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Ponderomotive Forces in Dielectrics*

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This article disputes the validity of the usually accepted expression for the volume force in fluid dielectrics, as given in most textbooks. It proposes that this volume force ought to be derived from the expression of the force in a point dipole. It also shows that the alternative force expression in terms of polarization charge densities leads to correct results concerning the total force.

Analisa-se criticamente a dedução da expressão da força de volume em um dielétrico sob a ação de um campo elétrico, apresentada pela maioria dos livros em Eletromagnetismo. Mostra-se que essa dedução parte de hipóteses inadequadas, o que invalida a expressão obtida. Propõe-se então que se adote para a força de volume aquela obtida da força sobre um dipolo pontual. Mostra-se também que as cargas "fictícias" de polarização podem ser usadas para o cálculo da força total sobre o dielétrico.

INTRODUCTION

Ponderomotive Forces in Dielectrics

When a dielectric is polarized, electrical volume forces-called ponde-

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romotive - develop inside it. These volume forces cause small displacemnts and deformations in the dielectric body - known as electrostriction - and sometimes the motion of its entire mass.

The theory of Elasticity allows us to infer those small deformations once the pattern of volume as well as surface forces are given. And within the approximation it usually employes, based on the smallness of the displacements, the changes in the force pattern caused by them may be ignored. This applies to solids and to almost incompressible liquids. In gases, the displacements may becom more important and the changes in the force pattern caused by them may be taken into account. It is under this frame that the subject of ponderomotive forces and electrostriction is presented in the textbooks and will be discussed here. However, the results obtained in this frame should reduce to those of the first one (va'lid for isotropic and incompressible liquids) in an obvious way, since the latter is a particular case of the first. We intend to show (see Section 1) that this is not the case for the expression of the volume force given in most textbooks.

Assume now a dielectric fluid under the action of a slowly increasing electric field. The derisity at each fixed point will change according to the changing electric: field. The same will be true for every small mass of the dielectric if we follow them through the small displacements under the action of the increasing field, owing to the increasing pressure. Since density changes induce dielectric constant changes, we conclude that the dielectric constant must be considered as a function of the electric. field. This dependence is not a kind of high field effect, that is, one that can be handled adding field correction terms to the normal dielectric constant. It is of a more involved nature but neverthless must be in principle considered. Therefore, the dielectric cannot be taken as linear, in the sense electrostatics gives to this term. We recall that a linear dielectric may have its dielectric constant as a function of the position but not of the electric field.

The considerations we have gone through are important for the forthcoming discussion and criticism of the expression usually accepted for the ponderomotive force.

Any small portion of a polarized dielectric is supposed to constitute a small dipole. Therefore, we would expect the expression for the volume force to be a straightforward generalization of the force $\vec{F}$ on a point dipole of the moment $\vec{p}$, in an electric field $\vec{E}$ ($\vec{F} = (\vec{p}.\nabla)\vec{E}$). This gives directly $\vec{f} = (\vec{P}.\vec{\nabla})\vec{E}$, where $\vec{P}$ is the polarization vector, that is, the dipole moment per unit volume (this expression will be referred to as E1). Instead, most textbooks present a different expression, first derived by Helmholtz (E2). Only a few of them[1] discuss the reasons for rejecting E1. When this is done, two expressions, E1 and another one, E3 - obtained by the product of the electric field $\vec{E}$ and the density of polarization charge ρ_P that is, $\vec{f} = \rho_P\vec{E} = \vec{E}(-\vec{\nabla}.\vec{P})$ - are presented as equally plausible. The antinomy thus created is resolved by making the expression of the volume force conform to the energy principle (work = energy change). According to the calculation then carried out, expression E2 obtains. It should be mentioned that E1 is sometimes accepted as correct[2] when referring to rigid polarization.

We now proceed to examine these arguments. Concerning the alleged antinomy, our contention is that only E1 is justified, since the dielectric does have dipoles. Hence, E1 should be chosen and E3 rejected. Besides, we believe that the expression for the volume force should equally apply to induced and rigid polarization as well. It will be seen that E1 does satisfy this criterium while E2 does not.

In this paper, we select E1 as the suitable expression for the electrical volume force. The only reason for this choice is that it satisfies the methodological procedure Physics often uses in going from the discrete to the continuum. However, this is not enough, since E2 is supposed to result also from a very general principle - the energy principle - to which any selected expression should conform. Therefore, Section 1 presents the Helmholtz approach, and the mistake it incorporates is localized. Assuming the correctness of E1, Section 2 derives the surface forces arising in the dielectric boundary due to the discontinuity of the polarization field. Section 3 shows that E3 may likewise be used when the total force on the dielectric is required. This result ensures the validity of Newton's Third Law for the volume and surface forces just proposed. Section 4 makes some comments on the electrostriction problem, and the article comes to an end with the final remarks of Section 5.

When the volume force is deduced from the energy principle, the expression of the energy must be assumed from the start. Such an expression is available for linear dielectrics and, among these, dielectrics under electrostriction are not included, as discussed before. Therefore, only for assumed linear dielectrics (ie., when electrostriction is neglected in calculating the force pattern) have we made the connection between EI and the energy principle. For the sake of simplicity, an infinite dielectric was considered in this case.

1. THE HELMHOLTZ DERIVATION

Helmholtz derived the ponderomotive force $\vec{f}$ from the energy variation δU incurred in an arbitrary small displacement $s(\vec{x})$ (a function of the position $\vec{x}$) of the dielectric:

$$\delta U = - \int \vec{s} \cdot \vec{f} \, dv . \qquad (1.1)$$

Assuming for the energy U the expression

$$U = \frac{1}{2} \int \frac{\vec{D}^2}{\varepsilon} dv , \qquad (1.2)$$

(where $\vec{D}$ is the displacement field and ε the permittivity), its variation is calculated in terms of the displacement $\vec{s}(\vec{x})$ and the result is cast in the form of Eq. (1.1), from which $\vec{f}$ is readily identified owing to the arbitrariness of $\vec{s}(\vec{x})$. Now following Ref.2, p. 190, one has from Eq. (1.2) above

$$\delta U = \frac{1}{2} \{ 2 \int \vec{E} . \delta\vec{D} \, dv - \int \vec{E}^2 \, \delta\varepsilon \, dv \} .$$

If the dielectric does not possess free charges, the first integral in the right of the above expression may be shown to vanish. Hence, for discharged dielectrics, as henceforth assumed in this article, we have

$$\delta U = - \frac{1}{2} \int \vec{E}^2 \delta\varepsilon \, dv . \qquad (1.3)$$

The variation $\delta\varepsilon$ is supposed to contain two terms. The first one, $\delta\varepsilon'$, comes from the displacement $\vec{s}(\vec{x})$ considered as locally rigid, and is given by

$$\delta\varepsilon' = -\vec{s}\,.\,\vec{\nabla}\varepsilon \quad . \tag{1.4}$$

The second component, $\delta\varepsilon''$, is related to volume changes induced by the displacement field $\vec{s}$, and for a fluid dielectric of density μ is given by

$$\delta\varepsilon'' = -\mu\,\frac{d\varepsilon}{d\mu}\,\vec{\nabla}.\vec{s} \quad . \tag{1.5}$$

Now, with $\delta\varepsilon = \delta\varepsilon' + \delta\varepsilon''$ used in Eq.(1.3), the following volume force is obtained:

$$\vec{f} = -\frac{\vec{E}^2}{2}\,\vec{\nabla}\varepsilon + \vec{\nabla}\left(\mu\,\frac{d\varepsilon}{d\mu}\,\frac{\vec{E}^2}{2}\right) \quad . \tag{1.6}$$

For further reference, we give the energy variation when $\vec{\nabla}\,.\,\vec{s} = 0$ (Eqs. 1.3 and 1.4):

$$\delta U = \frac{1}{2}\int \vec{E}^2\vec{s}.\vec{\nabla}\varepsilon\; dv \quad . \tag{1.7}$$

This completes the Helmholtz derivation.

Our criticism is directed to the use of Eq.(1.2) as the suitable expression for the energy. As already stressed, linearity cannot be assumed if electrostriction is included. Moreover, the result given in Eq. (1.6) is not such as to give in the limiting case of incompressible fluids the same result as if electrostriction were ignored. That is, what value of $d\varepsilon/d\mu$ should be assigned to an incompressible fluid?

2. VOLUME FORCES AND TENSION

In this Section, we assume the correctness of the expression E1 (that is, $\vec{f} = (\vec{P}.\vec{\nabla})\vec{E}$), and calculate what system of surface forces it leads to for dielectrics of induced polarization. Such surface forces do appear owing to the discontinuity of the polarization field at the dielectric boundary. Before doing this, words are needed to clarify the meaning of χ in the relation

$$\vec{P} = \chi\vec{E} \quad .$$

χ will be assumed to be a scalar and therefore only isotropic fluid dielectrics are embraced when electrostriction is considered. When this is neglected, the above relation applies also to isotropic solid dielectrics.

The above equation only says that $\vec{P}$ and $\vec{E}$ are parallel. χ ultimately depends on the electric field but what matters in the calculation below is the explicit dependence of χ on the position.

We write for the volume force the expression

$$\vec{f} = (\vec{P}.\vec{\nabla})\vec{E} = \chi(\vec{E}.\vec{\nabla})\vec{E} \ . \qquad (2.1)$$

Since $\vec{\nabla} \times \vec{E} = 0$, this equation turns into

$$\vec{f} = \frac{\chi}{2} \vec{\nabla} \vec{E}^2 \quad . \qquad (2.2)$$

In order to study surface forces, the susceptibility will be assumed to go rapidily to zero in a very narrow region near the dielectric surface. The direction of variation is that of its normal.

We now integrate Eq.(2.2) in this narrow region describing the dielectric boundary (see Fig.1), in which the susceptibility goes from its bulk value to zero. First, let us write Eq.(2.2) as

$$\vec{f} = \frac{1}{2} \left[\vec{\nabla}(\chi\vec{E}^2) - \vec{E}^2 \ \vec{\nabla}\chi\right] \ , \qquad (2.3)$$

and integrate it over a small cylinder of volume V' whose bases are in the regions where $\chi = 0$ and χ equal to its bulk value. We may write

$$\int_{V'} \vec{f} \, dv = \frac{1}{2} \left[\oint \chi \, E^2 \, \vec{n}' \, ds - \int_{V'} E^2\vec{\nabla}\chi \, dv\right] \quad . \qquad (2.4)$$

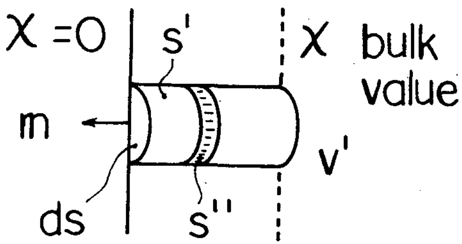

Fig.1

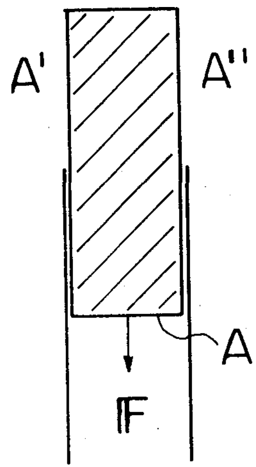

Fig.2

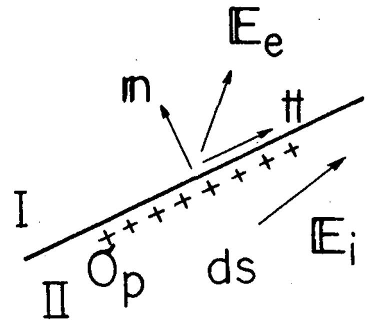

Fig.3

Fig.1 - The dielectric near its surface.

Fig.2 - The pulling in of a dielectric slab by a charged plane condenser.

Fig.3 - The dielectric surface with its surface polarization charge.

The contribution to the surface integral from any section of the lateral surface like S'' vanishes by symmetry. Likewise, the one due tothe base where $\chi = 0$. Therefore,

$$\oint \chi \vec{E}^2 \vec{n}' ds = -\vec{n} \chi \vec{E}^2 ds , \tag{2.5}$$

$\vec{n}$ being the outward normal to the dielectric. The volume integral in Eq.(2.4) will be evaluated in the following way. First, we write

$$\vec{E}^2 = \vec{E}_t^2 + \vec{E}_n^2 ,$$

where $\vec{E}_t$ and $\vec{E}_n$ are respectively the tangential and normal components of the electric field. Therefore,

$$\int_{V'} \vec{E}^2 \vec{\nabla} \chi \, dv = \int_{V'} \vec{E}_t^2 \vec{\nabla} \chi \, dv + \int_{V'} \vec{E}_n^2 \vec{\nabla} \chi \, dv .$$

Now, we know that the tangential conponent of the electric field, $\vec{E}_t$, and the normal component of the displacement vector, $\vec{D}_n$, remain constant as one crosses the boundary. Therefore, we write

$$\int_{V'} \vec{E}_t^2 \nabla \chi \, dv = \vec{E}_t^2 \int_{V'} \vec{\nabla} \chi \, dv = -\vec{n} \, \vec{E}_t^2 \chi \, ds ,$$

and (2.6)

$$\int_{V'} \vec{E}_n^2 \vec{\nabla} \chi \, dv = \vec{D}_n^2 \int_{V'} \frac{1}{\varepsilon^2} \vec{\nabla} \chi \, dv = \frac{-\vec{n} \, \vec{D}_n^2 \chi \, ds}{\varepsilon \, \varepsilon_0}$$

Since $\vec{D}_n = \varepsilon \vec{E}_{in}$, the last integral may be written as

$$\int_{V'} \vec{E}_n^2 \vec{\nabla} \chi \, dv = -\vec{n} \frac{\varepsilon \chi}{\varepsilon_0} \vec{E}_{in}^2 \, ds . \tag{2.7}$$

Of course, $\vec{E}_{in}$ is the normal component of the electric field inside the dielectric.

Substituting Eqs.(2.5), (2.6) and (2.7) into Eq.(2.4), we finally get

$$\int_{V'} \vec{f}\, dv = \frac{\vec{n}\, \chi^2\, \vec{E}^2_{in}}{2\varepsilon_0}\, ds \quad . \qquad (2.8)$$

This equation shows that the dielectric surface is subject to a tension, tangential forces being absent.

To summarize the discussion: the induced dielectric is subject to volume forces given by Eq.(2.1) and to a tension given by Eq.(2.8).

2.1. Examples

We remark that when the total force is required, and the susceptibility χ is taken as a constant (electrostriction in the homogeneous and isotropic dielectric being neglected), the integral of the body force may also be transformed into a surface integral:

$$\int_V \vec{f}\, dv = \frac{\chi}{2} \int \vec{\nabla}\, \vec{E}^2\, dv = \frac{\chi}{2} \oint_S \vec{E}^2 . \vec{n}\, ds \ , \qquad (2.9)$$

the surface S being taken inside but near the dielectric boundary. This expression gives directly the total force pulling in the dielectric to the interior of a charged plane capacitor (Fig.2). Here, the force coming from the pressure cancels out on the surfaces A' and A'' and is zero at the surface A. The remaining force source is the surface integral given by Eq. (2.9), whose net contribution comes from the surface A and gives the same result as usually deduced by the energy method.

Another test to the proposed volume force and pressure scheme is obtained by considering a point charge in front of a semi-infinite uniform dielectric with a plane boundary. Here, the force on the point charge may be calculated in advance and, owing to the Newton's Third Law, the total force on the dielectric is known. The calculation shows that the proposed scheme does indeed lead to the correct value. In the next Section, it will be shown that this scheme always ensures that Newton's Third Law will be satisfied.

2.2. Connection With the Energy Principle

Let us take the case of an infinite dielectric and show that Eq.(1.7) may be derived. The surface pressure need not be considered, and we give a small displacement $\vec{s}(\vec{x})$ satisfying $\vec{\nabla}.\vec{s} = 0$, keeping the free charges fixed. The energy variation δU is

$$\delta U = - \int (\vec{P}.\vec{\nabla})\vec{E}.\vec{s}\ dv\ .$$

Following Ref.2, p.207, this integral may be transformed into

$$\delta U = \int \vec{s}\ .\ (\vec{E}^2\ \vec{\nabla}\chi\ + \frac{\chi}{2}\ \vec{\nabla}\ \vec{E}^2)\,dv$$

or (2.10)

$$\delta U = \int \vec{s}\ \left[\frac{\vec{E}^2\ \vec{\nabla}\ \chi}{2} + \frac{\vec{\nabla}(\chi\ \vec{E}^2)}{2}\right] dv\ .$$

Now, since $\vec{\nabla}.\vec{s} = 0$, we have

$$\vec{s}.\vec{\nabla}(\chi\ \vec{E}^2)\ = \vec{\nabla}.(\chi\ \vec{E}^2\ \vec{s})\ .$$

Substituting this into Eq.(2.10), using the divergence theorem and throwing out the surface integral, the result in Eq.(1.7) is obtained.

If the constraint on the divergence of the displacement is relaxed, the energy variation is given by

$$\delta U = \frac{1}{2} \int \left[\vec{E}^2\ \delta\chi\ - \vec{E}^2(\chi\ -\ \mu\ \frac{d\chi}{d\mu})\vec{\nabla}.\vec{s}\ \right] dv\ .$$

3. POLARIZATION CHARGES AND THE TOTAL FORCE

In this Section, it will be shown that the volume and surface polarization charges may also be used to calculate the total force.

Now, the polarized dielectric as conceived of as a charge distrilbution in vacuum of volume density ρ_p and surface density σ_p, satisfying

$$\rho_p = -\vec{\nabla} \cdot \vec{P} , \quad \sigma_p = \vec{P} \cdot \vec{n} \quad . \tag{3.1}$$

It is clear that, owing to the continuity of the tangential component of the electric field, a tangential force would be expected. According to what has been said in Section 2, the dielectric boundary is subject only to a tension. Therefore, the mentioned tangential component may only result from a partia1 incorporation in the surface terms of contributions due to the bulk. Integrating the bulk force in the volume we get

$$\int_V (\vec{P}.\vec{\nabla})\vec{E} \, dv = \oint_S \vec{E}(\vec{P}.\vec{n}) \, ds \quad \int_V \vec{E} \, \vec{\nabla}.\vec{P} \, dv \, , \tag{3.2}$$

where the surface integral is taken inside but near the dielectric boundary. Then, we have to show that the integrand of the surface integral added to the tension previously found (Eq.2.8) gives the same force as the one deduced from the surface charge density. It is clear that the volume integral in Eq.(3.2) is already in the desired from, that is,

$$- \int_V \vec{E} \, \vec{\nabla}.\vec{P} \, dv = \int_V \vec{E} \, \rho_p \, dv \quad . \tag{3.3}$$

In order to calculate the surface force, conceiving the dielectrics as an ensemble of polarization charges, let us take a small surface element ds where the charge density is σ_p, with outward normal $\vec{n}$. The outside and inside electric fields are $\vec{E}_e$ and $\vec{E}_i$ and their normal and tangential components (in direction of $\vec{n}$ and $\vec{t}$, Fig.3) will be denoted by E_{in} and E_{it}.

Since the tangential component of the electric field is conserved, there will be a tangential component of the force, df_t, given by

$$df_t = ds \, \sigma_p \, E_{et} = ds \, \sigma_p \, E_{it} \quad . \tag{3.4}$$

We must be a little more careful in calculating the normal force, since we have to subtract from the normal fields the field created by the element itself. In the outer region (region I in Fig.3), the self field is $\sigma_p \vec{n}/2\varepsilon_0$. Therefore, the normal component of the force, df_n, is

$$df_n = ds\, \sigma_p (E_{en} - \frac{\sigma_p}{2\varepsilon_0}) \quad . \qquad (3.5)$$

Of course, the same value results when the inner normal component of the electric field is used, taking into account that the self field inside (Region II) is now $(-\sigma_p \vec{n}/2\varepsilon_0)$. To see this, we write df_n as

$$df_n = ds\, \sigma_p (E_{in} + \frac{\sigma_p}{2\varepsilon_0}) \quad . \qquad (3.6)$$

Now, Eqs.(3.5) and (3.6) are consistent because

$$E_{en} - E_{in} = \frac{\sigma_p}{\varepsilon_0} \quad . \qquad (3.7)$$

Hence, when the dielectric is conceived as an ensemble of surface and volume polarization charges the total force is given by the volume integral, Eq.(3.3), and surface integrals whose integrands are given by Eq.(3.4) in the tangential direction and by Eq.(3.5) in the normal one.

We show now that adding, to the tension found before, the surface integral in Eq.(3.2), the same value for the total force results.

The tangential component of the force obtained considering the integrand of the surface integral in Eq.(3.2) gives

$$(\vec{E}_i . \vec{t}) (\vec{P} . \vec{n}) ds = E_{it}\, \sigma_p\, ds$$

which is the same as Eq.(3.4).

Likewise, the normal component of the force to be compared with Eq.(3.

6) is the sum of Eq. (2.8), the tension, plus the normal component of the integrand in the surface integral in Eq.(3.2), that is,

$$\frac{\chi^2}{2\varepsilon_0} E_{in}^2 \, ds + E_{in} \, \sigma_p \, ds \quad , \tag{3.8}$$

and since $\chi E_{in} = a_p$ we see that the value of the expression in Eq. (3. 8) is the same as in Eq. (3.6).

Therefore, the description of the dielectric as an ensemble of charges may also be used to calculate the total force, as if they were free charges. This assures us that the Newton's Third Law is preserved for the proposed force scheme.

4. ELECTROSTRICTION

For homogeneous solid or liquid dielectrics, electrostriction would be calculated assuming the electric field pattern as given. Eq.(2.2) with χ as a constant, should feed the elastic equations as the given volume force.

For fluid dielectrics, including electrostriction, the pertinent equations are Eq.(2.2), together with the state equation of the fluid, and finally the relation between state variables and dielectric constant . The system should be solved consistently.

5. FINAL REMARKS

Many of the expressions here presented may be found scattered in the literature. We do not aim for originality, but rather to call attention to a kind of logical error usually found in the literature.

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