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Backlund Transformations as Canonical Transformations*

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Toda and Wadati as well as Kodama and Wadati have shown that the Backlund transformations for the exponential lattice equation, sine-Gordon equation, K-dv equation and modified K-dv equation, are canonical transformations. The purpose of this paper is to show that the Backlund transformations for the Boussinesq equation, for a generalized K-dv equation, for a model equation for shallow water waves, and for the nonlinear Schorddinger equation, are also canonical transformations.

Toda e Wadati assim como Kodama e Wadati mostraram que as transformações de Backlund para a equação do reticulado exponencial, equação de sine--Gorbon, equação K-dv e equação K-dv modificada, são transformações canônicas. A finalidade deste artigo é a de mostrar que as transformações de Backlund, para a equação de Boussinesq, para a equação de K-dv generalizada para uma equação modelo de ondas de águas rasantes, e para a equação de Schrddinger não linear, são também transformações canônicas.

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1. INTRODUCTION

Toda and Wadati[1] have shown that the Backlund transformation for the exponential lattice equation is a canonical transformation. Subsequently, Kodama and Wadati[2] exterided this result to the sine-Gordon equation and, more recently, these last authors[3] have established the same property for the Kortweg-de Vries and modified Kortweg-de Vries equations.

The purpose of this paper is to show that the same holds for other nonlinear equations like a generalized Kortweg-de Vries equation[4], for a model equation for shallow water waves[5], for the Boussinesq equation, and finally for the non linear Schrödinger equation.

In Section 2, we recall the basic formulae for the canonical transformations of field equations. In Section 3, we apply it to the Boussinesq equation, for which we write the generating functional. In Section 4, we consider equation of the type $\phi_{xt}+\Phi(\phi,\phi_x,\phi_{xx},\ldots)=0$ by making use of the results of Ref.3, and applying them to a generalized K-dV equation[4] and for a model equation foi- shallow water waves[5]. In Section 5, we write the generating functional for the nonlinear Schrödinger equation. In Section 6, we make some comments about the method.

2. CANONICAL TRANSFORMATIONS FOR FIELD EQUATIONS

We shall consider a system described by a Lagrangian of the form:

$$L(t) = \int L(\phi,\phi_x,\phi_t,\phi_{xt},\phi_{xx},\phi_{xxx},\ldots)dx, \tag{1}$$

where the Lagrangian density is assumed independent of time derivatives of the field higher than the first.

In (1), we used the notation

$$\phi_x = \frac{\partial\phi}{\partial x}, \quad \phi_t = \frac{\partial\phi}{\partial t}, \quad \phi_{xt} = \frac{\partial^2\phi}{\partial x\partial t}, \text{ etc.}$$

We write the following expansion[6]:

$$\phi(x,t) = \sum_n q_n(t) u_n(x) \quad , \tag{2}$$

where the $u_n(x)$ constitute a complete set of orthonormal functions, i.e.,

$$\int u_n(x) u_m(x)\, dx = \delta_{nm} \ . \tag{3}$$

$$\sum_n u_n(x') u_n(x) = \delta(x-x') \ . \tag{4}$$

We are, without any loss of generality, restricting ourselves to real functions.

From the Lagrange equations,

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_n} - \frac{\partial L}{\partial q_n} = 0 \ , \tag{5}$$

the following field equation obtains:

$$\frac{\partial L}{\partial \phi} - \frac{\partial}{\partial x} \frac{\partial L}{\partial \phi_x} - \frac{\partial L}{\partial t} \frac{\partial L}{\partial \phi_t} + \frac{\partial^2}{\partial x^2} \frac{\partial L}{\partial \phi_{xx}} + \frac{\partial^2}{\partial x \partial t} \frac{\partial L}{\partial \phi_{xt}} + \cdots \ . \tag{6}$$

From the expression for the momenta,

$$p_n(t) = \frac{\partial L}{\partial \dot{q}_n} = \int \pi(x,t) u_n(x)\, dx \tag{7}$$

it follows

$$\pi(x,t) = \frac{\partial L}{\partial \phi_t} - \frac{\partial}{\partial x} \frac{\partial L}{\partial \phi_{xt}} + \frac{\partial^2}{\partial x^2} \frac{\partial L}{\partial \phi_{xxt}} + \cdots \ . \tag{8}$$

The Hamiltonian of the system is

$$H = \sum_n p_n(t) \dot{q}_n(t) - L = \int \pi(x,t) \phi_t(x,t) - L = \int \mathcal{H}(x,t)\, dx, \tag{9}$$

where the Hamiltonian density is given by :

$$H(x,t) = \pi(x,t)\phi_t(x,t) - L(x,t) \ . \tag{10}$$

Given the Hamiltonian density, the Hamilton equations can be written as:

$$\pi_t(x,t) = -\frac{\delta H}{\delta\phi} , \tag{11}$$

$$\phi_t(x,t) = \frac{\delta H}{\delta\pi} , \tag{12}$$

where the functional derivative, $\frac{\delta}{\delta\phi}$, is given here by

$$\frac{\delta}{\delta\phi} = \frac{\partial}{\partial\phi} - \frac{\partial}{\partial x}\frac{\partial}{\partial\phi_x} + \frac{\partial^2}{\partial x^2}\frac{\partial}{\partial\phi_{xx}} + \dots , \tag{13}$$

the right hand side being applied to the corresponding density.

A transformation which takes ϕ, π into ϕ', π' is canonical wheneverit is possible to find a functional $W = \int W(\phi,\phi_x,\phi_{xx},\dots)dx$ such that

$$\int_{-\infty}^{+\infty}\left[L - L' - \frac{dW}{dt}\right]dt = 0 , \tag{14}$$

i.e.

$$\int\left[\pi\phi_t - \pi'\phi_t' + H' - H - \frac{\partial W}{\partial t}\right]dxdt = 0 . \tag{15}$$

From (15) one has the following transformation equations:

$$\pi = \frac{\delta W}{\delta\phi} , \quad \pi' = -\frac{\delta W}{\delta\phi'} , \quad H = H' - \frac{\partial W}{\partial t} . \tag{16}$$

W is the generating functional of the canonical transformation defined by Eq.(16).

3. THE BOUSSINESQ EQUATION

The Boussinesq equation writes as[7]

$$\phi_{xx} - \phi_{tt} + 6(\phi^2)_{xx} + \phi_{xxxx} = 0. \tag{17}$$

By putting $\phi = v_x$, and integrating once with respect to x, with the integration constant set equal to zero, we have

$$v_{xx} - v_{tt} + 12 v_x v_{xx} + v_{xxxx} = 0 \ . \tag{18}$$

The corresponding Lagrangian density is

$$L = \frac{v_x^2}{2} - \frac{v_t^2}{2} + 2v_x^3 - \frac{1}{2} v_{xx} \ , \tag{19}$$

while the Hamiltonian density is given by

$$H = \frac{\pi^2}{2} - \frac{v_x^2}{2} - 2v_x^3 + \frac{v_{xx}}{2} \ , \tag{20}$$

with

$$\pi = \frac{\partial L}{\partial v_t} = - v_t \ . \tag{21}$$

Introducing new field variables,

$$\begin{aligned} v_+ &= v + v' \ , \quad \pi_+ = \pi + \pi' \\ v_- &= v - v' \ , \quad \pi_- = \pi - \pi' \end{aligned} \tag{22}$$

we can write, from Eq.(20),

$$H' - H = \frac{\pi_+ \pi_-}{2} + \frac{v_{+x} v_{-x}}{2} + \frac{v_{-x}}{2} \left[3\, v_{+x}^2 + v_{-x}^2 \right] - \frac{1}{2} v_{+xx}\, v_{-xx} \ . \tag{23}$$

Putting

$$v_{-t} = a\left[v_{+xx} + 2v_{-}v_{-x}\right] = -\pi_{-} \tag{24-I}$$

$$v_{+t} = \frac{1}{a}\left[v_{-} + v_{-xx} - (a^2-3)v_{-}v_{+x} - (a^2-1)v_{-}^{3}\right] + \frac{\lambda}{a} = -\pi_{+}, \tag{24-II}$$

it follows that

$$H'-H = \frac{\partial}{\partial x}F(v_{-},v_{-x},v_{+x},\ldots) + 2(a^2+3)\left[v_{-}^{2}v_{-x}v_{+x} + \frac{1}{2}v_{-x}v_{+x}^{2}\right]. \tag{25}$$

By choosing $a^2 = -3$, we see that $H' - H$ reduces to a divergence. Choosing the generating functional

$$W = \frac{1}{2a}\int\left[\frac{3}{2}v_{+}v_{+xx} + 6v_{-}v_{-x}v_{+} - \frac{v_{-}^{2}}{2} - \frac{v_{-}v_{-xx}}{2} - v_{-}^{4} - \lambda v_{-}\right]dx, \tag{26}$$

we have, from Eq. (16), the expressions

$$\begin{aligned} \pi_{+} &= 2\frac{\delta W}{\delta v_{-}} = -v_{+t} = \frac{1}{a}\left[-4v_{-}^{3} - 6v_{-}v_{+x} - v_{-} - v_{-xx} - \lambda\right], \\ \pi_{-} &= 2\frac{\delta W}{\delta v_{+}} = -v_{-t} = \frac{1}{a}\left[3v_{+xx} + 6v_{-}v_{-x}\right], \end{aligned} \tag{27}$$

which are consistent with Eqs. (24-I) and (24-II) for $a^2 = -3$.

These are the Backlund transformations obtained by Chen[8] from the Zakharov equations[9].

4. A GENERALIZED K-dV EQUATION AND A MODEL EQUATION FOR SHALLOW WATER WAVES

We now consider equations of the type

$$\phi_{xt} = K(\phi,\phi_{x},\phi_{xx},\ldots), \tag{28}$$

where K does not depend neither on ϕ_t nor on its derivatives. In this case, it is simple to see that the corresponding Lagrangian density can be written as

$$L = \phi_x \phi_t - U(\phi, \phi_x, \phi_{xx}, \ldots) \ . \tag{29}$$

The corresponding equation of motion is

$$\phi_{xt} = -\frac{1}{2} \frac{\delta U}{\delta \phi} \ , \tag{30}$$

with

$$U = \int U(\phi, \phi_x, \phi_{xx}, \ldots) dx \ . \tag{31}$$

The corresponding Hamiltonian formalism is not uniquely defined, since

$$\pi(x,t) = \phi_x \ , \tag{32}$$

i.e., the momenta depend on the fields.

However, we can formally consider the transformation

$$H = \pi \phi_t - L \tag{33}$$

keeping only the equation

$$\pi_t = -\frac{1}{2} \frac{\delta H}{\delta \phi} \ , \tag{34}$$

and look for those canonical transformations for which

$$H' - H = \frac{\partial}{\partial x} J(\phi, \phi', \phi_x, \phi'_x, \ldots) \ , \tag{35}$$

and also

$$\phi_x = \frac{\delta W}{\delta \phi} \ , \quad \phi'_x = -\frac{\delta W}{\delta \phi'} \ . \tag{36}$$

Examples of such transformations are the Backlund transformations for the

K-dV and modified K-dV equations[3]. Here we shall consider the following generalized K-dV equation[4]:

$$\phi_t = -\phi_{xxxxx} + 10\phi\phi_{xxx} + 20\phi_x\phi_{xx} - 30\phi^2\phi_x \; . \tag{37}$$

Putting $\phi = v_x$, we can write the following Lagrangian density:

$$L = v_x v_t + v_{xxx}^2 + 5v_x^4 - 5v_x^2\, v_{xxx} \; , \tag{38}$$

the corresponding Hamiltonian density being

$$H = -v_{xxx}^2 - 5v_x^4 + 5v_x^2\, v_{xxx} \; . \tag{39}$$

Introducing $v_+ = v + v'$, $v_- = v - v'$, it follows, neglecting divergence terms, that

$$H'-H = -\frac{5}{2}\, v_{+xxx}v_{+x}v_{-x} - \frac{5}{4}\, v_{-xxx}\left[v_{+x}^2 + v_{-x}^2\right] + \frac{5}{2}\left[v_{+x}^3 v_{-x} + v_{-x}^3 v_{+x}\right]$$

$$+ v_{-xxx}\, v_{+xxx} \; . \tag{40}$$

Let us make the *Ansatz*[3]

$$v_{+x} = f(v_-) \; . \tag{41}$$

Then

$$v_{+xx} = \frac{\partial f}{\partial v_-}\, v_{-x}, \quad v_{+xxx} = \frac{\partial^2 f}{\partial v_-^2}\, v_{-x}^2 + \frac{\partial f}{\partial v_-}\, v_{-xx} \; . \tag{42}$$

It then follows that Eq.(40), neglecting again divergence terms, can be written as

$$H'-H = \frac{5}{2}\, v_{-x}^3\, f(v_-)\left[1 - \frac{\partial^2 f}{\partial v_-^2}\right] + v_{-xxx}\left[-\frac{5}{4}\, v_{-x}^2 + \frac{\partial^2 f}{\partial v_-^2}\, v_{-x}^2 + \frac{\partial f}{\partial v_-}\, v_{-xx}\right] . \tag{43}$$

Taking

$$\frac{\partial^2 f}{\partial v_-^2} = 1 \; , \tag{44}$$

it is easy to show that

$$H' - H = \frac{\partial}{\partial x} J(v_- , v_{-x} , \ldots).$$

It follows, from Eqs.(41) and (44), that

$$v_{+x} = - 2k^2 + \frac{v_-^2}{2} , \qquad (45\text{-I})$$

which is the first Backlund transformation (k is an arbitrary constant).

The generating functional is, in this case,

$$W = \int \left[-v_x v - 2k^2 (v-v')^2 + \frac{1}{6} (v-v')^3 \right] dx , \qquad (46)$$

which generates the transformations

$$\begin{aligned} \pi &= v_x = \frac{\delta W}{\delta v} = - v'_x - 2k^2 + \frac{1}{2} (v-v')^2 , \\ \pi' &= v'_x = - \frac{\delta W}{\delta v'} = - v_x - 2k^2 + \frac{1}{2} (v-v')^2 . \end{aligned} \qquad (47)$$

The two expressions (47) coincide with Eq.(45-I).

The second Backlund transformation is obtained from Eq. (45-I) and the equation of motion which follows from Eq.(38). It is given by

$$\begin{aligned} v_{+t} = v_{-x}\, v_{-xxx} - v_-\, v_{-xxxx} - \frac{1}{2} v^2_{-xx} + 5v_{-x}\left[k^2 + \frac{1}{4} v_-^2\right] \\ + 5v_-\, v_{-xx}\, v_{+x} + 20k^6 - \frac{5}{16} v_-^6 - k^4 v_-^2 + \frac{5}{4} k^2 v_-^4 . \end{aligned} \qquad (45\text{-II})$$

We now turn to a model equation for shallow water waves[5], which writes as

$$v_{xt} - v_{xxxt} + 4v_x v_{xt} + 2v_{xx}\, v_t = 0. \qquad (48)$$

This case is more involved than the former one.

Eq.(48) corresponds to the Lagrangian density

$$L = \frac{1}{2} v_x v_t + \frac{1}{2} v_{xx} v_{xt} + \frac{1}{2} v_x^2 + v_x^2 v_t , \tag{49}$$

from which we have

$$\pi = \frac{1}{2} v_x + v_x^2 - \frac{1}{2} v_{xxx} . \tag{50}$$

The Hamiltonian density reads

$$H = \pi v_t - L = -\frac{v_x^2}{2} . \tag{51}$$

Here, we have

$$H' - H = \frac{1}{2} v_{+x} v_{-x} , \tag{52}$$

which takes the form of a divergence if

$$v_{+x} = f(v_-) , \tag{53}$$

$f(v_-)$ being an arbitrary function. The one soliton solution of Eq. (48) suggests the following form of Eq. (53):

$$v_{+x} = -2k^2 + \frac{1}{2} v_-^2 . \tag{54-I}$$

Introducing it in Eq. (48), the second Backlund transformation is obtained:

$$v_{-t}\left[1 + 2v_{+x}\right] = v_{-xxt} - v_{-x}\left[1 + v_{+t}\right] . \tag{54-II}$$

By choosing the following functional generator

$$W = \int \Big[-\frac{1}{2} v_x' v - k^2 (v-v') + \frac{1}{12} (v-v')^3 + \frac{v v'_{xxx}}{2} \quad \frac{v v_x^2}{4} - \frac{v' v_x'^2}{4} + \frac{v v' v_{xx}}{4} - \frac{v v' v'_{xx}}{4} - \frac{v v_x v'_x}{4} + 3k^4 (v-v') + 2k^2 v_x' v - \frac{1}{6} v_x' (v-v')^3 + \frac{3}{80} (v-v')^5 - \frac{k^2}{2} (v-v')^3 \Big] dx \tag{55}$$

the transformations

$$\pi = \frac{\delta W}{\delta v} \ , \ \pi' = - \frac{\delta W}{\delta v'} \ ,$$

and Eq. (50) give us (54-I).

Therefore, in this case, the Backlund transformation is also a canonical transformation.

5. NONLINEAR SCHRÖDINGER EQUATION

In the case of the nonlinear Schrödinger equation

$$i \ u_t + u_{xx} + \bar{u}u^2 = 0 \ , \tag{56}$$

we have also to consider its complex conjugate equation

$$- i \ \bar{u}_t + \bar{u}_{xx} + u\bar{u}^2 = 0 \ , \tag{57}$$

taking u and $\bar{u}$ as independent fields. The corresponding Lagrangian density reads:

$$L = \frac{i}{2} \ (\bar{u}u_t - u\bar{u}_t) - u_x\bar{u}_x + \frac{|u|^4}{2} \ . \tag{58}$$

In order to avoid difficulties which come from the fact that

$$\pi = \frac{\partial L}{\partial u_t} = \bar{u} \ , \tag{59}$$

we can make the formal change $x \leftrightarrow t$, which corresponds to

$$\begin{aligned} \pi &= \frac{\partial L}{\partial u_x} = - \bar{u}_x \ , \\ \bar{\pi} &= \frac{\partial L}{\partial \bar{u}_x} = - \bar{u}_x \ . \end{aligned} \tag{60}$$

In this case, tha Hamiltonian density writes as

$$H = -\pi\bar{\pi} + \frac{i}{2}\left[-\bar{u}u_t + u\bar{u}_t\right] - \frac{|u|^4}{2}, \tag{61}$$

the Hamilton equations taking the form

$$\begin{aligned} \pi_x &= -\frac{\delta H}{\delta u}, \quad u_x = \frac{\delta H}{\delta \pi}, \\ \bar{\pi}_x &= -\frac{\delta H}{\delta \bar{u}}, \quad \bar{u}_x = \frac{\delta H}{\delta \bar{\pi}}, \end{aligned} \tag{62}$$

where $\mathbf{H} = \int H\, dt$.

By considering the generating functional

$$W = \int \left[\frac{ik}{2}(u_+\bar{u}_- - \bar{u}_+u_-) + \frac{\chi}{4}(|u_+|^2 + |u_-|^2 - 4k^2) + \frac{\chi^3}{12} + i\,\frac{u_{-t}}{u}\,\chi\right] dt, \tag{63}$$

where

$$\chi = \pm\sqrt{\lambda - 2|u_-|^2}, \quad u_\pm = u \pm u', \tag{64}$$

it follows that

$$\begin{aligned} \bar{\pi}_+ &= -u_{+x} = 2\frac{\delta W}{\delta \bar{u}_-} = iku_+ - \frac{u_-}{2\chi}\left[|u_+|^2 + |u_-|^2 - 4k^2\right] - \frac{2iu_{-t}}{\chi}, \\ \bar{\pi}_- &= -u_{-x} = 2\frac{\delta W}{\delta \bar{u}_+} = -iku_- + \frac{u_+}{2}\chi, \end{aligned} \tag{65}$$

and the corresponding complex conjugate expressions.

These equations are exactly the Backlund transformations which generate soliton type of solutions for the nonlinear Schrödinger equation[10].

6. FINAL COMMENTS

The procedure for obtaining, for a given equation, transformations which generate from given solutions other solutions of the same equation, is essentially the following one:

a) to find a Lagrangian and construct the corresponding Hamiltonian;
b) to write, from a relation between two solutions ϕ and ϕ', the difference $H-H'$ as a divergence;
c) to obtain a functional generator, W, from which the equations

$$\pi = \frac{\delta W}{\delta \phi} \quad , \quad \pi' = - \frac{\delta W}{\delta \phi'} \tag{66}$$

obtain.

Let us note that this last step is essential in order to obtain a canonical transformation which relates distinct solutions of the same equation. For instance, the modified long wave equation, proposed by Gibbon et $al.$[11], namely,

$$u_t + u_x + 12uu_x - u_{xxt} = 0 \ , \tag{67}$$

can be rewritten, by means of a change of scale, as

$$v_{xt} - 6v_x v_{xx} - v_{xxxx} - v_{xxxt} = 0 \ . \tag{68}$$

The corresponding Hamiltonian density is

$$H = v_x^3 + v_{xx}^2 \tag{69}$$

which is exactly the Hamiltonian density for the K-dV equation.

The transformation $v_{+x} = f(v_-)$, with $\frac{d^2 f}{dv_-^2} = 1$, turns $H'-H$ into a divergence. However, the transformation equation (66) do not hold, since here we have $\pi = (1/2)(v_x - v_{xxx})$.

REFERENCES

1. M. Toda and M. Wadati, J. Phys. Soc. Japan 39, 1204(1975).

2. Y. Kodama and M. Wadati, Prog. Theor. Phys. 56, 342(1976).
3. Y. Kodama and M. Wadati, Prog. Theor. Phys. *56,* 1740(1976).
4. K. Sawada and T. Kotera, Prog. Theor. Phys. 51, 1355(1974).
5. R. Hirota and J. Satsuma, J.Phys.Soc.Japan 40, 611(1976).
6. S.S. Schweber, *An Introduction to Relativistic Quantum Field Theory,* Row, Peterson and Co. (1961).
7. A.C. Scott, F.Y.F. Chu and D.W. Mc Laughlin, Proc. IEEI, 1443 (1973).
8. H.H. Chen, in *"Backlund transformations"* Nashville, Tennesee 1974 Proceedings, ed. by R.M. Miura, p.241 (1976).
9. V.E. Zakharov, Soviet Physics, JETP 38, 108(1974).
10. G.L. Lamb Jr., Journ.of Math.Phys. *15,* 2157(1974).
11. J.D.Gibbon, J.C. Eilbeck and R.K. Dodd, J.Phys. A. Math. Gen. 9, L 127 (1976).