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# A Note on the Josephson Tunneling*

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With a simple generalization of Bogoliubov's quasi-particle approach, the results for the Josephson current in a superconductor are recovered, without resorting to the usual pseudo-spin treatment.

Através de uma simples generalização do tratamento com quasi-partículas de Bogoliubov, obtêm-se os resultados para a corrente de Josephson em um supercondutor, isso sem se fazer uso do tratamento usual de pseudo-spin. Mostra-se também a equivalência entre as duas descrições.

## INTRODUCTION

The Josephson effect[1] occurs when tunneling currents flow between two superconductors separated by a thin isolating layer. This effect has been extensively analized and different alternative derivations of Josephson's results are available in the current literature The most common approach[4] is that in terms of the pseudo-spin formalism of Anderson[5]. A more elaborate calculation, along the same lines, is due to Lee and Scully[6] who included the interaction of the junction with the radiation field contained in the resonance cavity and the effect of the externa1 (normal) part of the circuit.

It has been remarked' that Bogoliubov's quasi-particle picture of superconductivity[7], in its original form, does not allow a description of the

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Josephson effect because the quasi-particle states do not have a definite phase factor. The difference in phase of the superconducting wavefunctions, on both sides of the junction, is basic to the explanation of the D.C. and A.C. Josephson effects. In fact the ill-definition of the number of electrons in the superconducting phase gives rise to the existence of a well defined phase for the quasi-particle states, which is also coherent[8]. This results in non-vanishing off-diagonal elements of the density matrix[9]. This is just a manifestation of the condensation in Cooper pairs. A simil'ar situation can be found in the case of condensation of the electrongas in states with magnetic ordering or with the formation of charge and/or spin density waves. These cases can be precisely described in terms of Bogoliubov's quasi-particles with appropriate coherent phases[10].

These considerations lead us to propose a simple and straightforward generalization of the Bogoliubov-Valatin transformation which allow us to recover the results for the Josephson current, within the usual Bogoliubov-BCS scheme, without resorting to the pseudo-spin algebra. We show the equivalence between the two representations.

## 1. OBTAINING THE JOSEPHSON CURRENT BY GENERALIZING THE BOGOLIUBOV TRANSFORMATIONS

The formalism for dealing with tunneling effects is well known. Bardeen[11]; Cohen, Falicov and Phillips[12], and Prange[13], have discussed the problem from a many-particle point of view. As previously stated, we are interested here in tunneling between superconducting junctions.

We use the effective Hamiltonian[14]

$$H_{BCS} = \sum_{\underset{\sim}{K}} E(\underset{\sim}{K})\,(N_{\underset{\sim}{K}\uparrow} + N_{-\underset{\sim}{K}\downarrow}) + \sum_{\underset{\sim}{K}\underset{\sim}{K}'} V(\underset{\sim}{K},\underset{\sim}{K}')\; C^{+}_{\underset{\sim}{K}'\uparrow} C^{+}_{-\underset{\sim}{K}'\downarrow}\, C_{-\underset{\sim}{K}\downarrow}\, C_{\underset{\sim}{K}\uparrow} \;, \qquad (1.1)$$

where $E(\underset{\sim}{K})$ is the kinetic (band) energy of electrons in Bloch States $|\underset{\sim}{K}\sigma>$, and the $V(\underset{\sim}{K},\underset{\sim}{K}')$ are the matrix elements of the atractive electron-electron interaction responsible for the ocurrence of superconductivity; the usual second quantization notation has been used.

Now consider two independent superconductors which are brought together to form the superconductor junction, and denote by $H_1$ and $H_2$ their respective Hamiltonians. The coupling energy,

$$H_{12} = - T \sum_{\mathbf{K},\mathbf{q}} (b^+_{\mathbf{K}1}\, b_{\mathbf{q}2} + b^+_{\mathbf{q}2}\, b_{\mathbf{K}1}) \ , \tag{1.2}$$

transfers Cooper pairs from one side to the other of the barrier. In (1.2), $T$ is the transfer integral, which we assume to be K-independent, and we have introduced the operators for the creation and annihilation of a Cooper pair, namely,

$$b^+_{\mathbf{K}\xi} = c^+_{\mathbf{K}\uparrow\xi}\, c^+_{-\mathbf{K}\downarrow\xi} \ , \quad b_{\mathbf{K}\xi} = c_{-\mathbf{K}\downarrow\xi}\, c_{\mathbf{K}\uparrow\xi} \ ; \quad \xi = 1,2 \ ;$$

each value of 5 refers to the superconductor on each side of the barrier. It is straightforward to calculate the tunneling current, since

$$J = - e \frac{dN_1}{dt} \tag{1.3}$$

is the operator representating the current flowing from side 1 to side 2 across the insulating barrier, and $N_1$ is the total number of electron on side 1.

Using the total Hamiltonian

$$H = H_1 + H_2 + H_{12} \ , \tag{1.4}$$

we easily find

$$\dot{N}_1 = \frac{1}{i\hbar} [N_1, H] = \frac{1}{i\hbar} \sum_{\mathbf{K}} [n_{\mathbf{K}\uparrow 1} + n_{-\mathbf{K}\downarrow 1}, H]$$
$$= \frac{1}{i\hbar} T \sum_{\mathbf{K}\mathbf{q}} (b^+_{\mathbf{K}1}\, b_{\mathbf{q}2} - b^+_{\mathbf{q}2}\, b_{\mathbf{K}1}) \ . \tag{1.5}$$

At this point we introduce the usual Bogoliubov operators of quasi-particles through the matrix equation

$$\begin{bmatrix} \alpha_{\underset{\sim}{K}} \\ \beta^{+}_{-\underset{\sim}{K}} \end{bmatrix} = \begin{bmatrix} u_{\underset{\sim}{K}} & v_{\underset{\sim}{K}} \\ -v^{*}_{\underset{\sim}{K}} & u^{*}_{\underset{\sim}{K}} \end{bmatrix} \cdot \begin{bmatrix} c_{\underset{\sim}{K}} \\ c^{+}_{-\underset{\sim}{K}} \end{bmatrix} , \tag{1.6}$$

with the condition

$$|u_{\underset{\sim}{K}}|^2 + |v_{\underset{\sim}{K}}|^2 = 1 , \tag{1.7}$$

where

$$u_{\underset{\sim}{K}} = \cos\theta_{\underset{\sim}{K}} , \; v_{\underset{\sim}{K}} = \sin\theta_{\underset{\sim}{K}} ,$$

and we verify that the Josephson effect does not occur since these operators do not add a definite phdse factor to the quasi-particle states.

We avoid this difficulty by identifying the Bogoliubov 2x2 transformation matrix with the 2x2 complex matrix associated with a rotation in the R(3) space, i.e., $u$ and $v$ ate taken as Cayley-Klein parameters[17]:

$$\begin{aligned} u_{\underset{\sim}{K}} &= \exp(\tfrac{1}{2}\, i\psi_{\underset{\sim}{K}}) \, . \, \cos(\tfrac{1}{2}\, \theta_{\underset{\sim}{K}}) \, . \, \exp(\tfrac{1}{2}\, i\gamma_{\underset{\sim}{K}}) \, , \\ v_{\underset{\sim}{K}} &= \exp(-\tfrac{1}{2}\, i\psi_{\underset{\sim}{K}}) \, . \, \sin(\tfrac{1}{2}\, \theta_{\underset{\sim}{K}}) \, . \, \exp(-\tfrac{1}{2}\, i\gamma_{\underset{\sim}{K}}) \, . \end{aligned} \tag{1.8}$$

We find that this new transformation matrix satisfies all the orthonormalization conditions of the preceding one. When we put $\psi = \gamma = 0$ for every $\underset{\sim}{K}$, we recover the transformation widely used to study the superconducting state.

To deal with the Josephson tunneling, we propose here a simple generalization of the quasi-particle operators $\alpha$ and $\beta$, choosing $\gamma_{\underset{\sim}{K}} = 0$ and $\psi_{\underset{\sim}{K}} = +$, for every $\underset{\sim}{K}$, in Eqs. (1.8). Just as in the quasi-spin formalism, a degeneracy appears in the ground state since we obtain linearly independent degenerate ground states for different values of $. The transformation is generated by the matrix

$$\begin{bmatrix} \cos\frac{\theta}{2}\cdot\exp(\frac{1}{2}i\psi) & \sin\frac{\theta}{2}\cdot\exp(-\frac{1}{2}i\psi) \\ -\sin\frac{\theta}{2}\cdot\exp(\frac{1}{2}i\psi) & \cos\frac{\theta}{2}\cdot\exp(-\frac{1}{2}i\psi) \end{bmatrix}. \quad (1.9)$$

With (1.9), we form the new operators $\alpha_{\underset{\sim}{K}}$ and $\beta_{\underset{\sim}{K}}$ in order to express the tunneling current (1.5). We take its average in the BCS ground state, using the properties

$$\beta_{-\underset{\sim}{K}}|\psi_{BCS}\rangle = 0 \ , \ \alpha_{\underset{\sim}{K}}|\psi_{BCS}\rangle = 0,$$

and obtain

$$\langle J\rangle = \frac{2eT}{\hbar}\sum_{\underset{\sim}{K}\underset{\sim}{q}} \sin\theta_{\underset{\sim}{K}1} \sin\theta_{\underset{\sim}{q}2} \sin(\psi_1 - \psi_2) \ . \quad (1.10)$$

This is the well known form of the DC - Josephson effect, and along similar lines one can also derive the AC - Josephson effect. We have thus shown that these effects and related questions can be studied within the scheme of Bogoliubov's quasi-particle description of the superconducting state.

## 2. EQUIVALENCE OF THE PSEUDO-SPIN AND QUASI-PARTICLE FORMALISM

Consider the two quantum mechanical states $|K\uparrow\rangle$ and $|-K\downarrow\rangle$, and the operators

$$c_{\underset{\sim}{K}\uparrow} \ ; \ c_{-\underset{\sim}{K}\downarrow} \ ; \ c^+_{\underset{\sim}{K}\uparrow} \ ; \ c^+_{-\underset{\sim}{K}\downarrow} \quad (2,1)$$

of annihilation and creation of an electron in those states. We use them to define the pseudo-spin operators:

$$S^+_{\underset{\sim}{K}} = c^+_{\underset{\sim}{K}\uparrow}\, c^+_{-\underset{\sim}{K}\downarrow} \ ,$$

$$S^-_{\underset{\sim}{K}} = c_{-\underset{\sim}{K}\downarrow}\, c_{\underset{\sim}{K}\uparrow} \ ,$$

$$S_{0K} = \frac{1}{2}\left(c^+_{K\uparrow}\, c_{K\uparrow} - c^+_{-K\downarrow}\, c_{-K\downarrow}\right). \quad (2.2)$$

With these operators, Anderson[9] has shown that the BCS Hamiltonian is equivalent to that of a system of spins in the presence of a magnetic field plus an anisotropic $(X,Y)$ exchange term. To solve this magnetic version of the problem, he uses the molecular field approximation, and then rotates each pseudo-spin K by an angle $\theta$ around the Y-axis (so that the new $S_{0\underset{\sim}{K}}$ remains in the XZ-plane and makes an angle $\theta_K$ with the old $S_{0K}$). The three components of the rotated pseudo-spin $\tilde{S}_{\underset{\sim}{K}}$ can be expressed, in terms of the Bogoliubov operators $\alpha_{\underset{\sim}{K}}$, $\alpha^+_{\underset{\sim}{K}}$, $\beta_{-\underset{\sim}{K}}$ and $\beta^+_{-\underset{\sim}{K}}$, in the same way as the initial components of the pseudo-spin $S_K$ are expressed in terms of the operators (2.2). In particular, the transformed BCS Hamiltonian is a sum of terms proportional to the $\tilde{S}_{0\underset{\sim}{K}}$.

Here, we shall prove that the same relationship holds for the more general Bogoliubov operators defined by (1.6) and (1.9). Instead of the simple rotation discussed above, (1.9) generates a rotation obtained by first rotating $\psi$ around Z, and then $\theta$ around the new Y-axis. The components of the rotated pseudo-spin are then

$$
\begin{aligned}
\tilde{S}_{0\underset{\sim}{K}} &= \cos\theta_{\underset{\sim}{K}} \cdot S_{0\underset{\sim}{K}} + \sin\theta_{\underset{\sim}{K}} \cos\psi_{\underset{\sim}{K}} \cdot S_x + \sin\theta_{\underset{\sim}{K}} \sin\psi_{\underset{\sim}{K}} \cdot S_y \ , \\
\tilde{S}_{x\underset{\sim}{K}} &= -\sin\theta_{\underset{\sim}{K}} \cdot S_{0\underset{\sim}{K}} + \cos\theta_{\underset{\sim}{K}} \cos\psi_{\underset{\sim}{K}} \cdot S_x + \cos\theta_{\underset{\sim}{K}} \sin\psi_{\underset{\sim}{K}} \cdot S_y \ , \qquad (2.3) \\
\tilde{S}_{y\underset{\sim}{K}} &= -\sin\psi_{\underset{\sim}{K}} \cdot S_x + \cos\psi_{\underset{\sim}{K}} \cdot S_y \ ,
\end{aligned}
$$

where we have used $\tilde{S}_x = \frac{1}{2}(\tilde{S}^+ + \tilde{S}^-)$ and $\tilde{S}_y = \frac{i}{2}(\tilde{S}^- - \tilde{S}^+)$ to better show the rotation.

Reverting to $\tilde{S}^+$ and $\tilde{S}^-$, we easily prove using (1.6) and (1.9) that

$$
\begin{aligned}
\tilde{S}^+_{\underset{\sim}{K}} &= \alpha^+_{\underset{\sim}{K}} \beta^+_{-\underset{\sim}{K}} \\
\tilde{S}^-_{\underset{\sim}{K}} &= \beta_{-\underset{\sim}{K}} \alpha_{\underset{\sim}{K}} \\
\tilde{S}_{0\underset{\sim}{K}} &= \frac{1}{2}(\alpha^+_{\underset{\sim}{K}} \alpha_{\underset{\sim}{K}} - \beta^+_{-\underset{\sim}{K}} \beta_{-\underset{\sim}{K}}) \ . \qquad (2.4)
\end{aligned}
$$

The pseudo spin operators (2.2) act on a two dimensional vector space generated by two states: in one of them both $|\underset{\sim}{K}\uparrow\rangle$ and $|-\underset{\sim}{K}\downarrow\rangle$ are occupied, while in the other both are empty. It is always possible to find a unitary transformation $P_{\underset{\sim}{K}}$ in this two dimensional space for which[18]

$$\tilde{S}_{0\underset{\sim}{K}} = P_{\underset{\sim}{K}} S_{0\underset{\sim}{K}} P^{+}_{\underset{\sim}{K}} ,$$

$$\tilde{S}^{+}_{\underset{\sim}{K}} = P_{\underset{\sim}{K}} S^{+}_{\underset{\sim}{K}} P_{\underset{\sim}{K}} ,$$

$$S^{-}_{\underset{\sim}{K}} = P_{\underset{\sim}{K}} S^{-}_{\underset{\sim}{K}} P^{+}_{\underset{\sim}{K}} \qquad (2.5)$$

This argument completes the proof of the equivalence between the two descriptions, one with pseudo-spiin and the other with the Bogoliubov operators. We remark that both (2.2) and (2.4) generate the Lie algebra of the group $SU(2)$; this group is homomorphic with the rotation group $O(3)$. This homomorphism provides the connection between the rotation of the pseudo-spin and the unitary transformation in the space of state vectors.

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