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On the Random Geometry of a Symmetric Matter-Antimatter Universe

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A statistical analysis is made of the random geometry of an early simmetric matter-antimatter universe model. Such a model is shown to determine the total number of the largest agglomerations in the universe, as well as of some special configurations. Constraints on the time development of the protoaggloromerations are also obtained.

E feita uma análise estatística da geometria aleatória de um modelo de universo primordial com simetria matéria-antimatéria. Mostra-se que um tal modelo fixa o número total das maiores aglomerações do universo, assim como o de algumas aglomerações especiais. Obtêm-se também restrições no desenvolvimento temporal das proto-aglomerações.

1. INTRODUCTION

In the early stages of a matter-antimatter symmetric standard universe, space is occupied by an emulsion composed of randomly shaped regions, each of them with a positive (matter) or negative (anti-matter) baryon number. Annihilation takes place in the contact layers, which are thin enough, as compared to the dimensions of the regions, to be taken simply as surfaces for most purposes.

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This qualitative description is supposed to be valid up to the recombination time, when there is a disruption of the system and the general expansion takes the regions apart to constitute the largest inhomogeneities in the universe. Hlthough it arises as a result of calculationsin Omnès model[1], which proposes detailed mechanisms for the origin and the growth of the regions, this picture is general enough as to be practically unavoidable in any symmetric model. A change in it would lead to unacceptable consequences, generally a too small or too large annihilation rate. In Omnès model, the word "emulsion" is justified because (1) the width of the contact layer is shown[2] to be small so that it can be considered as an interface; and (2) the pressure gap between the two sides at a point in the interface is shown[3] to obey the same Laplace-Kelvin equation as in ordinary emulsions. The growth of the average dimensions of the regions is not assured by the usual argument (minimization of the surface free-energy) of equilibrium thermodynamics because temperature gradients are present and the problem has to be handled in a more sophisticated way. The maintenance of a thin contact layer is essential because otherwise there would be too much annihilation and no matter would survive. Interpenetration of matter and antimatter regions would be Fatal to any symmetric model. On the other hand, a complete separation would cause much more matter to survive than is found in the universe today.

In this paper, we shall ignore any details about mechanisms for thebirth and the development of the regions, and only keep the geometrical picture of a primeval emulsion, trying to obtain some general results which should hold for a very general class of symmetric models. For want of better, we shall use words like "emulsion" and "maze", however inaccurate they may be in the case.

We present a statistical treatment in principle valid for any continuous random maze and apply it to the primeval emulsion. An important qualitative aspect is that a symmetric model will always, besides eventually describing the origin and evolution of the inhomogeneities, fix their number. It will be shown that good numbers are obtainable for clusters or galaxies, although the treatment by itseif is not able to decide which of them the regions wiil originate. Only the knowledge of their dimensions at recombination time would allow to enlighten this point.

Some general characteristics of random mazes are worth recalling here. Such systems appear in all branches of science[4] and have been the object of a large amount of attention since the pioneering work of Broadbent and Hammersley[5]. Although most of the results concern lattice models [6], some of them are believed to be valid for general systems, even the continuous ones[7]. These studies are in general named after the most conspicuous phenomenon exhibited by such random media, namely, percolation. Suppose we have a disordered system formed by two distinct substances. To fix the ideas, take a homogeneous medium of one material to which one adds, continuously and at random positions in space, small quantities of the second material. With growing concentration, this last material will occupy more and more space and constitute larger an larger regions. At a certain value of the relative concentrations of the two materials, percolation occurs: the probability to have an infinite (i.e., crossing all the system) region of the second material becomes different from zero. This critica1 phenomenom, according to the materials involved, is related to metal-semiconductor transitions[8], dilute ferromagnetism[9], transport through porous media and many other phenomena[5]. An important outcomes is that the critical concentration does not seem to depend upon the details of the system: only its dimension is important. For two-dimensional systems, it stays around 0.5, and for three-dimensional somewhere between 0.2 and 0.35.

In the case of a symmetric emulsion, the concentration is 0.5 and one conclusion is forced upon us: the primeval maze, having concentrations above the critical one, will be percolated. There will be regions (both of matter and antimatter) of infinite extension[10]. These regions will be submitted to much violence at the recombination time[11], and an extra hypothesis is necessary: that the infinite regions, which will forcibly be disrupted at that time, will break into regions of the same average size of the finite regions. This seems quite reasonable if we characterize this size as the linear dimension of a volume of matter (or antimatter) which is "visible" from a point inside it, as will be specified below.

There is all reason to believe that the regions keep something of their individuality after the general disruption. If the breaking preserves them, there will be large inhomogeneities each either of matter or anti-matter. If the breaking divides them into smaller pieces, there will be sets of neighbouring matter inhomogeneities. This last case has of course some appeal to the problem of the origin of galaxies and their tendency to join in clusters.

The statistical approach presented below has not been, to our knowledge, systematically applied to the problem of percolation. It has been used rather laterally to give a lower limit to the percolation critical concentration[8]. It cannot really explain percolation, as it does not consider local density fluctuations. It has been used more extensively in the analysis of diffusion in porous media[12], but only as a means to obtain bounds for the values of the diffusion coefficient[13]. For this reason, we are forced to review the subject in some detail in Section 2. In Section 3, we apply it to the calculation of the number of inhomogeneities. In Section 4, it is shown that the approach allows also to estimate the relative concentrations of some special kinds of inhomogeneities. Some constraints on the average size of the regions, coming from general arguments about the annihilation rate, are presented in Section 5.

2. RANDOM GEOMETRY

Let us consider an emulsion formed by two different immiscible substances, forming regions of random shape. The regions of one of the substances (e.g., matter) will be simulated by a bed of randomly werlapping spheres, all of them with one and the same radius, smaller than the minimum curvature radius of the interface. The random character of the emulsion is provided by allowing the sphere centres to have equal probability of being in any point of a large volume V. Let us take a large nomber $N = \bar{N}V$ of spheres of volume $v_S = \frac{4\pi}{3} a^3$. The probability that n centres be in a volume v, contained in V, is given by the binomial distribution

$$W_N^v(n) = \frac{N!}{n!(N-n)!} \left(\frac{v}{V}\right)^n \left(1 - \frac{v}{V}\right)^{N-n} . \qquad (1)$$

Notice that v/V is the probability for a centre to be inside v. The average number of centres** in v is

$$\langle n\rangle = N\frac{v}{V} = \bar{N}v \quad . \tag{2}$$

In the limit of large N and V, with $\bar{N}$ constant, the binomial distribution reduces to the Poisson distribution,

$$P_n(v) = \frac{(\bar{N}v)^n}{n!}e^{-\bar{N}v} \tag{3}$$

In particular, the probability that no centre be in v (i.e., that v be completely in the antimatter) is

$$P_0(v) = e^{-\bar{N}v} \quad . \tag{4}$$

An extremely important parameter is the relation Φ between the volume not covered by any sphere and the total volume V. In our case, this is the fraction occupied by antimatter. In the case of porous media, our "matter" corresponds to the occupied regions and our "antimatter" to the empty regions, and this parameter is called the *void fraction*. Obviously, it is the probability of any point in the medium to be outside any sphere, that is, to be at a distance larger than a of any centre. This is the probability of existing, around it, an empty region of volume v_S and so

$$\Phi = e^{-\bar{N}v_S} \quad . \tag{5}$$

In our symmetric case, $\Phi = 1/2$.

We can now calculate the probability $P(\varepsilon)d\varepsilon$ for a point in the antimatter to be at a distance, between ε and $\varepsilon+d\varepsilon$, from the nearest interface point or, in other words, to be at a distance between $(\varepsilon+a)$ and $(\varepsilon+d\varepsilon+a)$ from the nearest sphere centre. This is the product of the probability that no centre is in a sphere of radius $(\varepsilon+a)$ around the point by the probability that some centre is in the volume $4\pi(\varepsilon+a)^2d\varepsilon$ enveloping it:

$$P(\varepsilon)d\varepsilon = 4\pi(\varepsilon+a)^2\bar{N}d\varepsilon\, e^{-N\frac{4\pi}{3}(\varepsilon+a)^3} \quad . \tag{6}$$

This allows one to measure the are S of the interface as the probability of a point to be on it, i.e.,

$$\bar{S} \equiv \frac{S}{V} = P(0) = 4\pi a^2 \bar{N}\Phi = 2\pi a^2 \bar{N} \tag{7}$$

in our case.

The probability for a centre to stay at a distance r from its nearest centre is, in an analogous way,

$$w(r) = 4\pi\bar{N}r^2 dr\, e^{-\frac{4\pi}{3}\bar{N}r^3} \tag{8}$$

Two spheres are connected (i.e., they overlap) if their centres are at a distance smaller than 2a from each other. So, the probability that a given sphere is connected to at least another one is

$$\int_0^{2a} w(r)\,dr = 1 - e^{-8\bar{N}v_S} = 1-\Phi^8 \quad . \tag{9}$$

Hence, the probability P_i for a sphere to be isolated is Φ^8, or, in our case,

$$P_i = 3.9 \times 10^{-3} \quad . \tag{10}$$

The probability for a straight line with one end fixed in the antimatter to cross the interface at a distance in the interval $(\lambda, \lambda + d\lambda)$ is

$$P_\lambda d\lambda = e^{-\bar{N}(\frac{4\pi}{3}a^3 + 2\pi a^2\lambda)} \bar{N}4\pi a^2 d\lambda \cos\theta \ ,$$

where

a) the first term in the exponent excludes centres around the two ends of the straight-line;

b) the second excludes them from a cylinder of radius a all around the line, which therefore is entirely in the antimatter;

c) the remaining factor is the probability to have at least one centre in the interval $(a, a+d\lambda\cos\theta)$ enveloping the sphere around the crossing point at the interface (this guarantees that this point is on the surface);

d) finally, θ is the angle between the line and the normal to the interface at the meeting point; the average value of $d\lambda\cos\theta$ can be obtained[15] if we remember that only points in the matter are to be considered here:

$$\frac{d\lambda}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi/2} \cos\theta\sin\theta d\theta = \frac{d\lambda}{4} .$$

So,

$$P_\lambda d\lambda = \pi a^2\bar{N}d\lambda\ e^{-\pi a^2\bar{N}\lambda} . \tag{11}$$

The average distance from an antimatter point to the surface is

$$\ell = \langle\lambda\rangle = \int_0^\infty \lambda P_\lambda d\lambda = \frac{1}{\pi a^2\bar{N}} = 4\Phi\frac{V}{S} . \tag{12}$$

In terms of ℓ, Eq. (11) becomes

$$P_\lambda d\lambda = e^{-\lambda/\ell}\,\frac{d\lambda}{\ell} . \tag{13}$$

Eq. (12) can also be written

$$\ell = -\frac{4a}{3\ln\Phi} ;$$

for $\Phi = 1/2$, we have

$$\ell = 1.92\ a .$$

A good characterization of the average linear dimension of a region will be

$$L = 2\ell = 3.82\ a . \tag{14}$$

For a point in the matter, it is easy to show that the average distance to the interface will be

$$\frac{1-\Phi}{\Phi}\,\frac{1}{\pi a^2 \bar{N}} \;.$$

Of course, in our case, this coincides with R.

Recalling that the volume element for a cone of length X and angle $d\Omega$ is $(\lambda^3/3)d\Omega$, one can evaluate the average volume inside a region which is scanned by straight lines from a point:

$$V = 4\pi \int_0^\infty \frac{\lambda^3}{3}\, d\lambda\; e^{-\lambda/\ell} = 8\pi\ell^3 \doteq \pi L^3 \tag{15}$$

This "visible" volume gives a lower limit estimate for the volume of a region. For the reasons given in the introduction, we will take it as the typical volume.

The efficient volume of a sphere, v_e , the superpositions discounted, is defined by

$$N v_e = (1-\Phi)\, V \,;$$

this leads to

$$v_e = \frac{0-1}{\ln \Phi}\, v_S \;,$$

or, in our case, to

$$v_e = 0.72\; v_S = 3\; a^3 \;. \tag{16}$$

The comparison between (15) and (16), using (14), allows one to obtain the relation between the number density of regions $\bar{N}_c$ and the number density of spheres $\bar{N}$:

$$\bar{N} = 58.4\; \bar{N}_c \;. \tag{17}$$

3. THE NUMBER OF AGGLOMERATIONS

During the history of the symmetric universe, prior to recombination, matter and antimatter constitute a system which is precisely a geometrical emulsion as just described. At any instant' t, the system can be simulated by a bed of spheres as above. A model will give the average size $L(t)$. As $L(t)$ varies with time, so will vary the minimum curvature radius and we shall always be able to define spheres with radius $a(t)$ satisfying Eq.(14). In this way, it will be possible to simulate the primeval emulsion continuously in time. As $\Phi = 1/2$ is of course kept constant, the number density of centres will change with time according to

$$\frac{4\pi}{3}\bar{N}a^3(t) = 0.69 ,$$

or

$$\bar{N}(t) = \frac{9.3}{L^3(t)} .$$

The number density of regions will satisfy

$$\bar{N}_c(t) = \frac{0.16}{L^3(t)} . \qquad (18)$$

Let us consider now the emulsion at recombination time. Let us suppose the regions to give origin to clusters of galaxies. The present day number density of clusters will be

$$\bar{N}_c(t_0) = \frac{\bar{N}_c(t_R)}{(1+z_R)^3} ,$$

the index R indicating the values of the expansion t and of the red-shift z at recombination time, These are ill-defined in the symmetric model, in which recombination takes place in a very long period. Furthermore, using Eq. (18), the above expression becomes

$$\bar{N}_c(t_0) = \frac{0.16}{(1+z_R)^3L^3(t_R)} , \qquad (19)$$

and no model is known which gives a faithful value for $L(t_R)$. We shall estimate it from the mass of the Coma clusters, for which we take[16]

$$M_c \simeq 6 \times 10^{48} \text{ g} .$$

As accretion does not seem to be able to change this order of magnitude[17], we shall take this mass as the one of a region at recombination. At that time, the density inside the proto-cluster is the universal homogeneous average matter density[18]

$$\rho_R = 5.5 \times 10^{-30} \hat{\Omega}\ (1+z_R)^3 \text{ g. cm}^{-3} ,$$

where

$$\hat{\Omega} = \frac{\rho_0}{\rho_c} \left(\frac{H_0}{50}\right)^2 .$$

Using (15), Eq.(19) gives finally

$$\bar{N}_c(t_0) = 1.4 \times 10^{-5}\ \hat{\Omega}\ .\ \text{Mpc}^{-3} , \tag{20}$$

a value consistent with the present estimate[19] of $\bar{N}_c$ between 3×10^{-7} and 1.5×10^{-6} Mpc^{-3}. Also the Zwicki scale of maximum clustering[20] can be estimated

$$L_Z \approx \pi^{1/3}(1+z_R) L_R = \frac{33.5}{\hat{\Omega}^{1/4}} \text{ Mpc} . \tag{21}$$

Of course, these are, at best, order-of-magnitude calculations, intended only to show that good numbers can be obtained. Furthermore, these results are not at all unexpected: Eq.(18) only gives a correct geometrical factor to an unavoidable consequence of supposing that at the time of their formation the clusters were packed side-by-side. The argument only favours symmetric models as far as there is no reason to suppose such a packing in cluster formation in the usual non-symmetrical standard model.

The above considerations are not enough to decide whether the regions indeed originate clusters: Eq.(19) can also be applied to galaxies. As these are self-bound systems whose average size has probably not changed very much since the epoch of formation, one can take $L_R \approx 2 \times 10^{-2}$ Mpc. As z_R for a symmetric model[21] is $\approx$ 100,

$$\bar{N}_G(t_0) \approx 2 \times 10^{-2}\ \mathrm{Mpc}^{-3}\ , \tag{22}$$

which is a very good number. Whether clusters or galaxies are formed is to be decided by a dynamical detailed model which fixes $L(t_R)$. A reason for preferring to take the regions as protoclusters is simply that galaxies seem simpler to get formed by other mechanisms. If galaxies were originated by the regions, another way should be found to originate clusters. In this case, galaxies and antigalaxies would quite probably coabide in a same cluster. This would not cause much trouble from the observational point of view, as the usually accepted arguments based on the non observation of high-energy y-rays from clusters are to be revised[22].

4. THE NUMBER OF "QUASARS"

An interesting exercise is to reconsider an old proposal by Omnès of a scherne for a quasar model: quasars would come from a region of matter trapped by antimatter (or vice-versa) which remained so after the disruption at recombination. Annihilation would be a very efficient energy source, but the stability of such a system is a very complicated problem[23] and the model is not fashionable nowadays. Nevertheless, it is remarkable that the symmetric model fixes also the number of those proposed configurations.

A first idea is to use the probability P_l of a sphere to be isolated, Eq.(10). A fraction -4 × 10^{-3} of all the N spheres are isolated; from Eq. (17), the number of such systems would be $N_Q \approx 0.23\ N_c$. There would exist one "quasar" for every 4 or 5 large inhomogeneities. From (20), this would give

$$\bar{N}_Q = 3.2 \times 10^{-6}\ \hat{\Omega}\ \mathrm{Mpc}^{-3}\ . \tag{23}$$

The estimate of 10^7 detectable quasars[24] corresponds to

$$\bar{N}_Q \approx 5 \times 10^{-5}\ \hat{\Omega}^{3/2} \quad .$$

A more realistic estimate would come from the probability to have inside a volume V a number of spheres large enough to constitute a significant nucleus of matter, while small enough for this matter to remain surrounded by antimatter. Quasars with too large or too small nuclei would probably no more exist as such today, as the kind of particles in minority could have been completely annihilated. Of course, only a detailed analysis of the lifetime of these systems could tell the necessary volume for the nucleus, but the statistical approach gives a "spectrum" for its initial values. If we take a central region of volume ranging from v_S up to (say) $V/3$ (which correçponds to about 18 spheres), Eq.(3) gives

$$P \simeq \sum_{n=1}^{18} \frac{(\bar{N}v)^n}{n!}\, e^{-\bar{N}v} = 2 \times 10^{-2} \quad , \tag{24}$$

where we have used (14), (15) and $\bar{N}v_S = 0.69$. This, combined with (20), gives

$$\bar{N}_Q(t_0) \simeq 3 \times 10^{-7}\hat{\Omega}\ \mathrm{Mpc}^{-3}.$$

5. CONSTRAINTS ON THE MEAN SIZE

Let us now apply the above geometric statistical scheme to the discussion of the annihilation. Forgetting any details on how it takes place in the contact layer, we shall suppose simply that every particle crossing the interface is instantaneously annihilated. The interface specific surface area is, from (12) and (14),

$$\bar{S} = \frac{S}{V} = \frac{4}{L} \ . \tag{25}$$

In the time interval dt, all particles will cross the interface which are in a volume $(1/3)S\mathrm{v}dt$, and so their annihilated fraction will be

$$dF = \frac{1}{3}\frac{S}{V}\,\mathrm{v}dt = \frac{4}{3}\frac{\mathrm{v}dt}{L} \; . \tag{26}$$

The velocity v will be regulated by different phenomena at each epoch. During the annihilation period, in Omnès model, it will be fixed most of the time by the neutron diffusion, with a coefficient[10]

$$D = 1.3 \times 10^{8}\; t^{5/4}\ \mathrm{cm}^2\ \mathrm{s}^{-1} \; , \tag{27}$$

and the diffusion velocity will be

$$\mathrm{v}_D = \frac{d}{dt}\,(6Dt)^{1/2} = 3 \times 10^{4}\; t^{1/8} \; . \tag{28}$$

This is practically constant during the period, which goes from $t \approx 10^{-4}$ s to $t \approx 1$s, and Eq.(26) shows that annihilation is much more intense at its beginning (when $L(t)$ is smaller) than at its end. Unfortunately, there is no acceptable model for $L(t)$. It is simple to see from (26) that the once proposed[10] growth by diffusion,

$$L(t) = L_0 + \sqrt{6Dt} \; , \tag{29}$$

would lead to a very quick extinction of all matters. Nucleosynthesis considerations[25] lead to two lower bounds for L at $t \approx 1$s. If the standard He abundance is to be obtained,

$$L(t \approx 1\mathrm{s}) \gtrsim 3 \times 10^{6}\ \mathrm{cm}. \tag{30}$$

If, after nucleosynthesis is over, there is still enough annihilation to create deuterium by the disruption of He nuclei, then

$$L(t \approx 1\mathrm{s}) \gtrsim 1.5 \times 10^{8}\; \hat{\Omega}^{1/3}\ \mathrm{cm}. \tag{31}$$

Although detailed calculations should be done to allow a definite statement, one can say that Eq.(26) favours the first of the above lower limit, as it shows that annihilation will take place mainly at the beginning of the period. An interesting condition can be obtained for $L(t)$. From (26), the neutron number density variation with time will be given by $dN/dt = -4vN/L$, and so

$$N(t) = N_0 \exp\left[-\frac{3}{4}\int_{t_0}^{t}\frac{v(t)dt}{L(t)}\right] . \tag{32}$$

The baryon to photon ratio,

$$\eta(t) = \frac{N(t)}{N_\gamma(t)} , \tag{33}$$

has a present day value[18] of

$$\eta(t_0) = 7 \times 10^{-9}\,\hat{\Omega} , \tag{34}$$

up to now unexplained. One of the ambitions of symmetric models is precisely to account for it. Using

$$N_\gamma(t) = 1.1 \times 10^{37}\, t^{-2} \tag{35}$$

and the value $\eta=3$ for very small t, Eq. (32) yields

$$\eta(t) = 1.7 \times 10^{7}\, t^{3/2} \exp\left[-\frac{4}{3}\int_{t_i}^{t}\frac{v(t)dt}{L(t)}\right], \tag{36}$$

where t_i is the beginning of the annihilation period. The sirnple requirement, that $\eta(t)$ be a decreasing function at an instant t, implies that

$$L(t) \lesssim \frac{8}{9}\, vt , \tag{37}$$

or, if (28) is used for $v(t)$,

$$L(t) \lesssim 2.7 \times 10^{4}\, t^{9/8} . \tag{38}$$

Equation (37) is a quite intuitive result: if the regions grow too quickly as compared to vt, no annihilation will take place. Comparing

(38) and (30), one is faced with a puzzle: nucleosynthesis seems incompatible with the small present values of η. Really, the only possible issue is that $L(t)$ remain close to the diffusion length for some time (during which strong annihilation brings $\eta(t)$ to nearly its present value) and afterwards it grows very quickly to surpass the lower boundat $t \approx 1$ sec. This means that the annihilation era must shrink to a short period following the separation time and that $\eta(t)$ already had practically its present value at the nucleosynthesis epoch.

6. FINAL COMMENTS

There has been much discussion about the up-to-now unexplained value of the proton to photon ratio $\eta(t_0)$. The ill-feeling it causes comes from its status of an additional universal constant in the standard model. Much of the appeal of Omnès' model comes from the possibility it hints at to calculate it.

On the other hand, not much attention has been called upon the necessity that a cosmological model, besides providing mechanisms for the origin of the large agglomerations, also explain their numbers. We believe having made it apparent that knowledge is not needed, on the detailed formation process, to obtain such numbers. A symmetric model will fix their values, and also possibly the concentrations of some special inhomogeneities. Furthermore, simple statistical considerations impose severe constraints on the very history of the protoagglomerations.

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