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Triangulation in Friedmann's Cosmological Model*

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In Friedmann's model, physical 3-space has a curvature k=constant In the cases of greatest interest (K#O), triangulation for the measurement of great distances should be based on non-Euclidean geometries: Riemannian (or doubly elliptic) geometry for a closed universe, and Bolyai-Lobatchevsky's (or hyperbol ic) for an open universe.

No modelo de Friedrnann, o espaço fisico tridimensional tem curvatura k= constante. Nos casos de maior interesse (K#O), a triangulação para a medida de grandes distâncias deve basear-se nas geometrias não euclideanas: geometria de Riemann (ou duplamente el íotica) para o universo fechado, e geometria de Bolyai-Lobatchevsky (ou hiperbõl ica) para o universo aberto.

1. INTRODUCTION

Triangulation has been a very limited means for the measurement of astronomical distances[1]. However, recent advances in obtaining great resolution powers in observation instruments[2] makes one believe that eventually triangulation will be used for the determination of distances far greater than those mentioned in Ref.1.

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In this paper, we shall not pay much attention to these technical problems. Instead, we want to analyze how the curvature of space, as shown by Friedmann's model, affects the results of triangulation. A hypothetical numerical example is worked out, sharply illustrating the influence of the curvature signs.

2. FRIEDMANN'S METRICS

In an isochronous, co-moving reference system, with universal time t and spatial coordinates χ, θ, ϕ (Refs.3,4), the space-time line element in Friedmann's model is

$$ds^2 = c^2 dt^2 - dl^2 , \tag{1}$$

where

$$dl^2 = R^2(t)\left[d\chi^2 + g(\chi)(d\theta^2 + \sin^2\theta d\phi^2)\right] , \tag{2}$$

is the line element of 3-space

There are three cases to consider[3]:

(I) Flat space, which is Euclidean:

$$g(\chi) = \chi^2 ; \tag{3a}$$

(II) Closed space, isometric with a hyperspherical surface embedded in an abstract 4-dimensional Euclidean space:

$$g(\chi) = \sin^2\chi ; \tag{3b}$$

(III) Open space, isometric with a "hyperhyperbolic" surface embedded in an abstract *Minkowski* space[4]:

$$g(\chi) = \sinh^2\chi . \tag{3c}$$

In the derivation of these results, the question of space curvature is touched upon from the viewpoint of general tensor analysis[3]. Here, we shall examine directly the curvature of a plane in Friedmann's space. Let us choose

$$\phi = \{0, \pi\} ; \tag{4}$$

hence

$$d\phi = 0 \ . \tag{5}$$

If we define

$$u = R(t)\chi \ , \tag{6a}$$

$$v = R(t)\theta \ , \tag{6b}$$

and

$$G(u) = g(\chi) \ , \tag{7}$$

and substitute Eqs. (5), (6), (7) into Eq.(2), we get

$$dl^2 = du^2 + G(u)dv^2 \ . \tag{8}$$

Therefore, (u,v) are polar geodesic coordinates[5], and so the Gaussian curvature of the surface is given by[5]

$$K = -\frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial u^2} = \begin{cases} 0 \ , & \text{in case (I)} \quad (9a) \\ +\dfrac{1}{R^2} \ , & \text{in case (II)} \quad (9b) \\ -\dfrac{1}{R^2} \ . & \text{in case (III)} \quad (9c) \end{cases}$$

Thus, Eq. (4) describes different kinds of planes. In case (I), it is obviously an *Euclidean* plane. In case (II), it has the same geometry as the surface of a sphere of radius $R(t)$. For a plane, this geometry is called *Riemannian*, or *doubly elliptic*[6]. In case (III), the geometry is that of *Bolyai-Lobatchevsky*, or *hyperbolic*. This can be seen indirectly, from two facts[5]:

(a) the Euclidean surface called the pseudo-sphere has negative Gaussian curvature just of the form (9c), and therefore the same inner geometry as our plane; and (b) the inner geometry of the pseudo-sphere has been shown (by E. Beltrami, in 1868) to be that of a Lobatchevsky plane. Now, by suitable orientation of the coordinate axes, any place can be described by Eq. (4). Hence, in Friedmann's space, any plane has one of the geometries just specified for the three cases.

3. TRIANGULATION

In the preceding Section, differential geometry was used to show the nature of a plane in Friedmann's model of the universe. As a result, we find the geometries involved are simple enough, so that we can use *finite trigonometry* in all three cases.

We have to solve triangle ABC, with sides a, b, c (Fig.1). A is the (apparent) location of the source.

The angles B and C and the base distance a are measured directly, and we want to know the distance b (or c, since they are essentially equal, a being much smaller than both b and c).

Case *I*: $K=0$.
Trivially, by the law of sines,

$$b = \frac{\sin B}{\sin(\pi - B - C)} a \; . \qquad (10)$$

Case *II*: $K = +1/R^2$, Riemannian plane.
Here (as below, in case III), we have to assume that $R(t)$ is known, so that actually we deal with the ratios $\alpha = a/R$, $\beta = b/R$, and $\gamma = c/R$. From spherical trigonometry, we have

$$\frac{\tan[(\beta+\gamma)/2]}{\tan(\alpha/2)} = \frac{\cos[(B-C)/2]}{\cos[(B+C)/2]} \qquad (11)$$

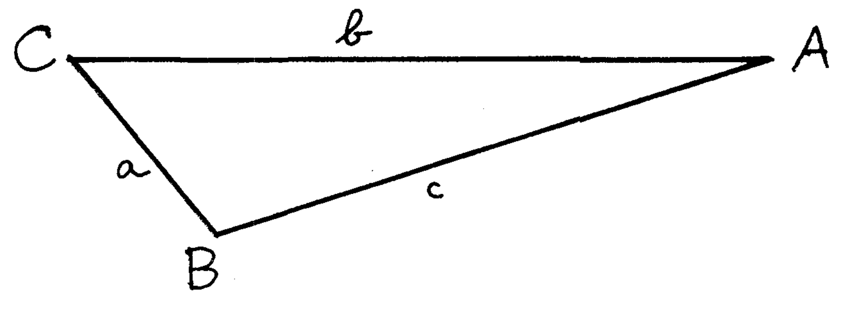

Fig. 1

Here, $\tan\frac{a}{2} \simeq \frac{\alpha}{2}$, $\beta \simeq \gamma$, and

$$\cos\frac{B+C}{2} \simeq \frac{\pi - B - C}{2} \quad ;$$

hence, Eq.(11) gives

$$\tan\beta = \frac{a}{T - B - C} \cos\frac{B-C}{2} \ , \tag{12}$$

or

$$b = R \arctan\left[\frac{a/R}{\pi - B - C}\cos\frac{B-C}{2}\right] \ . \tag{13}$$

Notice that b varies from 0 to πR, taking the value $\pi R/2$ when $B+C = \pi$.

Case III: $K = -1/R^2$, Lobatchevskyan plane.
The trigonometry for this case can be obtained from that of case II by the simple substitution (cf.Ref.6)

$$\xi \rightarrow i\xi \ , \tag{14}$$

for $\xi = \alpha, \beta, \gamma$. One gets

$$\tanh\beta = \frac{\alpha}{V - B - C} \cos\frac{B-C}{2} \ , \tag{15}$$

or

$$b = R \ \mathrm{arg\,tanh}\left[\frac{a/R}{\pi - B - C}\cos\frac{B-C}{2}\right] \ . \tag{16}$$

Here b varies from 0 to ∞ .

One word about the calculated distance b:
Given the nature of light propagation in Friedmann's model[3] and the expansion factor $R(t)$, the distances so obtained are *apparent* distances, that is, they give the position that the source would have in the occasion of the measurement, assuming that it had maintained the zero-velocity character of a coordinate body in the co-rnoving reference system.

4. NUMERICAL EXAMPLE

Let us assume the values

$$\alpha = 10^{-9} , \tag{17}$$

$$B = \frac{2\pi}{3} \text{ radian}, \tag{18}$$

$$C = \left(\frac{\pi}{2} - 10^{-9}\right) \text{ radian}, \tag{19}$$

for the basic quantities. The present value of $R(t)$ is of the orderof 10^{10} light-years, so that we have actually taken a=10 light-years. This may not be a realistic base for the triangulation, but, as rnentioned above, here we are primarily concerned with exhibiting the effect of curvature.

Indeed, leaving $R(t)$ indeterminate, we get frorn Eqs. (10), (13), (16), (17), (18), and (19),

$$b = 0.866\ R , \text{ for } K = 0 , \tag{20}$$

$$b = 0.714\ R , \text{ for } K = + 1/R^2 , \tag{21}$$

$$b = 1.317\ R , \text{ for } K = - 1/R^2 . \tag{22}$$

Therefore, for distances of the order of $R(t)$, the results are considerably different in the three cases.

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