} L^\rho - 2\theta_{\mu\nu}$
$[\chi_\mu, L_\nu] = [L_\mu, \chi_\nu] = -\varepsilon_{o\mu\nu\rho} \chi^\rho$
$[N_\mu, L_\nu] = [L_\mu, N_\nu] = -\varepsilon_{o\mu\nu\rho} N^\rho$
$[\theta_{\mu\nu}, \chi_\rho] = [\theta_{\mu\nu}, r] = [\theta_{\mu\nu}, \partial_r] = 0$
$[\theta_{\mu\nu}, N_\rho] = \theta_{\mu\rho}\chi_\nu + \chi_\mu \theta_{\nu\rho}$
$[\theta_{\mu\nu}, L_\rho] = \chi_\mu \varepsilon_{o\nu\rho\sigma} \chi^\sigma + \chi_\nu \varepsilon_{o\mu\rho\sigma} \chi^\sigma$
$[L^2, \chi_\mu] = 2(N_\mu - \chi_\mu)$, $[L^2, N_\mu] = -2\chi_\mu L^2$, $[L^2, L_\mu] = 0$

or, by rearrangement:

$$\partial_\mu = \sum_{c=0}^{3} \alpha_c P^{c\dagger}_\mu \tag{9b}$$

where

$$\alpha_0 = \partial_0 \ , \quad \alpha_1 = \partial_r + \frac{2}{r} \ , \quad \alpha_2 = -\frac{1}{r} \ , \quad \alpha_3 = 0 \tag{9c}$$

2.2. Antisymmetric Tensor Operators

The expansion of antisymmetric tensors, for **instance,** the electromagnetic field $F_{\mu\nu}$ as

$$F_{\mu\nu} = \sum_{c<d=0}^{3} Q^{cd}_{\mu\nu} f_{cd} \tag{10a}$$

is obtained from the definition $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, with

$$Q^{cd}_{\mu\nu} = P^c_{\ \mu} P^d_{\ \nu} - P^c_{\ \nu} P^d_{\ \mu} \quad (c<d) \neq (2,3) \tag{10b}$$

$$Q^{23}_{\mu\nu} = N_\mu L_\nu - N_\nu L_\mu - \chi_\mu L_\nu + \chi_\nu L_\mu \tag{10c}$$

and

$$f_{01} = a_{1,0} - a_{0,r} \ , \quad f_{02} = a_{2,0} - \frac{a_0}{r} \ , \ f_{03} = a_{3,0} \tag{11a}$$

$$f_{12} = \frac{1}{r}\,(ra_2)_{,r} - \frac{a_1}{r} \ , \quad f_{13} = \frac{1}{r}\,(ra_3)_{,r} \ , \ f_{23} = \frac{a_3}{r} \tag{11b}$$

where time and radial derivatives are indicated by commas (,0 and $,r$). Here we have used eqs. (9) and the following algebraic properties of the P^c_μ operators, obtained from Table III:

$$[P^0_\mu, P^c_\nu] = 0 \tag{12a}$$

$$[P^1_\mu, P^2_\nu] - [P^1_\nu, P^2_\mu] = 0 \tag{12b}$$

$$[N_\mu, N_\nu] = \chi_\mu N_\nu - \chi_\nu N_\mu \ . \tag{12c}$$

A term $\chi_\nu L_\mu - \chi_\mu L_\nu$ was added to the operator $N_\mu L_\nu - N_\nu L_\mu$ in eq. (10c) to turn it into $Q^{23}_{\mu\nu}$, which is orthogonal to all other operators Q^{cd}.

The six Q^{cd}'s are suitable for expansion of any antisymmetric tensor. Indeed, they are a complete set of orthogonal unnormalised "projection operators" as it is easily shown using Tables I-III:

$$Q^{cd\dagger} . Q^{ef} = Q^{cd\underline{\mu\nu}\dagger} . Q^{ef}_{\mu\nu} = \delta^{ce}\delta^{df}\Lambda^{cd} \tag{13}$$

We have:

$$\Lambda^{01} = 2 \quad , \quad \Lambda^{02} = \Lambda^{03} = \Lambda^{12} = \Lambda^{13} = -2L^2 , \; \Lambda^{23} = 2L^4 \tag{14}$$

As the Q^{cd}'s are essentialy commutators of the P^c's, expressions of the type

$$A_\mu B_\nu C^{\underline{\mu}} = B_\nu A_\mu C^{\underline{\mu}} + [A_\mu , B_\nu] C^{\underline{\mu}} \tag{15}$$

given in Table IV, are easily calculated, using Tables I and III. Many of the relations contained in Table IV result from others by taking the adjoint of both sides.

To agree with the usual nomenclature[1-3], field quantities a_3 and f_{c3} shall be called odd (or magnetic), while a_c and f_{cd}, with c,d $\neq$ 3, shall be called even (or electric).

We shall finish this section on antisymmetric tensor operators with the computation of the expansion of the dual tensor

$$\tilde{F}^{\underline{\mu\nu}} = \frac{1}{2} \varepsilon^{\underline{\mu\nu\rho\sigma}} F_{\rho\sigma} = \sum_{c<d=0}^{3} Q^{cd\underline{\mu\nu}} \tilde{f}_{cd} \tag{16}$$

where $\varepsilon^{\underline{\mu\nu\rho\sigma}} = \varepsilon_{\mu\nu\rho\sigma}$, so that $\varepsilon^{\underline{0123}} = +1$. For this, therefore, we must compute $\tilde{f}_{cd}$ in terms of f_{ab}. From (16) we can write

$$\tilde{F}^{\underline{\mu\nu}} = \sum_{c<d=0}^{3} \tilde{Q}^{cd\underline{\mu\nu}} f_{cd} \tag{17}$$

Tables IV. Contracted Products of Three Vector Operatòrs

$$\chi_\mu N_\nu \chi^\mu = -\chi_\mu N_\nu N^\mu = -\chi_\mu L_\nu L^\mu = N_\nu$$

$$\chi_\mu L_\nu \chi^\mu = -\chi_\mu N_\nu L^\mu = -\chi_\mu L_\nu N^\mu = -N^\dagger_\mu L_\nu \chi^\mu = L_\nu$$

$$N_\mu \chi_\nu L^\mu = -N_\mu N_\nu L^\mu = -N^\dagger_\mu N_\nu L^\mu = -N^\dagger_\mu \chi_\nu L^\mu = L_\nu$$

$$N_\mu \chi_\nu N^\mu = -N^\dagger_\mu \chi_\nu N^\mu = N_\nu + \chi_\nu L^2$$

$$N_\mu L_\nu N^\mu = L_\nu (L^2 - 1)$$

$$N_\mu N_\nu \chi^\mu = 3N_\nu - 2\chi_\nu$$

$$L_\mu N_\nu \chi^\mu = -L_\mu \chi_\nu N^\mu = -L_\mu N_\nu N^\mu = -L_\mu N^\dagger_\nu N^\mu = -L_\mu N^\dagger_\nu \chi^\mu = L_\nu$$

$$L_\mu \chi_\nu L^\mu = \chi_\nu L^2 - N_\nu$$

$$L_\mu N_\nu L^\mu = -N^\dagger_\mu N_\nu N^\mu = N_\nu (L^2 + 1) - \chi_\nu L^2$$

where

$$\tilde{Q}^{cd\mu\nu} = \frac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}\,Q^{cd}_{\rho\sigma}\ . \qquad (18)$$

Hence:

$$\tilde{Q}^{1j\mu\nu} = \varepsilon^{\mu\nu\rho\sigma}\,P^1_\rho P^j_\sigma \quad , \quad j = 1,\ 2,\ 3$$

$$\tilde{Q}^{23\mu\nu} = \varepsilon^{\mu\nu\rho\sigma}\,(N_\rho L_\sigma - \chi_\rho L_\sigma)\ .$$

Since $P^i_0 = 0$, for $i = 1,2,3$, the above quantities vanish unless μ or $\nu = 0$. Thus we compute:

$$\tilde{Q}^{120\ell} = L^\ell \quad , \quad \tilde{Q}^{130\ell} = -N^\ell\ ,$$

$$\tilde{Q}^{23\ell \underline{\ell}} = [(\vec{N} - \vec{\chi}) \wedge \vec{L}]^{\underline{\ell}} = \chi^{\underline{\ell}} L^2 .$$

Therefore:

$$\tilde{Q}^{12}_{\mu\nu} = Q^{03}_{\mu\nu} \ , \ \tilde{Q}^{13}_{\mu\nu} = - Q^{02}_{\mu\nu} \ , \ \tilde{Q}^{23}_{\mu\nu} = Q^{01}_{\mu\nu} L^2 \qquad (19)$$

Also, since $\tilde{\tilde{Q}}^{cd} = + Q^{cd}$ (euclidean metric), we find

$$\tilde{Q}^{03}_{\mu\nu} = Q^{12}_{\mu\nu} \ , \quad \tilde{Q}^{02}_{\mu\nu} = - Q^{13}_{\mu\nu} \ , \ \tilde{Q}^{01}_{\mu\nu} = \frac{1}{L^2} Q^{23}_{\mu\nu} \ . \qquad (20)$$

Taking these last relations into (17) we finally get:

$$\begin{aligned} &\tilde{f}_{12} = f_{03} \quad \tilde{f}_{13} = - f_{02} \quad \tilde{f}_{23} = \frac{1}{L^2} \hat{f}_{01} \\ &\tilde{f}_{03} = f_{12} \quad \tilde{f}_{02} = - f_{13} \quad \hat{\tilde{f}}_{01} = L^2 f_{23} \end{aligned} \qquad (21)$$

As the operator $1/L^2$ is singular, these relations can be applied only if f_{01} does not include a spherically symmetric term ($\ell = 0$). Thus, in (21) above, $\hat{f}_{01}$ and $\hat{\tilde{f}}_{01}$ stand for

$$\hat{f}_{01} = f_{01} - \frac{1}{4\pi} \int f_{01} d\Omega \ , \ \hat{\tilde{f}}_{01} = \tilde{f}_{01} - \frac{1}{4\pi} \int \tilde{f}_{01} d\Omega \qquad (22)$$

2.3. Symmetric Tensor Operators

In order to obtain a complete set of symmetric tensor operators we proceed in a heuristic way, similar to that used for the antisymmetric operators. Thus, we expand the symmetric quantity $A_{\mu,\nu} + A_{\nu,\mu}$ as

$$A_{\mu,\nu} + A_{\nu,\mu} = \sum_{c \le e = 0}^{3} P^{ce}_{\mu\nu} d_{ce}(\vec{x}, t) \ . \qquad (23)$$

We find:

$$A_{\mu,\nu} + A_{\nu,\mu} = 2P^{00}_{\mu\nu}a_{0,0} = 2P^{11}_{\mu\nu}a_{1,r} + P^{01}_{\mu\nu}(a_{0,r} + a_{1,0}) + P^{02}_{\mu\nu}(a_{2,0} + \frac{a_0}{r})$$

$$+ P^{03}_{\mu\nu}a_{3,0} + P^{12}_{\mu\nu}(a_{2,r} + \frac{a_1}{r}) + P^{13}_{\mu\nu}a_{3,r} + 2P^{33}_{\mu\nu} \frac{a_1}{r}$$

$$+ (N_\mu N_\nu + N_\nu N_\mu) \frac{a_2}{r} + (N_\mu L_\nu + N_\nu L_\mu) \frac{a_3}{r} . \tag{24}$$

where the orthogonal set of operators $P^{cd}_{\mu\nu}$, $c \leq d$ is:

$$P^{00}_{\mu\nu} = \tau_\mu\tau_\nu , \quad P^{11}_{\mu\nu} = \chi_\mu\chi_\nu , \quad P^{33}_{\mu\nu} = \theta_{\mu\nu}$$

$$P^{ab}_{\mu\nu} = P^a_\mu P^b_\nu + P^a_\nu P^b_\mu , \quad a = 0,1 < b = 1,2,3 \tag{25a}$$

with P^{22} and P^{23} yet to be defined.

The operator $\theta_{\mu\nu}$ appeared in the expansion (24) from reordering $P^2_\mu P^1_\nu = N_\mu\chi_\nu$ as

$$N_\mu\chi_\nu = \chi_\nu N_\mu + \theta_{\mu\nu} \tag{26}$$

In the expansion (24) the two following operators also appeared:

$$P^2_\mu P^2_\nu + P^2_\nu P^2_\mu = N_\mu N_\nu + N_\nu N_\mu \tag{27a}$$

and

$$P^2_\mu P^3_\nu + P^2_\nu P^3_\mu = N_\mu L_\nu + N_\nu L_\mu \tag{27b}$$

As we had to do in the case of Q^{23} we have to add terms to these operators so that they become orthogonal to each other and to all the remaining P^{cd}'s. We find

$$P^{23}_{\mu\nu} = N_\mu L_\nu + N_\nu L_\mu + \chi_\mu L_\nu + \chi_\nu L_\mu$$

$$P^{22}_{\mu\nu} = N_\mu N_\nu + N_\nu N_\mu + \chi_\mu N_\nu + \chi_\nu N_\mu - \theta_{\mu\nu}L^2 \tag{25b}$$

In this manner, we have all the ten P^{cd}'s ($c \leq d$) orthogonal to each other. Indeed, using Tables I and IV, and equations (5) and (6) we obtain

$$P^{cd\dagger}\cdot P^{ef} = P^{cd\underline{\mu\nu}\dagger}\cdot P^{ef}_{\mu\nu} = \delta^{ce}\delta^{df}K^{cd} \tag{28a}$$

with:

$$\begin{aligned} 2K^{00} &= 2K^{11} = K^{01} = K^{33} = 2 \\ K^{02} &= K^{03} = K^{12} = K^{13} = -2L^2 \\ K^{23} &= K^{22} = 2L^2\,(L^2+2) \end{aligned} \tag{28b}$$

Obviously, from symmetry properties, the P^{cd}'s are all orthogonal to the Q^{ef}'s:

$$P^{cd\dagger}\cdot Q^{ef} = Q^{ef\dagger}\cdot P^{cd} = 0\ . \tag{29}$$

This set of ten symmetric, orthogonal operators is complete. Thus, we may expand any simmetric covariant tensor function of $(\vec{x},t)$, as, for instance, the metric field $g_{\mu\nu}$ or the perturbations of the background field $h_{\mu\nu} = g_{\mu\nu} - \overset{0}{g}_{\mu\nu}$, as

$$h_{\mu\nu} = \sum_{c\leq d}^{3} P^{cd}_{\mu\nu}\, a_{cd}(\vec{x},t) \quad \text{(cartesian coordinates)} \tag{30}$$

where a_{cd} is given by

$$a_{cd}(\vec{x},t) = K^{-1}_{cd}\, P^{cd\dagger}\cdot h \tag{31}$$

Analogously to the antisymmetric case, the symmetric quantities a_{cd} with $d \neq 3$, and a_{33} are even (or electric); the remaining ones with $d = 3$ are odd (or magnetic).

It is obvious that P^{cd} for $c = 0,1$, $d = 2,3$ kills any $\ell = 0$ part of a_{cd}, and it can be shown that P^{22} and P^{23} kill both any $\ell = 0$ or $\ell = 1$ parts of a_{22} and a_{23} respectively.

3. VECTQR AND TENSQR SPHERICAL HARMONICS

Vector and tensor operators can be written in spherical coordinates $y^{\alpha} = (t,r,\theta,\phi)$ as

$$P^{c}_{(\alpha)} = \frac{\partial x^{\mu}}{\partial y^{\alpha}} P^{c}_{\mu} \ , \quad Q^{cd}_{(\alpha)(\beta)} = \frac{\partial x^{\mu}}{\partial y^{\alpha}} \frac{\partial x^{\nu}}{\partial y^{\beta}} Q^{cd}_{\mu\nu} \tag{32}$$

and similarly for $P^{cd}_{(\alpha)(\beta)}$. We have the following useful expressions (i = 1,2,3):

$$\frac{\partial x^{i}}{\partial r} = \chi_{i} = \chi^{i} \ , \quad \frac{\partial x^{0}}{\partial y^{\alpha}} = \delta^{0}_{\alpha} \ , \quad \frac{\partial x^{i}}{\partial y^{0}} = 0$$

$$\frac{\partial x^{i}}{\partial \theta} = \frac{r}{\sin\theta} (\vec{\chi} \cos\theta - \vec{k})_{i} \ , \quad \frac{\partial x^{i}}{\partial \phi} = r(\vec{k} \wedge \vec{\chi})_{i} \tag{33}$$

$$L_{0} = L_{r} = 0 \ , \quad L_{\theta} = -\frac{r}{\sin\theta} \frac{\partial}{\partial\phi} \ , \quad L_{\phi} = r \sin\theta \frac{\partial}{\partial\theta}$$

$$N_{0} = N_{r} = 0 \ , \quad N_{\theta} = r \frac{\partial}{\partial\theta} \ , \quad N_{\phi} = r \frac{\partial}{\partial\phi} \tag{34}$$

Hence, for vector $A_{(\alpha)} = \sum P^{c}_{(\alpha)} a_{c}$, we find:

$$\begin{pmatrix} A_{0} \\ A_{r} \\ A_{\theta} \\ A_{\phi} \end{pmatrix} = \begin{pmatrix} a_{0} \\ a_{1} \\ r\, a_{2,\theta} \\ r\, a_{2,\phi} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\frac{r}{\sin\theta} a_{3,\phi} \\ +r \sin\theta\, a_{3,\theta} \end{pmatrix} \tag{35}$$

when the "scalars" a_c are expanded in spherical harmonics, we obtain the well known representation in vector harmonics, given, for instance, by Ruffini, Tiomno and Vishveshwara[4], to be referedas RTV hereafter, with the correspondence in notation:

$$a_0 \to f \quad , \quad a_1 \to h \quad , \quad ra_2 \to k \quad , \quad ra_3 \to -a \tag{36}$$

Similarly, for antisymnetric terms, we have:

$$F_{(\alpha)(\beta)} = - F_{(\beta)(\alpha)} = \sum_{c<d=0}^{3} Q^{cd}_{(\alpha)(\beta)} f_{cd}$$

$$= \begin{pmatrix} 0 & f_{01} & rf_{02,\theta} & rf_{02,\phi} \\ \cdot & 0 & rf_{12,\theta} & rf_{12,\phi} \\ \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -\frac{r}{\sin\theta} f_{03,\phi} & r\sin\theta f_{03,\theta} \\ \cdot & 0 & -\frac{r}{\sin\theta} f_{13,\phi} & r\sin\theta f_{13,\theta} \\ \cdot & \cdot & 0 & r^2\sin\theta f_{23} \\ \cdot & \cdot & \cdot & 0 \end{pmatrix}, \tag{37}$$

where the dots indicate antisymmetric terms. The first group of terms above are electric or even (no index 3), and the second group are magnetic or odd (one index 3).

The f_{cd} quantities expanded in spherical harmonics coincide with the corresponding quantities of Zerilli[8] (Table II of that work). However, the present notation is unambiguous, since the magnetic terms f_{03} and f_{13} (represented by $\tilde{f}_{02}/r$ and $\tilde{f}_{12}/r$ in that paper) cannot be mistaken with the electric terms f_{02} and f_{12} (also represented there as $\tilde{f}_{02}/r$ and $\tilde{f}_{12}/r$). Thus, the correspondence between the present notation and Zerilli's [8] is

$$\begin{aligned} f_{03} &\to -\tilde{f}^m_{02}/r \quad , \quad f_{13} \to -\tilde{f}^m_{12}/r \quad , \quad rf_{23} \to \tilde{f}_{23} \\ f_{01} &\to \tilde{f}_{01} \quad , \quad f_{02} \to \tilde{f}^e_{02}\, r \quad , \quad f_{12} \to \tilde{f}^e_{12}\, r \end{aligned} \tag{38}$$

If we take equation (11) into (37) and expand a_c in spherical harmonics, we obtain expressions (7) and (9) of RTV[4] (except for factors $g^{\mu\nu}$ for raising components) with the difference in notation expressed in (36) and the following corrections for misprints in RTV:

$$F^{\theta\phi}_{\ell m} \to F^{\theta\phi}_{\ell m}/Y_{\ell m} \qquad \text{(Eq. (7) of RTV)}$$

$$\sin\theta \to \sin^2\theta \qquad \text{(Eq. (9) of RTV)} \tag{39}$$

Finally, we write $h_{\mu\nu}$ in spherical coordinates:

$$h_{(\alpha)(\beta)} = \sum_{c \leq d=0}^{3} P^{cd}_{(\alpha)(\beta)}\, a_{cd}(\vec{x},t)$$

$$= \begin{pmatrix} a_{00} & a_{01} & ra_{02,\theta} & ra_{02,\phi} \\ * & a_{11} & ra_{12,\theta} & ra_{12,\phi} \\ * & * & r^2(a_{33} + Wa_{22}) & r^2 Xa_{22} \\ * & * & * & r^2\sin^2\theta(a_{33} - Wa_{22}) \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 & 0 & -\frac{r}{\sin 0} a_{03,\phi} & r \sin 0\, a_{03,\theta} \\ 0 & 0 & -\frac{r}{\sin 0} a_{13,\phi} & r \sin 0\, a_{13,\theta} \\ * & * & -\frac{r^2}{\sin 8} Xa_{23} & r^2 \sin 8\, Wa_{23} \\ * & * & * & r^2 \sin 0\, Xa_{23} \end{pmatrix} \tag{40a}$$

where the operators X and W are

$$X = 2 \frac{\partial}{\partial\phi} \left(\frac{\partial}{\partial\theta} - \text{cotg}\,\theta \right)$$

$$W = \frac{\partial^2}{\partial\theta^2} - \text{cotg} \frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} = 2 \frac{\partial^2}{\partial\theta^2} - L^2 . \tag{40b}$$

and the asterisks denote the elernents obtained by symmetrization. The first group of terms are "electric" (even number of indices 3) and the second group consists of "magnetic" terms (odd number of indices 3). The above expression for the field $h_{\mu\nu}$, Eq. (40), is obtained frorn the expansion of the symrnetric operator $P^{cd}_{(\alpha)(\beta)}$ in rpherical coordinates:

$$P^{cd}_{(\alpha)(\beta)} = P^{cd}_{(\beta)(\alpha)} = x^{\mu}{}_{,\alpha} x^{\nu}{}_{,\beta} P^{cd}_{\mu\nu} \tag{41}$$

We have:

$$\begin{aligned}
&P^{00}_{(0)(0)} = P^{01}_{(0)(r)} = P^{11}_{(r)\ (r)} = 1 \\
&P^{02}_{(0)(\theta)} = P^{12}_{(r)(\theta)} = rN_{\theta},\ P^{02}_{(0)(\phi)} = P^{12}_{(r)(\phi)} = rN_{\phi} \\
&P^{03}_{(0)(\theta)} = P^{13}_{(r)(\theta)} = rL_{\theta},\ P^{03}_{(0)(\phi)} = P^{13}_{(r)(\phi)} = rL_{\phi} \\
&P^{22}_{(\theta)(\theta)} = r^2 W\ ,\ P^{22}_{(\phi)(\phi)} = -\ r^2 \sin^2\theta W,\quad P^{22}_{(\theta)(\phi)} = r^2 X \\
&P^{23}_{(\theta)(\theta)} = -\frac{r^2}{\sin\theta} X\ ,\ P^{23}_{(\phi)(\phi)} = r^2 \sin\theta\ X,\ P^{23}_{(\theta)(\phi)} = r^2 \sin\theta\ W \\
&P^{33}_{(\theta)(\theta)} = r^2\ ,\quad P^{33}_{(\phi)(\phi)} = r^2 \sin^2\theta\ .
\end{aligned} \tag{42}$$

Except for the symmetry $P^{cd}_{(\alpha)(\beta)} = P^{cd}_{(\beta)(\alpha)}$, the remaining operator components not listed in (42) vanish.

If the "scalar" quantities $a_{cd}(\vec{x},t)$ are expanded in spherical harmonics $a^{\ell m}_{cd}(r,t)$ we obtain Zerilli's[3,14] expansions in tensor harmonics, except for normalization factors. Regge and Wheeler[2] use a different expansion, where the operator $\frac{1}{2}(N_{\mu}N_{\nu} + N_{\nu}N_{\mu} + \chi_{\mu}N_{\nu} + \chi_{\nu}N_{\mu})$ takes the place of $P^{22}_{\mu\nu}$. The correspondence between our notation and Regge-Wheeler's is, for a given (ℓ,m):

$$\begin{aligned}
&a_{00} = -g_0 H_0\ ,\quad a_{01} = H_1\ ,\quad ra_{02} = h_0^{(e)}\ ,\quad ra_{03} = h_0^{(m)} \\
&a_{11} = g_1 H_2\ ,\quad ra_{12} = h_1^{(e)}\ ,\quad ra_{13} = h_1^{(m)} \\
&a_{22} = G/2\ ,\quad r^2 a_{23} = -h_2/2\ ,\quad a_{33} = K - \ell(\ell+1)G/2
\end{aligned} \tag{43}$$

Actually, this Regge-Wheeler expansion is the one later written by Zerilli[8] for the $h_{\mu\nu}$'s (Table I of that paper).

Notice that a_{cd} for $c = 0,1$, $d = 2,3$ have no $R = 0$ component, while a_{22} and a_{23} have neither $R = 0$ or $R = 1$ components.

If, similarly to $h_{\mu\nu}$, we expand

$$16\pi T_{\mu\nu} = \sum_{c\leq d=0}^{3} P_{\mu\nu}^{cd}\, t_{cd} \tag{44}$$

the results for the expansion of $-16\pi T_{(\alpha)(\beta)}$ with coefficients A_{bc}, given in Table III of reference 8, are obtained with the correspondence in notation:

$$\begin{aligned} &t_{00} = A_{00}\ , \quad t_{01} = -A_{01}\ , \quad t_{11} = -A_{11} \\ &rt_{02} = -A^{e}_{02}\ , \quad rt_{12} = -A^{e}_{12}\ , \\ &r^2 t_{22} = -\frac{1}{2}A_{23}\ , \quad r^2 t_{33} = -A^{e}_{22}\ , \\ &rt_{03} = A^{m}_{02}\ , \quad rt_{13} = A^{m}_{12}\ , \quad r^2 t_{23} = \frac{1}{2}A^{m}_{22} \end{aligned} \tag{45}$$

We notice again the advantage of our notation, as the magnetic (odd) components are distinguished from the electric (even) ones by the fact that they have respectively, an odd or an even number of indices 3. Here, as in Eqs. (38) and (43) we added the superçcripts (e) or (m) to avoid further confusion.

4. EXPANSION OF CONTRAVARIANT VECTORS AND TENSORS

Up to here we have expanded only covariant quantities as A_μ, $F_{\mu\nu}$, and $h_{\mu\nu}$. In general, we have to deal with equations for these quantities in a curved background space, with a spherically symmetric metric $\overset{0}{g}_{\mu\nu}$, which from now on shall be called $g_{\mu\nu}$. We take $g_{\mu\nu}$ in the form (cartesian coordinates):

$$g_{\mu\nu} = g^{\underline{\mu\nu}} = g_0(r,t)\tau_\mu\tau_\nu + g_1(r,t)\chi_\mu\chi_\nu + g_2(r,t)\theta_{\mu\nu} \tag{46a}$$

For the **Reissner-Nordström** solution, for **instance,** we may have $g_2 = 1$ (schwarzschild coordinates), $g_1 = g_2$ (isotropic coordinates, etc. From now on we shall use Schwarzschild cartesial coordinates ($g_2 = 1$). For the **external Reissner-Nordström** solution: $g = g_0 g_1 = -1$.

In order to avoid **confusion** of the cartesian **"metric"** $\delta_{\mu\nu} = \delta^{\underline{\mu\nu}}$, used for contraction in the algebra, with the gravitational metric $g^{\mu\nu}$, we raise tensor indices with $\gamma^{\underline{\mu\nu}} \equiv g^{\mu\nu}$, so that the contravariant metric tensor is given by:

$$\gamma^{\underline{\mu\nu}} \equiv g^{\mu\nu} = g_0^{-1}\tau^{\underline{\mu}}\tau^{\underline{\nu}} + g_1^{-1}\chi^{\underline{\mu}}\chi^{\underline{\nu}} + \theta^{\underline{\mu\nu}} \neq g^{\underline{\mu\nu}} \qquad (46b)$$

where we have already considered $g_2 = 1$.

All contravariant tensors should then be expressed in terms of covariant ones as:

$$A^{\mu\nu} = \gamma^{\underline{\mu\rho}}\gamma^{\underline{\nu\sigma}}A_{\rho\sigma} \ . \qquad (47a)$$

Thus contravariant tensors can be dealt with by first lowering and then, after the expansion, raising their tensor indices

$$V^{\mu}(\equiv g^{\mu\nu}V_{\nu}) = \sum_{c=0}^{3} \gamma^{\underline{\mu\nu}}P^{c}_{\nu}v_{c} \ . \qquad (47b)$$

As an example, we consider the current density of a point charge:

$$J^{\mu}(\vec{x},t) = \frac{q}{\sqrt{-g}}\,\delta(\vec{x} - \vec{z}(t))\,\dot{z}^{\mu}$$

$$J_{\mu} = g_{\mu\nu}J^{\nu} \qquad (48)$$

$$(z^0 = t \ , \quad g = g_0 g_1)$$

where the dot represents derivation with respect to **t**.

Thus, with $J_{\mu} = \sum_{c=0}^{3} P^{c}_{\mu}\, j_c$ and $j_c = K_c^{-1}\, P^{c\underline{\mu}\dagger}\, J_{\mu}$, we get, **putting** $\dot{\vec{z}} = \vec{v}$, $R = |\vec{z}|$:

$$j_0 = \tau^{\underline{\mu}} J_\mu = -\, q\sqrt{-g_0/g_1}\;\delta(\vec{x} - \vec{z})$$

$$j_1 = \chi^{\underline{\mu}} J_\mu = q\sqrt{g_1/-g_0}\;\dot{R}\delta(\vec{x} - \vec{z})$$

$$j_2 = \frac{-1}{L^2} N^{\underline{\mu}\dagger} J_\mu = \frac{-q}{\sqrt{-g}\;L^2}\;\vec{v}.\vec{N}^\dagger\;\hat{\delta}(\vec{x} - \vec{z}) \tag{49}$$

$$j_3 = \frac{-1}{L^2} L^{\underline{\mu}\dagger} J_\mu = \frac{q}{\sqrt{-g}\;L^2}\;\vec{v}.\vec{L}\;\delta(\vec{x} - \vec{z})$$

where

$$\hat{\delta}(\vec{x}) \equiv \delta(\vec{x}) - \frac{\delta(r)}{4\pi r^2} \tag{49a}$$

We notice that in these equations, $1/L^2$ is well defined, since both $\vec{N}^\dagger\,\hat{\delta}(\vec{x} - \vec{z})$ and $\vec{L}\delta(\vec{x} - \vec{z})$ have no R = 0 term. Therefore

$$J^\mu = \gamma^{\underline{\mu\nu}} J_\nu = g_0^{-1}\tau^{\underline{\mu}} j_0 + g_1^{-1}\chi^{\underline{\mu}} j_1 + N^{\underline{\mu}} j_2 + L^{\underline{\mu}} j_3 \tag{50}$$

The coefficients of the expansion in spherical harmonics

$$j_c(\vec{x},t) = \sum_{\ell,m} j_c^{\ell m}(r,t)\;Y_{\ell m}(\theta,\phi) \tag{51a}$$

are (as $K_c \quad K_c^\dagger \quad K_c^*$)

$$j_c^{\ell m}(r,t) = \int d\Omega\;Y^*_{\ell m}(\theta,\phi)\;\frac{1}{K_c}\;P^{c\underline{\mu}\dagger} J_\mu(t,r,\theta,\phi)$$

$$= \int d\Omega\;J_\mu\Big(P^{c\underline{\mu}}\;\frac{1}{K_c}\;Y_{\ell m}\Big)^* = \int d\Omega\;J_\mu P^{c\underline{\mu}}\,\frac{1}{K_c}\,Y^*_{\ell m} \tag{51b}$$

snd we easily find

$$j_0^{\ell m}(r,t) = -\sqrt{\frac{-g_0}{g_1}}\ \frac{q}{r^2}\,\delta(r-R)\ \ Y^*_{\ell m}\ (\Theta(t),\Phi(t))$$

$$j_1^{\ell m}(r,t) = \sqrt{\frac{g_1}{-g_0}}\ \frac{q}{r^2}\,\dot{R}\delta(r-R)\ \ Y^*_{\ell m}(\Theta(t),\Phi(t))$$

$$j_2^{\ell m}(r,t) = \frac{1}{\ell(\ell+1)}\ \frac{q}{r}\ \frac{\delta(r-R)}{\sqrt{-g}}\ \frac{d}{dt}\ Y^*_{\ell m}(\Theta(t),\ \Phi(t))\ ,\ (\ell \neq 0) \tag{51c}$$

$$j_3^{\ell m}(r,t) = -\frac{1}{\ell(\ell+1)}\ \frac{q}{r}\ \frac{\delta(r-R)}{\sqrt{-g}}\ \left(\frac{\dot{\Theta}}{\sin\Theta}\ \frac{\partial}{\partial\Phi} - \dot{\Phi}\ \sin\Theta\frac{\partial}{\partial\Theta}\right)Y^*_{\ell m},\ (\ell \neq 0)$$

These expressions agree with the ones given by RTV[4] and by Zerilli[8], with the following correspondence*:

$$j_0 \to \psi/4\pi\ ,\ j_1 \to \eta/4\pi\ ,\ rj_2 \to \chi/4\pi\ ,\ rj_3 \to -\alpha/4\pi \quad \text{(RTV)}$$

$$j_0 \to v\ ,\ j_1 \to u\ ,\ rj_2 \to w\ ,\ rj_3 \to -y \quad \text{(Zerilli)}$$

As a second example, we consider the stress tensor $T^{\mu\nu}$ of a particle of mass m_0:

$$16\pi T^{\mu\nu}(\vec{x},t) = 16\pi\,\frac{m_0}{\sqrt{-g}}\,\delta(\vec{x} - \vec{z}(t))\ \dot{z}^\mu\ \dot{z}^\nu\ \frac{dt}{d\tau} \tag{52}$$

with $\frac{d}{dt} = \sqrt{-g_{\mu\nu}\dot{z}^\mu\dot{z}^\nu}$. Thus, using eqs. (30) and

$$t_{cd} = \frac{16\pi}{K^{cd}}\ P^{cd\dagger}\cdot T = \frac{16\pi m_0}{\sqrt{-g}}\ \frac{dt}{d\tau}\ \tilde{t}_{cd} \tag{53a}$$

---

* In RTV the following misprints are notices: $(\psi,\eta,\alpha) \to (\psi,\eta,\alpha)/4\pi$ in Eq. (15) and $T^r{}_0 \to 4\pi T^r{}_0$ in Eq.(16b).

we find

$$\tilde{t}_{00} = g_0^2\delta(\vec{x} - \vec{z}) \quad , \quad \tilde{t}_{01} = g_0 g_1 \dot{R}\delta(\vec{x} - \vec{z}) \quad , \quad \tilde{t}_{11} = g_1^2 \dot{R}^2\delta(\vec{x} - \vec{z})$$

$$\tilde{t}_{02} = \frac{-1}{L^2} g_0 \vec{v}\cdot\vec{N}^{\dagger}\hat{\delta}(\vec{x} - \vec{z}) \quad , \quad \tilde{t}_{03} = \frac{1}{L^2} g_0 \vec{v}\cdot\vec{L}\ \delta(\vec{x} - \vec{z})$$

$$\tilde{t}_{12} = \frac{-1}{L^2} g_1 \dot{R}\vec{v}\cdot\vec{N}^{\dagger}\hat{\delta}(\vec{x} - \vec{z}) \quad , \quad \tilde{t}_{13} = \frac{1}{L^2} g_1 \dot{R}\vec{v}\cdot\vec{L}\delta(\vec{x} - \vec{z}) \tag{53b}$$

$$\tilde{t}_{33} = \frac{1}{2} (\vec{v}^2 - \dot{R}^2)\ \delta(\vec{x} - \vec{z}) \ , \quad \tilde{t}_{23} = \frac{-1}{L^2 + 2} (\tilde{t}_{13} + \frac{1}{L^2} \vec{v}\cdot\vec{L}\vec{v}\cdot\vec{N}^{\dagger}\delta(\vec{x} - \vec{z})$$

$$\tilde{t}_{22} = \frac{-1}{L^2 + 2} (\tilde{t}_{12} + \tilde{t}_{33} - \frac{1}{L^2} \vec{v}\cdot\vec{N}^{\dagger}\vec{v}\cdot\vec{N}^{\dagger}\hat{\delta}(\vec{x} - \vec{z}))$$

The coefficients of the expansion in spherical harmonics, corresponding to (53a), are now ($K^{\dagger}_{cd} = K_{cd} = K^{*}_{cd}$):

$$\begin{aligned}\tilde{t}^{\ell m}_{cd}(r,t) &= \int d\Omega T_{\mu\nu} P^{cd\underline{\mu}\underline{\nu}} \frac{1}{K_{cd}} Y_{\ell m}(\theta,\phi)^{*} \frac{\sqrt{-g}}{m_0\ dt/d\tau} \\ &= \int d\Omega T_{(\alpha)(\beta)} P^{cd(\underline{\alpha})(\underline{\beta})} \frac{1}{K_{cd}} Y_{\ell m}(\theta,\phi)^{*} \frac{\sqrt{-g}}{m_0\ dt/d\tau} \\ &= \int d\Omega \frac{\delta(r - R)}{r^2} \delta(\theta - \Theta)\ (\phi - \Phi)\ g_{(\alpha)} v^{(\underline{\alpha})} g_{(\beta)} v^{(\underline{\beta})} \times \\ &\quad \times P^{cd}_{(\alpha)(\beta)} \frac{1}{K_{cd}} Y^{*}_{\ell m}\end{aligned} \tag{54}$$

(summatinn on α,β), with $g_{(\alpha)} = (g_0, g_1, r^2, r^2\sin^2\theta)$, $v^{(\underline{\alpha})} = (1, \dot{R}, \dot{\Theta}, \dot{\Phi})$.

From (41) and (42) we obtain the values of $\tilde{t}^{\ell m}_{cd}$ given in Table V.

Taking t_{cd} into (45) we obtain $A_{\alpha\beta}$, which coincide with the expressions listed by Zerilli[8] in his Tables IV and V, after misprints in those tables are corrected by the substitutions:

$$A_{00} \to -A_{00} \qquad A^{e}_{02} \to -A^{e}_{02}$$
$$A^{m}_{12} \to -A^{m}_{12} \qquad A^{m}_{22} \to -A^{m}_{22}/2\gamma \tag{55}$$

Table V. Harmonic coefficients for point particle stress tensor $(\tilde{t}^{\ell m}_{cd})$.

$$(\tilde{t}^{\ell m}_{00}, \tilde{t}^{\ell m}_{01}, \tilde{t}^{\ell m}_{11}) = (g_0^2, g_0 g_1 \dot{R}, g_1^2 \dot{R}^2) \frac{\delta(r - R(t))}{R^2} Y^*_{\ell m}(\Theta(t), \Phi(t))$$

$$(\tilde{t}^{\ell m}_{02}, \tilde{t}^{\ell m}_{12}) = (g_0, g_1 \dot{R}) \frac{1}{\ell(\ell + 1)} \frac{\delta(r - R)}{R} \frac{d}{dt} Y^*_{\ell m}\ , \quad (\ell \neq 0)$$

$$(\tilde{t}^{\ell m}_{03}, \tilde{t}^{\ell m}_{13}) = -(g_0, g_1 \dot{R}) \frac{1}{\ell(\ell+1)} \frac{\delta(r-R)}{R} \left(\frac{\dot{\Theta} Y^*_{\ell m,\Phi}}{\sin\Theta} - \dot{\Phi} \sin\Theta\ Y^*_{h,\Theta}\right), \quad (\ell \neq 0)$$

$$\tilde{t}^{\ell m}_{22} = \frac{(\dot{\Theta}^2 - \dot{\Phi}^2 \sin^2\Theta) W^*_{\ell m} + 2\dot{\Theta}\dot{\Phi} X^*_{\ell m}}{2(\ell - 1)\ \ell(\ell + 1)\ (\ell + 2)} \delta(r - R)\ , \quad (\ell \neq 0,1)$$

$$\tilde{t}^{\ell m}_{23} = \frac{(-\dot{\Theta}^2 + \dot{\Phi}^2 \sin^2\Theta) X^*_{\ell m}/\sin\Theta + 2\dot{\Theta}\dot{\Phi} \sin\Theta\ W^*_{\ell m}}{(\ell - 1)\ \ell(\ell + 1)(\ell + 2)} \delta(r - R)\ , \quad (\ell \neq 0,1)$$

$$\tilde{t}^{\ell m}_{33} = \frac{1}{2} (\dot{\Theta}^2 + \dot{\Phi}^2 \sin^2\Theta) Y^*_{\ell} \ \delta(r - R)$$

where: $\overset{*}{X}_{\ell m} \equiv \overset{*}{XY}_{\ell m}(\Theta, \Phi)$, $\overset{*}{W}_{\ell m} \equiv \overset{*}{WY}_{\ell m}$

5. SEPARATION OF MAXWELL EQUATIONS

As an example of the application of the present method we shall separate Maxwell equations in a spherically symmetric time dependent curved space. They are

$$\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma,\nu} \equiv 2\tilde{F}^{\mu\nu}{}_{,\nu} = 0 \tag{56}$$

$$(\sqrt{-g}\, \gamma^{\mu\rho} \gamma^{\nu\sigma} F_{\rho\sigma})_{,\nu} = 4\pi\sqrt{-g}\, \gamma^{\rho\mu} J_{\rho} \tag{57}$$

Using (9) and (21), the homogeneous equations (56) may be written as

$$H^{\mu} \equiv \partial_{\nu} \tilde{F}^{\mu\nu} = \sum_{c=0}^{2} \alpha_c P^{c\dagger}_{\nu} \tilde{F}^{\mu\nu} = 0 \tag{58}$$

$$\tilde{F}^{\mu\nu} = \sum_{d<e=0}^{3} Q^{de\mu\nu} \tilde{f}_{de} . \tag{59}$$

The vector equation (58) may be expanded as

$$H^{\mu} = \sum_{a=0}^{3} P^{a\mu} h_a = 0 \tag{60}$$

leading to four equations for the $\tilde{f}$ quantities:

$$P^{a\dagger} \cdot H = \sum_{c=0}^{2} \alpha_c P^{a\dagger}_{\mu} P^{c\dagger}_{\nu} \tilde{F}^{\mu\nu} = 0 \quad , \quad a = 0,1,2,3 . \tag{61}$$

From eqs. (10), we easily find that

$$P^{a\dagger}_{\mu} P^{c\dagger}_{\nu} - P^{a\dagger}_{\nu} P^{c\dagger}_{\mu} = \begin{cases} Q^{ca\dagger}_{\nu\mu} & \delta^{c}_{2}\, \delta^{a}_{3}\, Q^{13\dagger}_{\nu\mu} \quad \text{if } c < a \\ -Q^{ac\dagger}_{\nu\mu} & \delta^{a}_{2}\, \delta^{c}_{2}\, Q^{12\dagger}_{\nu\mu} \quad \text{if } c \neq 3 , a \leqslant c \end{cases} \tag{62}$$

(putting $Q^{cc} = 0$). Taking (62) into (61), we get

$$\sum_{c=0}^{a-1} \alpha_c (Q_{\nu\mu}^{ca\dagger} + \delta_2^c \delta_3^a \, Q_{\nu\mu}^{13\dagger}) \tilde{F}^{\underline{\mu\nu}} - \sum_{c=a}^{2} \alpha_c (Q_{\nu\mu}^{ac\dagger} - \delta_2^c \delta_2^a \, Q_{\nu\mu}^{12\dagger}) \tilde{F}^{\underline{\mu\nu}} = 0 \tag{63}$$

From (59) and (13-14), we have

$$Q_{\nu\mu}^{bd} \, \tilde{F}^{\underline{\mu\nu}} = \Lambda_{bd} \tilde{f}_{bd} \, . \tag{64}$$

Hence

$$\sum_{c=0}^{a-1} \alpha_c \Lambda_{ca} \tilde{f}_{ca} + \alpha_2 \delta_3^a \Lambda_{13} \tilde{f}_{13} - \sum_{c=a+1}^{2} \alpha_c \Lambda_{ac} \tilde{f}_{ac} + \alpha_2 \delta_2^a \Lambda_{12} \tilde{f}_{12} = 0. \tag{65}$$

Finally, using expressions (9) and (21), and considering successively $a = 0, 1, 2, 3$, we obtain

$$\partial_r (r^2 f_{23}) - r f_{13} = 0 \tag{66a}$$

$$r \partial_0 f_{23} - f_{03} = 0 \tag{66b}$$

$$\partial_0 r f_{13} - \partial_r (r f_{03}) = 0 \tag{66c}$$

$$\partial_0 r f_{12} - \partial_r (r f_{02}) + \hat{f}_{01} = 0 \tag{66d}$$

It should be mentioned that an alternative way of computing (61) is to express it as

$$\sum_{c=0}^{2} \alpha_c P_\mu^{a\dagger} \sum_{d<e=0}^{3} (P_\nu^{c\dagger} Q^{de\underline{\mu\nu}}) \tilde{f}_{de} \tag{61a}$$

where the vector operators $P_\nu^{c\dagger} \, Q^{de\underline{\mu\nu}}$ above can be immediately calculated from the definitions (10) and the use of equations (6) and Table IV.

In order to separate eq. (57) we use expression (46b) for $Y^{\underline{\mu\nu}}$ and take into account that the operators Q^{cd} comute with any function of r, t. We have:

$$\bar{F}^{\underline{\mu\nu}} = \sqrt{-g}\ \gamma^{\underline{\mu\rho}}\gamma^{\underline{\nu\sigma}}F_{\rho\sigma} = \sum_{c<d} Q^{cd\underline{\mu\nu}}\bar{f}_{cd} \tag{67a}$$

with

$$f_{0i} = \frac{\sqrt{-g}}{g_0} f_{0i} \quad , \quad \bar{f}_{1i} = \frac{\sqrt{-g}}{g_1} f_{1i} \qquad i = 2,3$$
$$f_{01} = \frac{\sqrt{-g}}{g} f_{01} \quad , \quad \bar{f}_{23} = \sqrt{-g}\ f_{23} \tag{67b}$$

Using (9), we write

$$\partial_\nu \bar{F}^{\underline{\mu\nu}} = \sum_{e=0}^{2} \sum_{c<d=0}^{3} \alpha_e P_\nu^{e\dagger} Q^{cd\underline{\mu\nu}}\bar{f}_{cd} \tag{68}$$

and then, computing $P_\nu^{e\dagger}Q^{cd\underline{\mu\nu}}$ as just indicated, we find

$$\partial_\nu \bar{F}^{\mu\nu} = -\alpha_0 (P^{1\underline{\mu}}\bar{f}_{01} + P^{2\underline{\mu}}\bar{f}_{02} + P^{3\underline{\mu}}\bar{f}_{03})$$

$$+ \alpha_1 (P^{0\underline{\mu}}\bar{f}_{01} - P^{2\underline{\mu}}\bar{f}_{12} - P^{3\underline{\mu}}\bar{f}_{13})$$

$$- \alpha_2 \left[P^{0\underline{\mu}}L^2\bar{f}_{02} + (P^{2\underline{\mu}} + P^{1\underline{\mu}}L^2)\bar{f}_{12} + P^{3\underline{\mu}}\bar{f}_{13} - P^{3\underline{\mu}}L^2\bar{f}_{23}\right]. \tag{69}$$

From $P_\mu^{a\dagger}\partial_\nu \bar{F}^{\underline{\mu\nu}} = 4\pi P_\mu^{a\dagger}\gamma^{\underline{\mu\nu}}J_\nu \sqrt{-g}$, using equations (6), (9), (49), (50) and (67b), and considering successively a = 0,1,2,3, we obtain:

$$-\frac{1}{r}\ \partial_r\left(\frac{r^2 f_{01}}{\sqrt{-g}}\right) + \frac{1}{r}\frac{\sqrt{-g}}{g_0} L^2 f_{02} = 4\pi\sqrt{-g}\ \frac{j_0}{g_0} \tag{70a}$$

$$\partial_0\left(\frac{f_{01}}{\sqrt{-g}}\right) + \frac{\sqrt{-g}}{rg_1} L^2 f_{12} = 4\pi\sqrt{-g}\ \frac{j_1}{g_1} \tag{70b}$$

$$-\partial_0\left(\frac{\sqrt{-g}\ f_{02}}{g_0}\right) - \frac{1}{r}\ \partial_r\left(\frac{r\sqrt{-g}\ f_{12}}{g_1}\right) = 4\pi\sqrt{-g}\ j_2 \tag{70c}$$

$$- \partial_0 \left(\frac{\sqrt{-g}\, f_{03}}{g_0}\right) - \frac{1}{r}\, \partial_r \left(\frac{r\sqrt{-g}\, f_{13}}{g_1}\right) - \frac{\sqrt{-g}}{r}\, L^2 f_{23} = 4\pi\sqrt{-g}\, j_3 \qquad (70d)$$

with j_c given by (49).

Finally, if we proceed to the separation of Maxwell equations (66) and (70), we observe that the rnagnetic equations (involving indices 3) are (66a-b), and (70d), eq. (66c) being a consequence of (66a) and (66b). Hence, we may express two of the quantities f_{03}, f_{13}, f_{23} in terms of the remaining one, and thus obtain a higher order equation. Indeed, taking f_{13} and f_{03} from (66a) and (66b), respectively, and substituting in (70d), we get the following second order equation for $f_{23}(\vec{x},t)$:

$$- \partial_0 \left[\sqrt{\frac{g_1}{-g_0}}\; \partial_0 (r^2 f_{23}) \right] + \partial_r \left[\sqrt{\frac{-g_0}{g_1}}\; \partial_r (r^2 f_{23}) \right] + \sqrt{-g}\, L^2 f_{23} =$$

$$= -4\pi r\, \sqrt{-g}\, j_3 = -4\pi q\, \frac{1}{L^2}\, r\vec{v}\cdot\vec{L}\; \delta(\vec{x} - \vec{z}(t)) \qquad (71)$$

On the other hand, the electric equations (not involving indices 3) are (70a-b), and (66d) - eq. (70c) being a consequence of (70a) and (70b). Here one could guess that - correspondingly to the fact that the magnetic equation (71) involves the quantity f_{23} - the electricequation should be expressed in terms of f_{01}, which is the "dual" of $L^2 f_{23}$ (eqs.(21)). Indeed, taking $L^2 f_{02}$ and $L^2 f_{12}$ from (70a) and (70b) into (66d) multiplied by L^2, we get the following second order equation for $\hat{f}_{01}(\vec{x},t)$:

$$- \partial_0 \left[\sqrt{\frac{g_1}{-g_0}}\; \partial_0 \left(\frac{r^2 \hat{f}_{01}}{\sqrt{-g}}\right) \right] + \partial_r \left[\sqrt{\frac{-g_0}{g_1}}\; \partial_r \left(\frac{r^2 \hat{f}_{01}}{\sqrt{-g}}\right) \right] + L^2 \hat{f}_{01}$$

$$= -4\pi(\partial_0 (r^2 \hat{j}_1) - \partial_r (r^2 \hat{j}_0)) = -\partial_0 \left[4\pi q \sqrt{\frac{g_1}{-g_0}}\; r^2 \dot{R} \hat{\delta}(\vec{x} - \vec{z}(t)) \right] -$$

$$- \partial_r \left[4\pi q \sqrt{\frac{-g_0}{g_1}}\; r^2 \hat{\delta}(\vec{x} - \vec{z}(t)) \right] \qquad (72)$$

and we observe that $f_{01}/\sqrt{-g}$ satisfies the same equation as f_{23}, except for the inhomogeneous terms.

Thus, we succeded in separating Maxwell equations without making any expansion in spherical harmonics. Only at the final stage of solving eq.(71) (for odd fields) or (72) (for even fields) it is convenient to make the expansion. For this, it is sufficient to make in these equations the substitutions:

$$f_{cd}(\vec{x},t) \to f_{cd}^{\ell m}(r,t)$$

$$L^2 \to -\ell(\ell+1) \tag{73}$$

$$j_c(\vec{x},t) \to j_c^{\ell m}(r,t)$$

These equations (71) and (72) coincide with equations (8) and (13) of RTV, respectively, with the **correspondence** in notation indicated in (36), and after correcting for the following misprints:

$\alpha^{\ell m} \to -\alpha\ell m$ (eq. (8) of RTV)

$b^{\ell m}{}_{,0} \to b^{\ell m}{}_{,r}$ (eq. (12a) of RTV)

It should be noticed that no condition for time independence of the metric coefficient g_0, g_1 were used in the separation of Maxwell equations. Also, $g = g_0 g_1$ was not taken equal to -1. Therefore, eq. (72) holds even in the interior region of a Reissner-Nordström pulsating metric (of course, in the external region $g = -1$, and g_0, g_1 are time independent).

On the other hand, we could have expressed all electric fields in terms of either f_{02} or f_{12}, instead of f_{01}. We choose f_{12} to be able to compare with the results of Zerilli.[8] However, equations (70a), (70b), and (66d) can now be expressed in terms of $f_{12}(\vec{x},t)$ only if g_0, g_1 are time independent. In this case, we easily find

$$\partial_0^2\left(\sqrt{\frac{-g_0}{g_1}}\, rf_{12}\right) - \sqrt{\frac{-g_0}{g_1}}\,\partial_r\left(\sqrt{\frac{-g_0}{g_1}}\,\partial_r\left(\sqrt{\frac{-g_0}{g_1}}\, rf_{12}\right)\right) +$$

$$+ g_0\frac{L^2}{r^2}\left(\sqrt{\frac{-g_0}{g_1}}\, rf_{12}\right) = 4\pi\left(\sqrt{-g}\,\hat{j}_1\,\frac{g_0}{g_1} - \sqrt{\frac{-g_0}{g_1}}\,\partial_r(g_0 j_2 r)\right) =$$

$$= 4\pi q\left[\left(g_0\dot{R}\hat{\delta}(\vec{x}-\vec{z}) + \frac{\vec{v}\cdot\vec{N}^\dagger}{L^2}\sqrt{\frac{-g_0}{g_1}}\,\partial_r\left(\sqrt{\frac{-g_0}{g_1}}\, r\hat{\delta}(\vec{x}-\vec{z})\right)\right)\right] \tag{74}$$

where, as said before (eq. (49a)), $\hat{\delta}(\vec{x}-\vec{z})$ represents the δ-function without the $\ell = 0$ part; and similarly for $\hat{j}_1$. This is obviously necessary since f_{12} (or any f_{cd} with c or d equal to 2 or 3) has no R = 0 part. This was also the reason for the appearance of $\hat{f}_{01}$, instead of f_{01} in (66d).

For the case of the external Reissner-Nordström metric equations (71) and (74), after expansion in spherical harmonics, coincide with eqs.(21) and (35) of Zerilli[8] (without the terms $R(m)$, A_{12}, A_{23} and K which represent perturbations of the gravitational energy and gravitational potential) with the correspondence in notation:

$$g_0 \to -e^{2\nu}\ ,\quad g = g_0 g_1 = -1$$

$$q\dot{R}\delta\ \vec{x}-\vec{z}\ \ \sqrt{\frac{-g_0}{g_1}}\ j_1 \to e^{\nu}u \tag{75}$$

$$-\frac{q}{L^2}\,\vec{v}\cdot\vec{N}^\dagger\hat{\delta}(\vec{x}-\vec{z}) = j_2\sqrt{-g} \to \frac{w}{r}$$

6. CONCLUDING REMARKS

The operators P^c, Q^{cd}, $\tilde{Q}^{cd}$ and P^{cd}, introduced in section 2, have simple cartesian algebraic properties, not involving functions, but only constant coefficients (Tables I, III and IV).

These operators were used in sections 2-4 to expand both covariant and contravariant vector, symmetric and antisymmetric tensor functions of $\vec{x}$ and t (cartesian coordinates) in terms of "scalar" (rotationaly invariant) functions of $\vec{x}$ and t.

In section 5, Maxwell equations in a time dependent spherically symmetric metric, expressed as contracted products of operators applied to "scalar" functions, were simplified by the use of the algebra or by contraction with further operators leading to "scalar" differential equations (eqs. (66) and (70)). These equations were then separated for solutions of both parities (i.e. electric and magnetic), leading to equations (71),(72) and (74).

The main advantages of the method and the essential results presented here may be summarized as follows.

i) We succeded in separating Maxwell equation in a time dependent spherically symmetric background without any previous expansion, both in spherical harmonics and in a Fourier time integral as is usually done.

ii) The equations for the fields f_{01} (electric) and f_{23} (magnetic) are more general than the previous ones of RTV[4] (for f_{01} and f_{23}) and Zerilli[8] (for f_{12} and f_{23}) since they are now neither restricted to a static metric nor to the condition $g = g_0 g_1 = -1$ (external Reissner-Nordström metric). The equation for f_{12}, corresponding to Zerilli's choice, was obtained only for static metrics but still without the further restriction of $g = -1$, thus being applicable also to internal metric solutions, which is not the case of Zerilli's equations.

iii) The expansion in spherical harmonics and time Fourier analysis can be done at any stage of the present method, using the results of section 3, where the correspondence between previous notations and ours were introduced. However, it is preferable to expand only after the final separation is completed, when it is then sufficient for the homogeneous equations to make the substitutions

$$f^{cd}(r,t) \to f^{cd}_{\ell m}(r,\omega)$$

$$\partial_0 f^{cd}(r,t) \to -i\omega f^{cd}_{\ell m}(r,\omega)$$

$$L^2 f^{cd}(r,t) \to -\ \ell(\ell+1) f^{cd}_{\ell m}(r,\omega) \ .$$

For the source terms one must explicitly perform the Fourier transform of the spherical harmonic expansion.

iv) The present method can be clearly extended to equations of higher tensor character, for instance, to the Einstein-Maxwell equations. Obviously, in that case, the symmetric operators P^{cd} and the algebraic properties of Tables I-IV are much more extensively used.

v) The simplicity of the algebra of operators allows, for more complicated metrics, the use of computational rnethods for the whole procedure of separation of the equations."

vi) The notation introduced for the quantities a_c, f_{cd}, a_{cd}, t_{cd}, etc., is simpler than the ones previously employed[2,3,8,14], as can be immediately recognized. The meaning of each component is obvious as is its parity (even or odd, according to the number of indices 3 it contains). The present notation also avoids possible confusions since it never involves the use of the same symbol for quantities of different parities.

vii) Finally, as all a_c, f_{cd}, and a_{cd} have the same dimension: r^{-1}, r^{-2}, r^0, respectively, it is easier to track down mistakes by dimensional inspection than when using spherical coordinates.

The present method has been applied to the separation of the Einstein-Maxwell equations in a spherically symmetric curved background[15], and these equations have been employed to compute coupled gravitational and electric radiation emitted in the presence of a Reissner-Nordström black-hole.[16]

REFERENCES

1. J.A. Wheeler, *Geometrodynamics*, Academic Press, New York, 1962.
2. T. Regge and J.A. Wheeler, Phys. Rev. *108*, 1063 (1957).
3. F.J. Zerilli, Phys. Rev. *D2*, 2141 (1970).
4. R. Ruffini, J. Tiomno, and C.V. Vishveshwara, Lett. Nuovo Cimento 3, 211 (1972).
5. R. Wald, private communication (1972).
6. J. Tiomno, Lett. Nuovo Cimento *5*, 851 (1972).
7. R. Ruffini, On the Energetics of Black Holes, in C. De Witt and B. S. De Witt (eds.), *Black Hole*, Les *Astres Occlus*, Gordon and Breach, New York, 1973, and references quoted therein.
8. F.J. Zerilli, Phys. Rev. *D9*, 860 (1974).
9. D.M. Chitre, R.H. Price, and V.D. Sandberg, Phys. Rev. Lett. *31*, 1018 (1973).
10. V. Moncrief, Phys. Rev. *D10*, 1057 (1974), and Phys. Rev. *D12*, 1526 (1975).
11. L.A. Disher de Sá (to be published).
12. J. Mathews, J. Soc. Ind. Appl. Math. *10*, 768 (1962).
13. J. Stachel, Nature 220, 779 (1968).
14. F. Zerilli, J. Math. Phys. *11*, 2203 (1970).
15. J. Tiomno and A.L.L. Videira (to be published).
16. L.A. Disher de Sá, J. Tiomno, and A.L.L. Videira (to be published).