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Light Propagation Properties of Graded Index Glass Fibers for Optical Communication

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Using the model of ray optics, we introduce the most important light propagation properties of multimode optical fibers. The group delay times of both skew rays and meridiohal rays are calculated and it is shown how near equalization of all group delay times can be achieved by proper choice of refractive index profile. The impulse responsehas been calculated to all orders of A and graphical results are show that include the effects of leaky ray losses.

As propriedades mais importantes de propagação de fibra's Õpticas de multimodos são apresentadas, empregando-se o modelo da Õptica geométrica. Calculam-se os retardamentos de grupo tanto de raios reversos como de raios meridionais, e mostra-se como se pode conseguir igualdade aproximada de todos os retardamentos de grupo por escolha conveniente do perfil do índice de refração. Foi calculada a resposta impulsiva em todas as ordem em A, e são mostrados os resultados gráficos que incluem os efeitos de perdas devidas a raios amortecidos.

INTRODUCTION

The ray optic model presents a very convenient tool for investigating the most essential properties of optical fibers where core diameter is large compared to the wavelength of light. Such fibers have the advan-

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tage of having a greater light carrying capacity than small corefibers and in addition are easier to connect and manufacture. Until the development of the graded index fiber, which has proprties of focusing light, the multimode large core fiber was nevertheless not competitive with the single-mode, small core fiber as an intermediate to long distance transmission medium. This is because high intermodal dispersion would limit the information carrying capacity. With the graded index fiber, it is possible to achieve a near equalization of the group velocities of all modes, thereby greatly increasing the maximum possible transmission rate of information (bit rate). With these fibers, it becomes possible to approach the low dispersion of single mode fibers and all the advantages of a large core diameter.

In what follows, we shall treat the fiber as a cylindrical structure having a central "core" region of radius a and an outer region (called "cladding") whose radius in our ray optic treatment need not be specified. The refractive index is assumed to have zero gradient in the direction of the axis of the fiber and inside the core to be a constant or monotonically decreasing function of the distance ρ from the axis. In the cladding region, it is a constant which we denote by n_c. The inequality $dn(\rho)/d\rho \leq 0$ is thus satisfied for all ρ.

In Sections I and II, we present the basic solutions of the ray equation, including the expression for the group delay time. In Section III, the group delay time of meridional rays is calculated for a particular index profile, while in the following Section it is demonstrated that this expression is satisfied also by skew rays. In Section V, we derive by a physical argument the attenuation constant of leaky rays. In Section VI, we calculate the impulse response of a fiber excited by a Lambertian source and present results that include the effects ofleaky rays. In the last two Sections, we present a brief overview of various causes of loss and mode coupling effects.

1. THE GROUP DELAY TIME OF MERIDIONAL RAYS

We consider an optical fiber of refractive index profile as discussed above. By Snell's law of light refraction', the cosine of the angle a ray makes at any point with the fiber axis multiplied with the local refractive index is a constant. If the ray is assumed to start its trajectory at a point on the axis, we express this relationship by the equation

$$n(\rho)\cdot\frac{dz}{ds} = n(0)\cos\theta_0 \ , \tag{1}$$

in terms of the angle θ_0 of the ray with the fiber axis, at the point of intersection. If $n(\rho)$ is a uniformly decreasing function of p, for $\rho<a$, then the ray motion is periodic. This is seen by substitution of the element of arc length along the ray trajectory,

$$ds = \sqrt{d\rho^2 + dz^2} \ , \tag{2}$$

into (1), and solving for $d\rho/dz$. We find

$$\frac{d\rho}{dz} = \pm\frac{\sqrt{n(\rho)^2 - n(0)^2\cos^2\theta_0}}{n(0)\cos\theta_0} \tag{3}$$

Equation (3) shows that at the value $\rho = \rho_t$ which satisfies

$$n(\rho_t) = n(0)\cos\theta_0 \ , \tag{4}$$

the function $\rho(z)$ has a maximum, so that the ray has reached a turning point here. We further see from (3) that the curve is symmetric about the turning point. The trajectory is therefore periodic and is symmetric about any of its turning points. This result can also be shown to be valid for rays which do not pass through the fiber axis (skew rays).

There are two quantities that are useful. The first is the ray period which is defined as the difference in the z-coordinate between two consecutive turning points. We may find it by solving equation (3) for dz, and integrating ρ between the limits 0 and ρ_t. The full ray period, h , is therefore given by

$$\Lambda = 2\int_0^{\rho_t} \frac{n(0)\cos\theta_0 \, d\rho}{\sqrt{n(\rho)^2 - n(0)^2\cos^2\theta_0}}$$

The second useful quantity is the time delay over one ray period. To calculate this we make use of the relation for the group velocity of a monochromatic wave at wavelength λ

$$v_g = \frac{c}{n - \lambda(dn/d\lambda)} \ . \qquad (6)$$

The term $\lambda dn/d\lambda$ is contributed by the material dispersion, and the expression in the denominator is called the "group index" n_g. The time delay per ray period for a monochromatic wave is then given by

$$P = \int \frac{ds}{v_g} \ , \qquad (7)$$

where the integration is over one full ray period.

Using equations (1) and (2), we express equation (7) as an integral over ρ. Substituting from equation (6) for the group velocity, we obtain

$$P = \frac{2}{c}\int_0^{\rho_t} \frac{n(\rho)\left[n(\rho) - \lambda \frac{dn(\rho)}{d\lambda}\right]}{\sqrt{n(\rho)^2 - n(0)^2\cos^2\theta_0}} \, d\rho \ . \qquad (8)$$

A final quantity of interest is the group delay time for a fiber of length L. If L is chosen to be an integral number of ray-periods, this delay time is given by the product of P with the number of ray periods in L. The latter quantity is given by the quotient L/Λ. Therefore, the **delay time,** for a monochromatic wave at wavelength λ in traversing a filber length L, is given by

$$T(L) = \frac{\dfrac{L}{c}\displaystyle\int_0^{\rho_t} \frac{n(\rho)\left[n(\rho) - \lambda \frac{dn(\rho)}{d\lambda}\right]}{\sqrt{n(\rho)^2 - n(0)^2\cos^2\theta_0}} \, d\rho}{\displaystyle\int_0^{\rho_t} \frac{n(0)\cos\theta_0}{\sqrt{n(\rho)^2 - n(0)^2\cos^2\theta_0}} \, d\rho} \qquad (9)$$

If the fiber length is not equal to an integral number of wave periods, the correct expression for $T(L)$ include a periodic term added to equation (9). From equation (9), we see, however, that the non periodic contribution is proportional to the fiber length L, so that by choosing L sufficiently large, the relative contribution of the periodic term can be made arbitrarily small.

2. THE GROUP DELAY TIME OF SKEW RAYS

Rays which do not pass through the fiber axis, at any point of their trajectory, are called skew rays. The results of the previous discussion are easily generalized to include skew rays. This is done first by generalizing the statement of Snell's law to two dimensions. Equation (1) is altered to the form

$$n(\rho)\,\frac{dz}{ds} = n(\rho_0)\cos\theta_0 \ , \qquad (10)$$

where the reference point at $\rho = \rho_0$ is now located a distance ρ_0 from the axis. If we include the application of Snell's law to the transverse direction normal to the radial and axial directions, we obtain the statement that the product of the projection of the unit tangent vector to the ray trajectory with the local refractive index and radial distance p is a constant (see Fig.1):

$$n(\rho)\rho^2\,\frac{d\psi}{ds} = n(\rho_0)\rho_0\cos\phi_0\sin\theta_0 \quad . \qquad (11)$$

Equations (10) and (11) may be thought of as expressing the constancy of the z- components of "momentum" and "angular momentum" respectively. In addition, equation (2) which expresses the length of arc is generalized to

$$ds = \sqrt{d\rho^2 + dz^2 + \rho^2 d\psi^2} \quad . \qquad (12)$$

The previous procedure for calculating the ray period is then repeated. We only write the result for the group delay time:

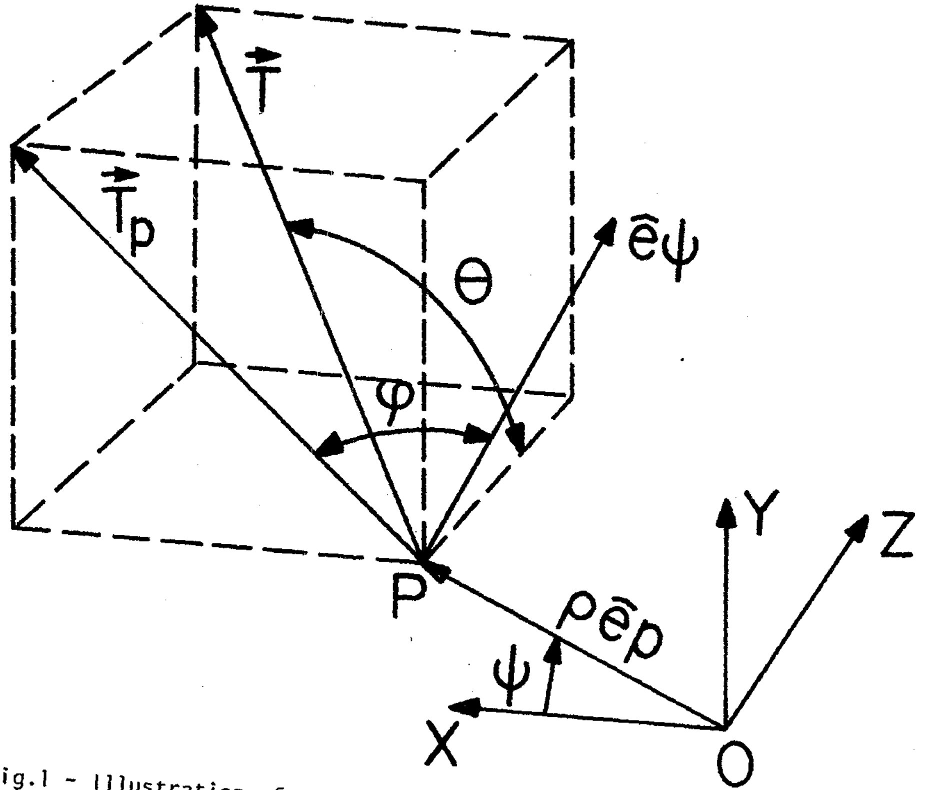


Fig.1 - Illustration of ray coordinates. $\vec{T}$ tangent to ray at point P. $\vec{T}_p$ is the projection of $\vec{T}$ onto the x-y plane (normal to fiber axis). The vector $\hat{e}_\psi$ is normal to OP and lies in x-y plane with sense as indicated

$$
T(L) = \frac{\dfrac{L}{c}\displaystyle\int_{\rho_t}^{\rho'_t} \frac{n(\rho)\left[n(\rho) - \lambda \frac{dn(\rho)}{d\lambda}\right] d\rho}{\left[n(\rho)^2 - n(\rho_0)^2\cos^2\theta_0 - n(\rho_0)^2\rho_0^2\sin^2\theta_0\cos^2\phi_0/\rho^2\right]^{1/2}}}{\displaystyle\int_{\rho_t}^{\rho'_t} \frac{n(\rho_0)\cos\theta_0 \; d\rho}{\left[n(\rho)^2 - n(\rho_0)^2\cos^2\theta_0 - n(\rho_0)^2\rho_0\sin^2\theta_0\cos^2\phi_0/\rho^2\right]^{1/2}}} \tag{13}
$$

3. A PARTICULAR REFRACTIVE INDEX PROFILE

For a given refractive index profile, described by the function $n(\rho)$, equations (9), (10) and (11) provide a complete set of differential equations which may be solved to provide the trajectory of a ray defined by its initial conditions ρ_0, θ_0, ψ_0. The solution in its most general form can only be performed in a few cases however; these include $n(\rho)$ equal to a constant within the core region of the fiber, and a parabolic function for $n(\rho)$. For other profiles, complete solutions may be found only if one restricts the calculation to meridional or skew rays of a particular kind. We shall consider the following profile, first introduced by Gloge and Marcatili[2]:

$$
\begin{aligned}
n(\rho)^2 &= n_0^2\left[1 - 2\Delta(\rho/a)^{\alpha}\right], \quad \rho \le a \\
&= n_0^2\left[1 - 2\Delta\right], \quad \rho \ge a .
\end{aligned} \tag{14}
$$

This profile includes the step-index and parabolic-index profiles as special cases, a being infinite for the first, equal to *2* for the latter. The general ray trajectories are not known, but for meridional rays we may readily perform the integrals in equations (4), (7) and (8). Neglecting the material dispersion term $dn/d\lambda$, we find[3]:

$$
\Lambda = \frac{4a\cos\theta_0}{(2\Delta)^{1/\alpha}} (\sin\theta_0)^{\frac{2-\alpha}{\alpha}} \sqrt{\pi}\, \frac{\Gamma(\frac{1}{\alpha})}{\Gamma(\frac{\alpha+2}{2\alpha})} , \tag{15}
$$

$$P = \frac{n_0}{c} \left(1 - \frac{2 \sin^2\theta_0}{\alpha + 2}\right) \Lambda/\cos\theta_0 \ , \tag{16}$$

$$T = \tau_0 \left(1 - \frac{2 \sin^2\theta_0}{\alpha + 2}\right) / \cos\theta_0 \ , \tag{17}$$

where the Γ's are Gamma functions, and $\tau_0 = n_0 L/c$ is the group delay time of the axial ray.

In Fig.(2), we have plotted the function T/τ_0 against θ_0 for various values of α. We note that for α=2 the curve has a relative minimum at α=0, and for α<2 the minimum shifts to the value of θ_0 given by

$$\cos\theta_m = \sqrt{\alpha/2} \ . \tag{18}$$

From equations (14) and (15), we find that at this value of θ_0, Λ has a relative maximum and P a relative minimum. This shows that the condition of equal delay times is equivalent to ideal focusing. The non axial ray which has the same delay time as the axial ray satisfies $\theta_0 = \theta_e$, where

$$\cos\theta_e = \alpha/2 \ . \tag{19}$$

From Eqs. (4) and (14), we observe that rays for which θ_0 exceeds the value θ_c defined by

$$\sin\theta_c = \sqrt{2\Delta} \tag{20}$$

would have turning radii greater than the fiber core radius a. These rays are refracted out of the core. In order that those rays with minimal delay time not be refracted, α must satisfy the inequality

$$\alpha \geq 2\ (1 - 2\Delta) \ . \tag{21}$$

On the other hand, for the non axial rays that have the same delay time as the axial ones (Eq.(19)) not to be refracted, we have the condition

$$\alpha \geq 2 \sqrt{1-2\Delta} \ . \tag{22}$$

Consider now all non-refracting rays, i.e. all rays for which θ_0 satisfies

$$0 < \theta_0 < \theta_c \ . \tag{23}$$

Inspection of Fig.(2) reveals then that the delay time of all these rays are confined to a fixed time interval which is determined by the delay times of the fastest ray subject to the condition expressed in equation (23), and the slowest ray. Let ΔT be this difference. Equation (17) then yields for ΔT:

$$\Delta T/\tau_0 = \left| \frac{1 - \frac{4\Delta}{\alpha+2}}{[1 - 2\Delta]^{1/2}} - 1 \right| \quad , \ \alpha \geq 2 \quad \text{or} \quad 0 \leq \alpha \leq 2(1-2\Delta) \ ,$$

$$= \left| \frac{2\sqrt{2\alpha}}{\alpha + 2} - 1 \right| \quad , \ 2(1 - 2\Delta) \leq \alpha \leq 2\sqrt{1-2\Delta} \ , \tag{24}$$

$$= \left| \frac{1 - \frac{4\Delta}{\alpha+2}}{\alpha + 2} - \frac{2\sqrt{2\alpha}}{\alpha + 2} \right| \ , \ 2\sqrt{1-2\Delta} \leq \alpha \leq 2 \ .$$

A graph of $\Delta T/\tau_0$ is plotted in Fig. (3). The interesting result is a remarkable near-equalization of all group delay times when α takes the value defined by

$$\alpha_0 = 2\sqrt{1-2\Delta} \tag{25}$$

which yields, for ΔT,

$$\frac{\Delta}{\tau_0} = 1 - \frac{2(1 - 2\Delta)^{1/4}}{1 + (1-2\Delta)^{1/2}} \ . \tag{26}$$

The profile for which α is given by Eq.(25) will be referred to as the "optimal" profile α_0.

Although the results of this Section were derived only for meridional rays, we shall show in the next Section that they are equally valid for all skew rays.

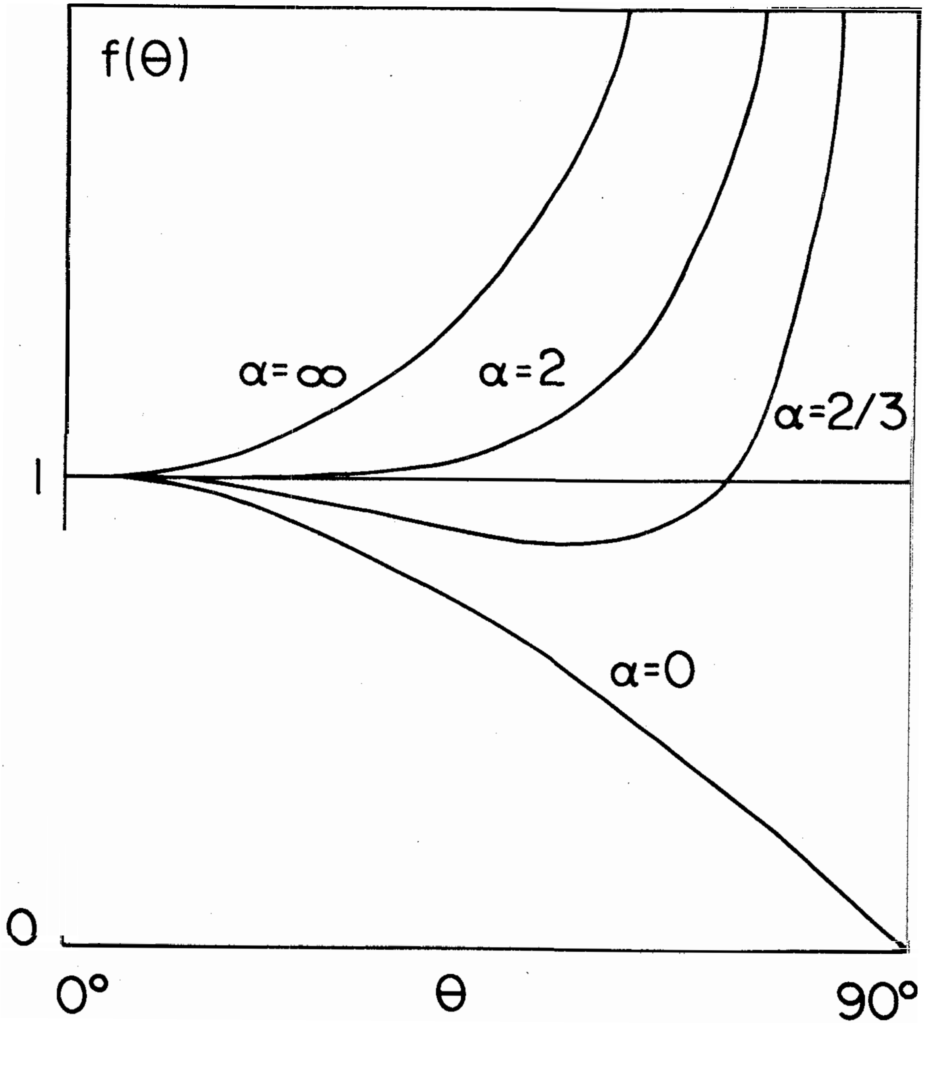


Fig.2 - Group delay time as a function of the angle θ for various values of alpha.

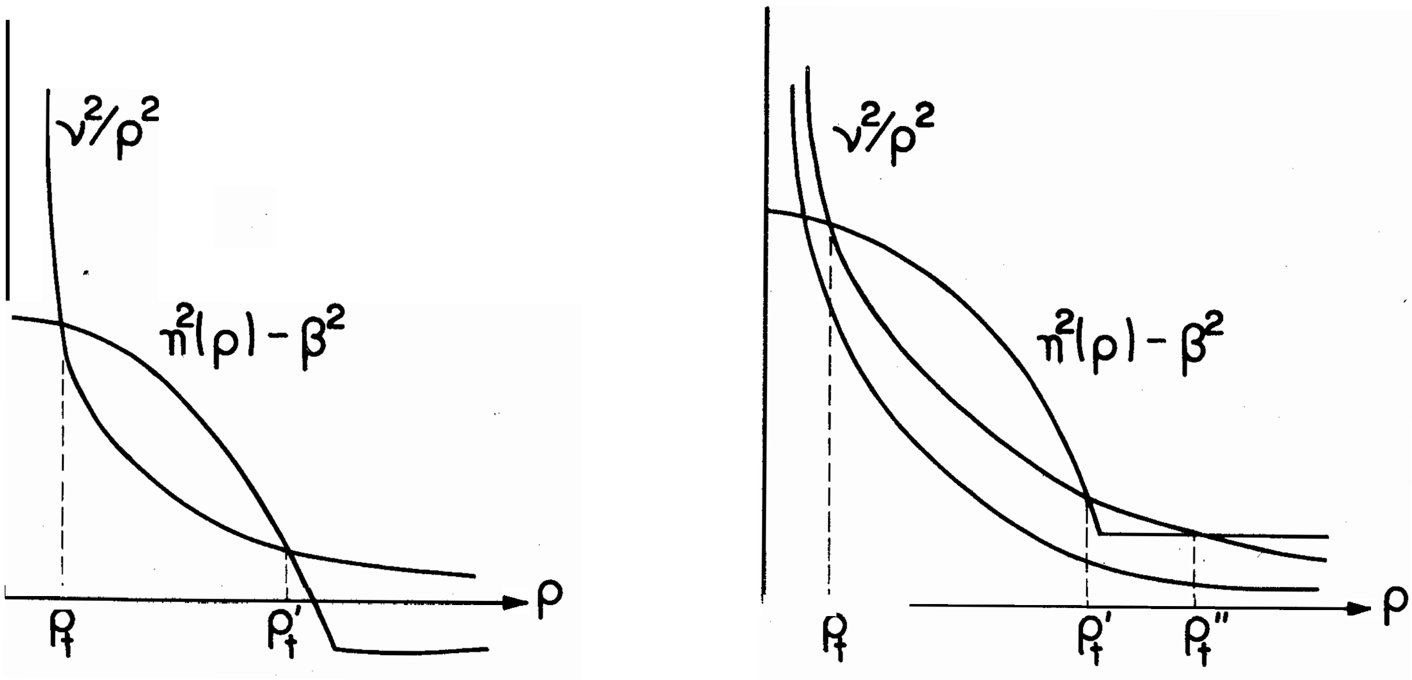


Fig.3 - Width of impulse response as function of alpha.

4. THE GROUP DELAY TIMES OF SKEW RAYS[4]

Introducing the abreviations

$$\beta = n(\rho_0)\cos\theta_0 \quad , \tag{27}$$

$$v = \rho_0 \; n(\rho_0)\sin\theta_0 \cos\phi_0 , \tag{28}$$

$$A(v) = \frac{L}{c}\int_{\rho_t}^{\rho_t'} \frac{n(\rho)^2 \, d\rho}{\left[n(\rho)^2 - \beta^2 - v^2/\rho^2\right]^{1/2}} \quad , \tag{29}$$

$$B(v) = \beta \int_{\rho_t}^{\rho_t'} \frac{d\rho}{\left[n(\rho)^2 - \beta^2 - v^2/\rho^2\right]^{1/2}} , \tag{30}$$

then by equation (12) the group delay time is written as the ratio

$$T(v) = A(v)/B(v) \; . \tag{31}$$

It is not possible to perform the integrals in equations (28) and (29) with $n(\rho)$ given by equation (13) and non-zero. We shall show, however, that

$$T(v)^{(n)} \Big|_{v=0} = 0 \; , \quad n = 1,2,3,\ldots, \tag{32}$$

where the superscript (n) denotes differentiation with respect to v^2. Equation (32) then states that the group delay times of skew rays for the profile given by equation (14) are independent of v. To prove equation (32), we note that the integrals

$$A^{(n)}(0) = \frac{1}{2}\cdot\frac{3}{2}\cdot\ldots\cdot\frac{(2n-1)}{2}\frac{L}{c}\int \frac{n^2(\rho)}{\left[n(\rho)^2 - \beta^2\right]^{(2n+1)/2}} \frac{d\rho}{\rho^{2n}} , \tag{33}$$

and

$$B^{(n)}(0) = \frac{1}{2}\cdot\frac{3}{2}\cdot\ldots\cdot\frac{(2n-1)}{2}\frac{L}{c}\int \frac{1}{\left[n(\rho)^2 - \beta^2\right]^{(2n+1)/2}} \frac{d\rho}{\rho^{2n}} , \tag{34}$$

can be evaluated. We find their ratio to be independent of n, and given by

$$\frac{A^{(n)}(0)}{B^{(n)}(0)} = \frac{A(0)}{B(0)} = \frac{\tau_0 \left(1 - \frac{2\sin^2 \xi}{\alpha+2}\right)}{\cos\xi} , \qquad (35)$$

where the angle ξ is defined by

$$\cos\xi = \beta/n(0) \qquad (36)$$

as in the meridional-ray case. Equation (32) then follows from equation (35) by a simple proof which is based on induction.

In summary, all rays propagating in an optical fiber described by the profile of equation (14) have group delay times given by equations (35) and (34). Insofar as equation (23) is accepted to define the skew rays under consideration, the equations (24) through (26) remain valid. However, we shall show that there exists a class of skew rays for which $\xi > \theta_c$, these are called "leaky rays".

5. LEAKY RAYS

From the above discussion, it is evident that not all regions of space are capable of transmiting a ray which is defined by given constants β and v. This condition is expressed by the inequality

$$n(\rho)^2 - \beta^2 - \frac{v^2}{\rho^2} \geq 0 . \qquad (37)$$

If $\beta < n(a)$, there may in fact be two regions of space that satisfy the inequality (36). This is best illustrated graphically in Fig.(4) where the functions $n^2(\rho) - \rho^2$ and v^2/ρ^2 are plotted.

The region satisfying

$$\rho_t < \rho < \rho_t' \qquad (38)$$

represents a cylinder within which the rays are trapped. The region that satisfies

$$\rho_t' < P < \rho_t'' \qquad (39)$$

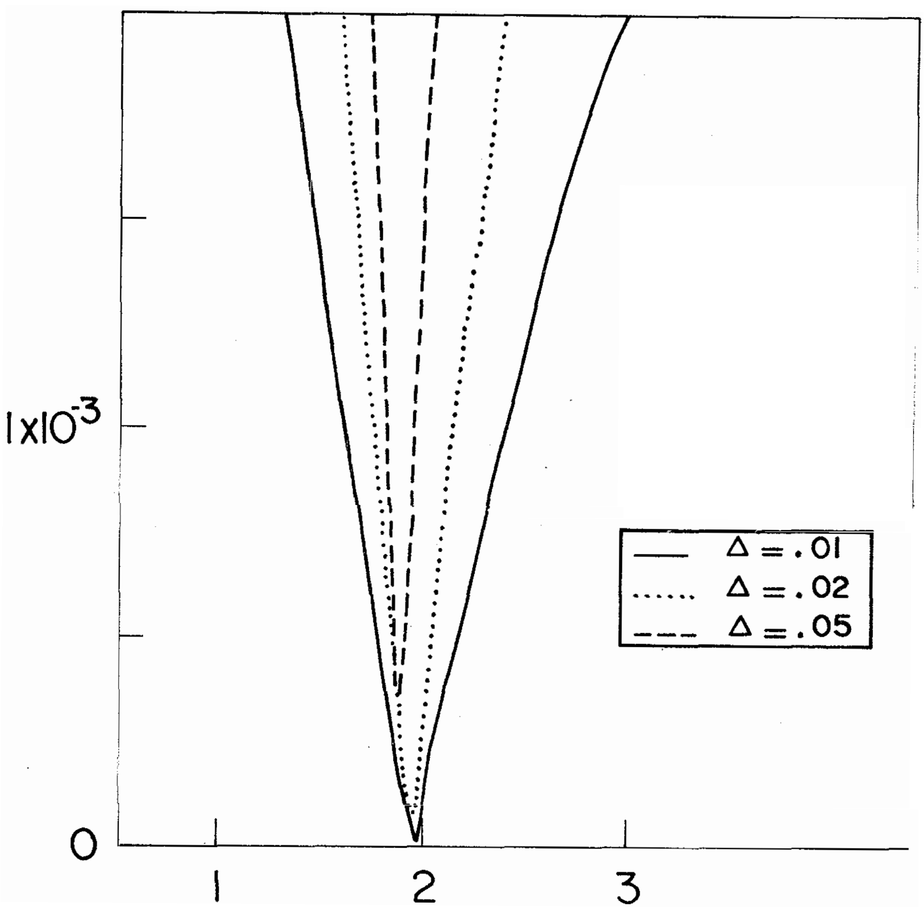


Fig.4 - Propagation **regions** for trapped, leaky and refracted rays.
(a) trapped ray ($\beta<n_c$).
(b) leaky ray (upper curve $\beta<n_c$ and $n_c^2-\beta^2-v^2/a^2<0$) and refracted ray (lower curve, $\beta<n_c$ and $n_c^2-\beta^2-v^2/a^2>0$).

is a cylindrical shell which is inaccessible to all rays with given constants v and p. Finally, the region

$$\rho > \rho_t'' \tag{40}$$

which extends to infinity, is again accessible. The regions between the radii ρ_t' and ρ_t'' becomes smaller when v decreases, and the condition for its existence, aside from $\beta<n(a)$, is given by

$$n(a)^2 - \beta^2 - \frac{v^2}{a^2} < 0 \ . \tag{41}$$

If the inequality (41) is not satisfied, the ray is no longer confined, and is refracted out of the core region. Such rays are of little interest for communication purposes.

Considering the situation depicted in Fig.4, we conclude from the above discussion based on ray optics that any ray defined by constants ρ and v and obeying Eq. (41) is completely trapped. In wave optics, this is not entirely the case: each time the ray meets the turning point ρ_t' it radiates some energy out of the guide. To see why this happens, we calculate the phase velocity of radiation in the region between ρ_t' and ρ_t''. Using equations from Section 2, we find this velocity to be

$$\begin{aligned} v^2 &= v_z^2 + v_\psi^2 \\ &= \frac{c^2}{n(\rho)^4}\left[\beta^2 + \frac{v^2}{\rho^2}\right], \quad \rho_t' < \rho < \rho_t'' \ . \end{aligned} \tag{42}$$

Now we solve for the radius ρ_m at which $v = c/n(\rho)$, the phase velocity of a free wave. By (41) we find this radius to satisfy

$$n(\rho_m)^2 - \beta^2 = v^2/\rho_m^2 \ . \tag{43}$$

But equation (41) is exactly the defining equation of the turning radius ρ_t''. At the radius ρ_t'' the phase velocity would begin to exceed the phase velocity of the free wave. This is not possible. The problem is resolved by an emission of radiation which emerges from a cylinder at $\rho = \rho_t''$ in the directions lying on a cone which is co-axial to the fiber axis and with apex angle γ given by

$$\cos \frac{1}{2}\gamma = \beta/n(a) \quad . \qquad (44)$$

The attenuation of these leaky rays may be estirnated in the following way. Consider a length dz of fiber. Let the fractional change in intensity of the light at each reflection be denoted by F. If $F \ll 1$, then after $dN = dz/\Lambda$ reflections the net loss is

$$dI(z) \simeq -\frac{dz}{\Lambda} F\, I(z). \qquad (45)$$

Integrating equation (45), we find that for a single ray an exponential attenuation law is satisfied:

$$I(z) = e^{-Fz/\Lambda}\, I(0) \quad . \qquad (46)$$

The factor F may be evaluated by a WKB solution of the scalar wave equation. It is found to be[5,6]

$$F \simeq \exp\left[-\frac{4\pi}{\lambda} \int_{\rho_t'}^{\rho_t''} \sqrt{\beta^2 + v^2/\rho^2 - n(\rho)^2}\; d\rho \right] \qquad (47)$$

in terms of the wavelength A of the light in air. From equation (47), we see that, in the limit $\lambda \to 0$, F vanishes, resulting in the ray optics prediction of zero attenuation.

6. TRANSMISSION OF SIGNALS: THE IMPULSE RESPONSE

In this Section, we shall study the excitation of the fiber by means of an incoherent light source. This source is assumed to be a light emitting surface which is pressed flush against the fiber end face. The distribution of intensity is supposed to be uniform over the source surface and isotropic in direction. Such a source is called "Lambertian", and is a realistic representation of the light-emmiting diode (LED). Let $W(\cos\theta)$ be the light intensity distribution inside the fiber, and $W'(\cos\theta')$ be that outside the fiber where by Snell's law, in terms of the index n_s of the source:

$$n_s \sin\theta' = n(\rho)\sin\theta \quad . \tag{48}$$

$W(\cos\theta)$ and $W'(\cos\theta')$ are related through the requirement that the energy flux is conserved:

$$W'(\cos\theta')\cos\theta'.\sin\theta' d\theta' = W(\cos\theta)\cos\theta\sin\theta d\theta \ . \tag{49}$$

Using (48) to eliminate θ', we obtain W in terms of W':

$$W(\cos\theta) = \frac{n(\rho)^2}{n_s^2} W'\left(\sqrt{1 - \frac{n(\rho)^2}{n_s^2}\sin^2\theta}\right). \tag{50}$$

Equation (50) relates the angular distribution of intensity inside the fiber to that of the source. Let further the intensity of the source have a time dependence given by the function $F(t)$, which we shall assume to be independent of angle. A ray which is incident at the point ρ, ψ and has direction defined by the angles 0, ϕ (see Fig. 1) will suffer a delay time in traversing the fiber length L according to the function $T(\rho,\psi,\theta,\phi)$, which for the profile of equation (13) we have found to be expressable in terms of a single parameter 8, given by

$$\beta = n(\rho)\cos\theta \ . \tag{51}$$

We postulate that the relationship between the output of the fiber and the input for a single ray is obtained by a sirnple time translation $T(\beta)$ and multiplication by an exponential attenuation factor due to various losses:

$$R(t) = e^{-\gamma(\rho,\psi,\theta,\phi)L} F(t - T(\beta)). \tag{52}$$

Integrating Eq.(52) over all rays gives the total flux at the exit end of the fiber:

$$R(t) = \int d\rho\, \rho \int d\psi \int d(\cos\theta) \int d\phi \cos\theta \ . \ W(\cos\theta)\ e^{-\gamma L}\ F(t - T(\beta)\ ). \tag{53}$$

Substituting equations (50) for W, with $W' = 1$ by the assumption of a Lambertian source, and assuming negligible loss, equation (53) simplifies to

$$R(t) = 2\pi \int d\rho \; n(\rho)^2 \int d(\cos\theta) \int d\phi \cos\theta \; . \; F(t - T(\beta)) \; , \tag{54}$$

ignoring constant factors other than the 2π due to the ψ integration. The impulse response $S(t)$ is obtained by assuming a delta function input:

$$F(t) = \delta(t) \; , \tag{55}$$

so that

$$S(t) = 2\pi \int d\rho \; . \; \rho n(\rho)^2 \int d(\cos\theta) \int d\phi \cos\theta \; . \; \delta(t - T(\beta)) . \tag{56}$$

The impulse response is a useful characterization of the fiber because the response $R(t)$ to an arbitrary input $F(t)$ may be found by calculating the convolution[7] of $S(t)$ with $F(t)$:

$$R(t) = S(t) \otimes F(t) \tag{57}$$

which is a basic property of stationary linear systems.

We note in passing that equations (53) through (57) are based on the assumptions of complete incoherence of the source as well as a perfect fiber. A coherent or partially coherent source will produce interferente which will disappear only after having passed a certain "coherence length" which depends on the coherence time of the source as well as index profile of the fiber. In addition, even if the source is completely incoherent, mode coupling effects due to imperfections in the fiber will invalidate the hypothesis of equation (52).

Returning to equation (54), we calculate the contribution to the impulse response due to trapped rays (i.e., $\beta > n(a)$) to be given by

$$S_T(t) = \int_{n(a)}^{n(0)} d\beta \; D_T(\beta) \; \delta(t - T(\beta)) \; , \tag{58}$$

where $D_T(\beta)$ is the energy flux due to rays in the interval β to $\beta+d\beta$:

$$D_T(\beta) = \frac{\pi^2}{2}\,\beta\, R(\beta)^2 \tag{59}$$

and $R(\beta)$ is defined by

$$n(R) = \beta\ . \tag{60}$$

Equations (58) through (60) were arrived at by integration of ϕ in equation (54) over the range 0 to $\pi/2$, using equation (51) to replace $\cos\theta$ by β as variable of integration, and interchanging the pand β integral~.The radial integration is then performed over the limits 0 to $R(\beta)$, defined by equation (60). For the profile of equation (13) solution of equation (60) yields:

$$R(\beta) = a\left(\frac{n(0)^2 - \beta^2}{2\Delta}\right)^{1/\alpha} \tag{61}$$

Performing the final integration gives from Eq. (58):

$$S_T(t) = \sum_n D_T(\beta_n)\left|\frac{d\beta_n}{dt}\right| , \tag{62}$$

where the sumnation is over all solution β_n of the equation $t = T(\beta_n)$, where $n(a) < \beta_n < n(0)$. The number of these solutions may be zero, one, or two (see Fig.2).

Expressions for the impulse response due to leaky rays, that is, rays for which $\beta<n(a)$ and $n(a)^2 - \beta^2 - v^2/a^2 < 0$, are also easily derived provide we neglect their attenuation. For these rays we find as in equation (58):

$$S_L(t) = \int_{\beta_m}^{n(a)} d\beta\,.\,D_L(\beta)\ \delta(t - T(\beta))\ , \tag{63}$$

where

$$D_L(\beta) = 2\beta\int_{\rho_m}^{a} d\rho\,.\,\rho\cos^{-1}\sqrt{\frac{a^2(n(a)^2 - \beta^2)}{\rho^2(n(\rho)^2-\beta^2)}}\ , \tag{64}$$

$$\beta_m = \sqrt{1-(\alpha+2)\Delta} \quad \text{if} \quad (\alpha+2)\Delta < 1 \;,$$
$$= 0 \qquad \text{otherwise} \;. \tag{65}$$

and ρ_m satisfies

$$a^2(n(a)^2 - \beta^2) = \rho_m^2(n(\rho_m)^2 - \beta^2) \;. \tag{66}$$

The expression for $S_T(t)$ in the step index case (α=∞) is particularly simple. We find from Eq.(62) $S_T(t)=0$ for all t, except

$$S_T(t) = \frac{1}{2} \; \frac{(\pi\tau_0 n_0 a)^2}{t^3} \quad \text{for} \quad \tau_0 \frac{n(0)}{n(a)} < t < \tau_0 \;. \tag{67}$$

For the parabolic profile (α=2) and the "optimum" profile ($\alpha=2\sqrt{1-2\Delta}$), we may find simple expressions by treating A as a small parameter. We then obtain approximately

$$S_T(t) = \frac{(\pi n_0 a)^2}{4\tau_0\Delta} \quad \text{for} \quad \tau_0 < t < \frac{\tau_0}{2}\left[\frac{n(0)}{n(a)} + \frac{n(a)}{n(0)}\right] \;, \tag{68}$$

for the parabolic profile, and

$$S_T(t) = \frac{(\pi n_0 a)^2}{2\tau_0}\left[\frac{1-\frac{2}{\alpha}}{2\Delta}\right]^{2/\alpha}\left[2\left(\frac{t}{\tau_m}-1\right)\right]^{-1/2} , \text{ for } \tau_m < t < \tau_0 \;, \tag{69}$$

for the "optimum" profile. It is interesting to note that to lowest order in A, only, the parabolic profile impulse response is altered by the presence of skew rays.

Closed form expressions for the impulse response due to leaky rays are also easily found. The formulas are somewhat unwieldy, however, so that they will be omitted. Numerical plots of the trapped and leaky ray impulse responses for various profiles are shown in Fig.(5).

We may also compute the total energy carried by the fiber in the form of trapped rays and in the form of leaky rays by integrating over ti-

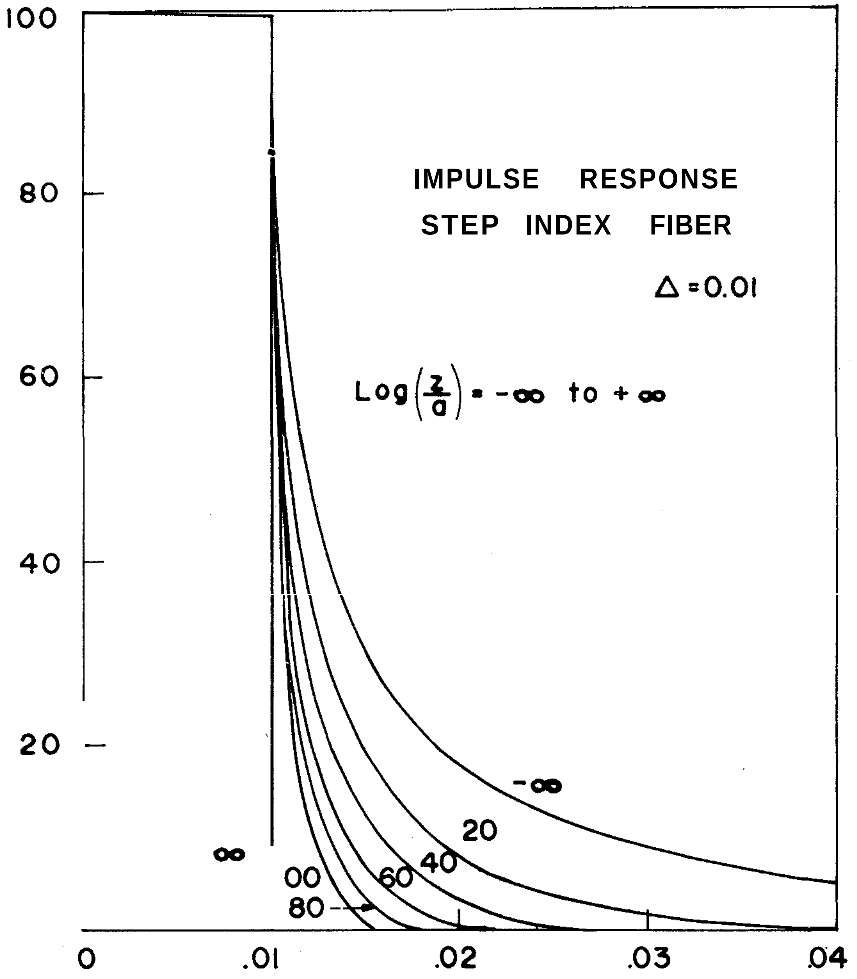
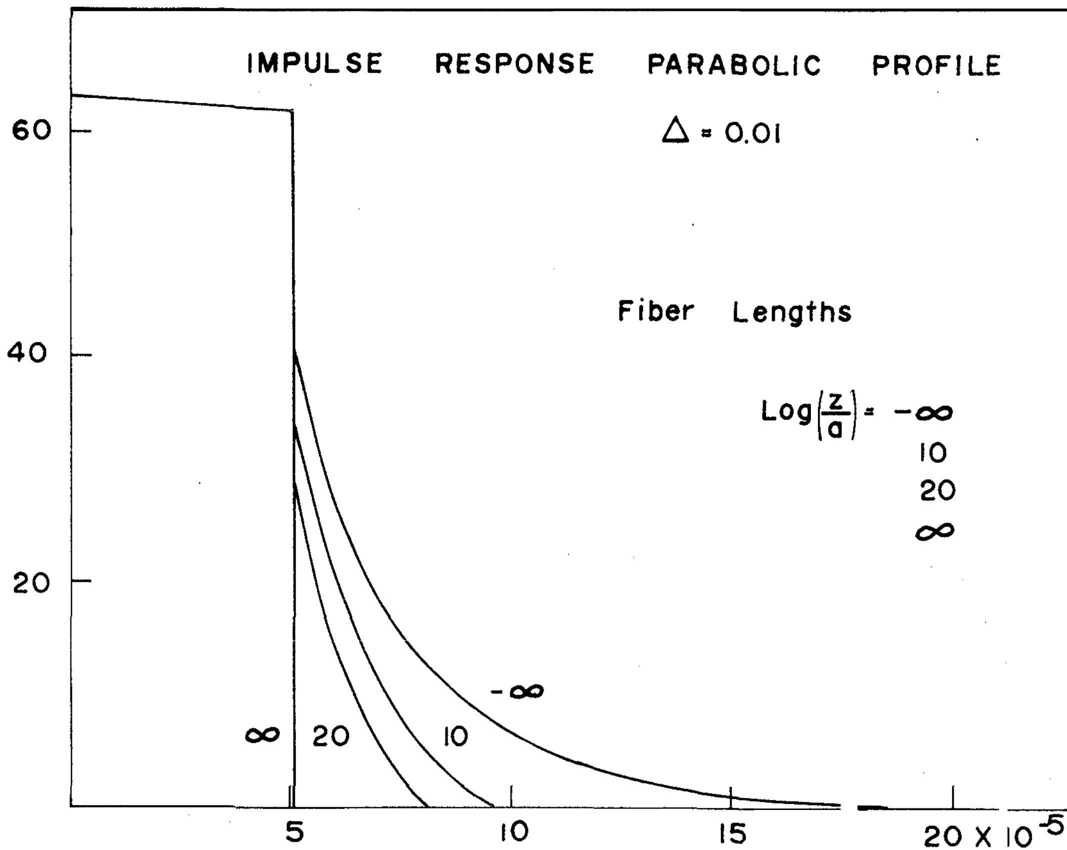


Fig.5 - Impulse response of various fiber types. Shown also is attenuation effect due to leaky rays. The latter was calculated with $2\pi n_0/\lambda = 1000$.

(a) Step index, (b) Parabolic index, (c) "Optimum" index, (d) Index with $\alpha=1$.



IMPULSE RESPONSE PARABOLIC PROFILE
$\Delta = 0.01$
Fiber Lengths
$\mathrm{Log}\left(\frac{z}{a}\right) = -\infty$
10
20
∞
$-\infty$
∞
20
10
60
40
20
5
10
15
20×10^{-5}

Fig. 5b

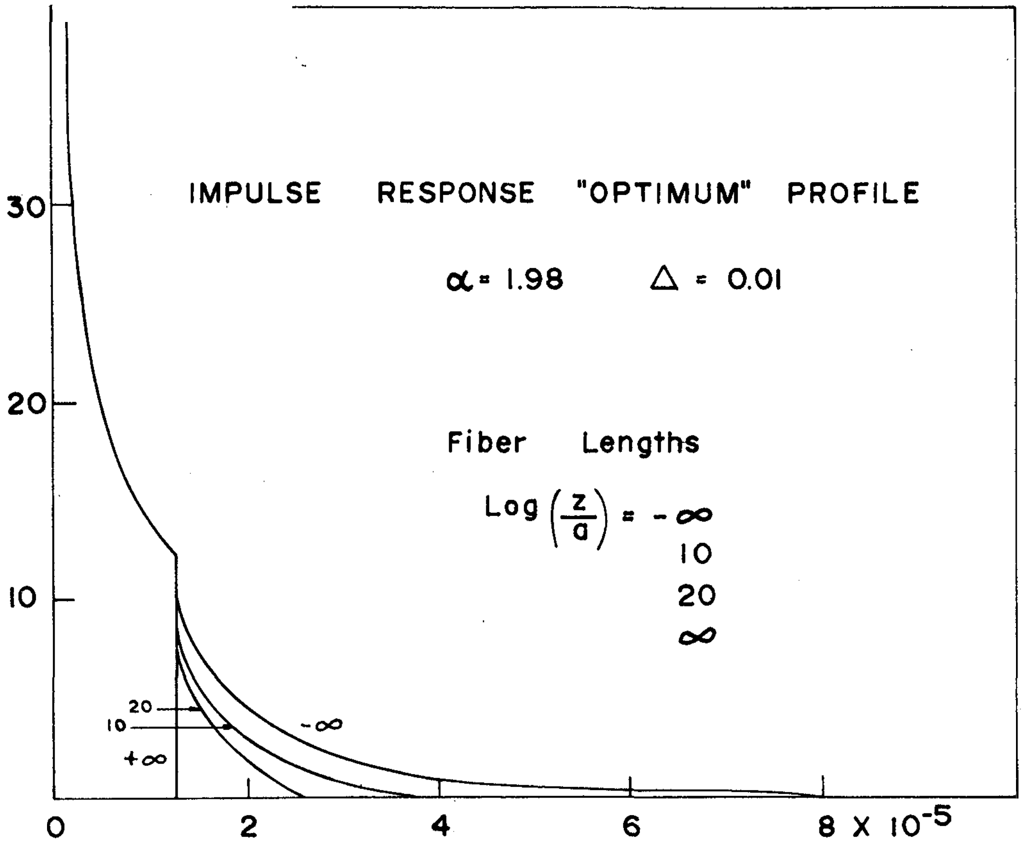

IMPULSE RESPONSE "OPTIMUM" PROFILE
α = 1.98 Δ = 0.01
Fiber Lengths
Log (z/a) = -∞
10
20
∞
30
20
10
20
10
+∞
-∞
0
2
4
6
8 X 10^-5

Fig. 5c

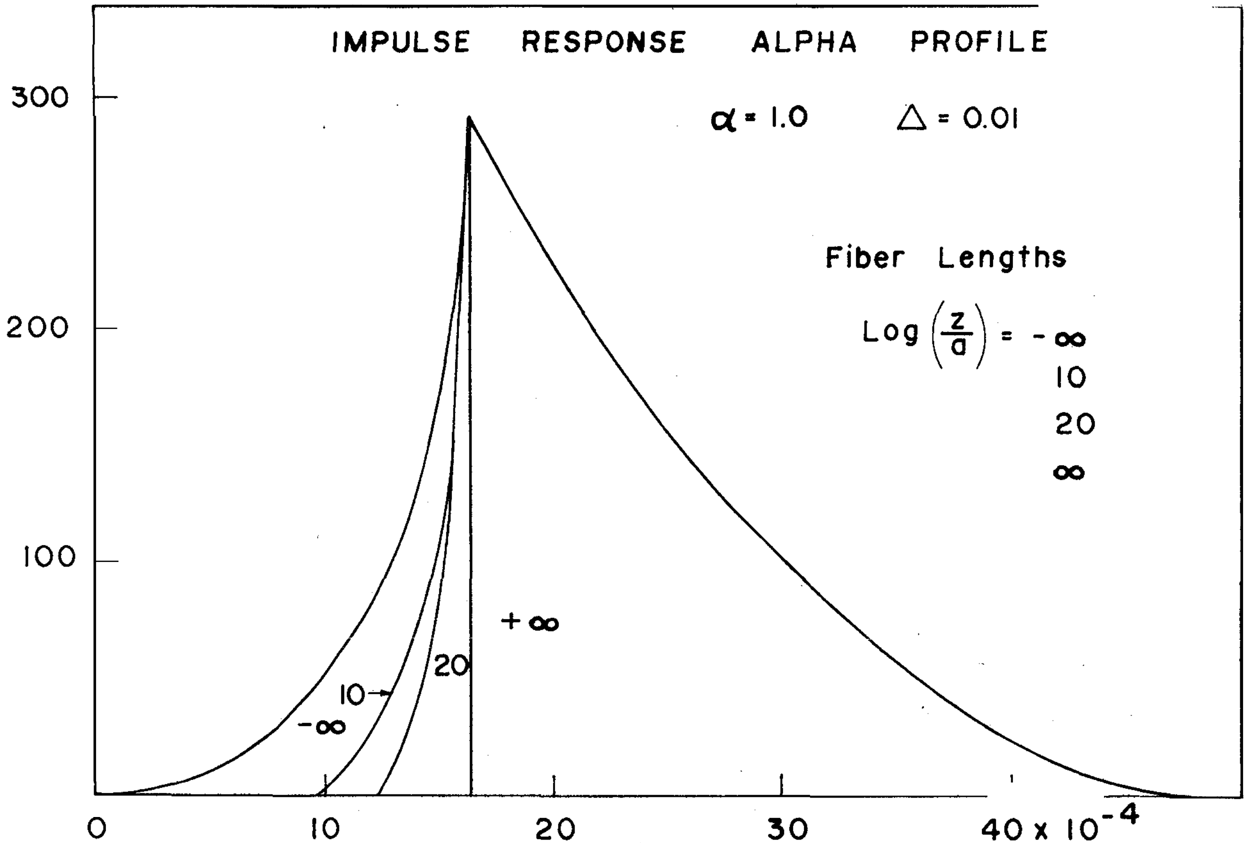

IMPULSE RESPONSE ALPHA PROFILE
α = 1.0 Δ = 0.01
Fiber Lengths
Log (z/a) = -∞
10
20
∞
300
200
100
0
10
20
30
40 x 10^-4
-∞
10
20
+∞

Fig. 5d

me the respective quantities $S_T(t)$ and $S_L(t)$, respectively. We find for the fraction of trapped rays the expression

$$\frac{W_T}{W_{T,L}} = \frac{\int_0^a \rho\left[n(\rho)^2 - n(a)^2\right] d\rho}{\int_0^a \rho\left[n(\rho)^2 - n(a)^2\right] / \sqrt{1-(\rho/a)^2}\ .\ d\rho}, \tag{70}$$

which for the profile of equation (13) is found to be

$$\frac{W_T}{W_{T,L}} = \frac{\alpha}{\alpha+2} \Big/ \left[2 - \frac{\Gamma(\frac{1}{2})\ \Gamma(\frac{\alpha}{2}+1)}{\Gamma(\frac{\alpha}{2}+\frac{3}{2})}\right]. \tag{71}$$

Equation (71) tells us that for the step-index fiber half of the energy in the impulse response is delivered by trapped rays, while for the parabolic fiber this ratio is 3/4.

The excitation of leaky rays represents a problem of economic significance in designing a communication system. As we have shown in Eq.(71), a Lambertian source wastes from 25 to 50% of its emitted power solely in this fashion. Actually, the loss is much greater because most rays of the Lambertian source are refracted out of the guide. If we count these refracted rays, we obtain for the fraction of guided power:

$$\frac{W_T}{W_{T,L,R}} = \frac{\alpha\ \Delta}{\alpha + 2(1 - \Delta)} \tag{72}$$

Because $\Delta \ll 1$, this ratio is very small, so that from the view point of coupling efficiency a directed source such as a laser is preferable.

7. PROPAGATION LOSSES

Of the various loss mechanisms, the simplest to describe is the material absorption loss. In our ray-optic description we assume the loss per unit path length of a ray to be given by the product of intensity $I_{\beta v}(s)$ and local absorption constant $\gamma(s)$:

$$\frac{d}{ds} I_{\beta v}(s) = -\gamma(s)\, I_{\beta v}(s) \quad . \tag{73}$$

Integration of equation (73) yields

$$I_{\beta v}(s) = I_{\beta v}(0) \cdot \exp\left[-\int_0^s \gamma(s)\,ds\right] , \tag{74}$$

where the integral is performed along the ray trajectory. A more useful representation of the loss factor may be found in terms of the modal absorption constant, defined in terms of fiber length L by

$$\gamma_{\beta v} = \frac{1}{L}\int_0^s \gamma(s)\ ds \quad . \tag{75}$$

By expressing the integral over path length as the product of the number of ray periods in the length L and the integral over one ray period, we find in the manner of Section 2:

$$\gamma_{\beta v} = \frac{2}{\Lambda_{\beta v}} \int_{\rho_t}^{\rho_t'} \frac{\gamma(\rho)n(\rho)}{\sqrt{n(\rho)^2 - \beta^2 - v^2/\rho^2}}\, d\rho \ . \tag{76}$$

The difficulty in applying equation (76) lies in our ignorance of the function $\gamma(\rho)$. The simplest assumption we can make is that $\gamma(\rho)$ is proportional to $n(\rho)$:

$$\gamma(\rho) = \gamma_0 n(\rho) \quad . \tag{77}$$

With this assumption the absorption becomes proportional to the delay time, leading to a simple expression for the impulse response:

$$S(t) = e^{-c\gamma_0 t}\ S_0(t) \ , \tag{78}$$

where $S_0(t)$ is the impulse response without loss. Because practical fibers have a loss of no more than a few decibels/km, equation (78) predicts very little alteration in the pulse shape. Greater effects onthe pulse shape will take place if $\gamma(\rho)$ has a greater spatial dependence than predicted by equation (77). In actuality, this may be the case because other mechanisms of loss such as. those that are due to micro-bending, variations in core diameter and defects in the cladding, protective jacket (in mono-mode fibers) or core-cladding interface ,will primarely cause losses in those rays which have $\beta \simeq n(a)$ and low "angular momentum" v.

The above mentioned imperfections of fibers may cause additional losses which one may classify into perhaps three categories. The first is that a ray may be scattered directly out of the fiber encountering the imperfection. This may be the case of defects such as tiny cracks in the cladding or core-cladding interface, or when there are very rapid variations in the core diameter or radius of curvature of fiber bends. Another mechanism is due to imperfections of low spatial frequency, which cause mode coupling between guided modes as well as radiative modes. In the language of ray optics, this means that a ray described by constants β and v gradually splits up into rays of different constants. This process continues until some of the energy has gone into radiative or leaky rays. The net effect is then not only an excess but an alteration of pulse shape. Gloge has shown[a] that after a minimum length of fiber has passed, an "equilibrium" state is arrived at in which the pulse attenuates uniformly, and broadens not in direct proportion to the fiber length but to square root of the fiber length. A final loss mechanism, discussed by Olshansky and Nolan[9], is caused by imperfections which have spatial. frequencies so low as to produce no mode coupling.

In this case there exists two adiabatic invariants, of which v is one, that remain unaltered during the motion of a ray. Loss is caused however by alteration of the ray's character from trapped to radiative which may happen if, for example, the core diameter is reduced. In the next Section, a greatly simplified discusssion of "mode-coupling" loss will be given.

8. MODE COUPLING LOSS

Limiting ourselves to meridional rays, we postulate that the power $P(\theta, z, t)$ in a ray characterized by the angle θ satisfies the following differential equation, in terms of the group velocity $v(\theta)$ and loss $\gamma(\theta)$:

$$v(\theta)\ \frac{\partial P}{\partial t} = -\ \gamma(\theta)P - \frac{\partial P}{\partial z} + D\,\frac{\partial^2 P}{\partial z^2}\ . \tag{79}$$

Ignoring the last terrn on the right hand side, we see that equation (53) is indeed a solution of equation (79). The last term represents a diffusion term which allows power to gradually distribute itself over all angles $\theta < \theta_c$. In the following treatment we shall make the assumption that the main extra loss of power comes from diffusion or rays to angles greater than θ_c. in this we differ from Gloge[8], who assumed the main extra loss to arise from greater loss when θ increases, with the angular dependence of γ as origin. We set γ equal to a constant, and ignore its θ dependence.

In steady state $\frac{\partial P}{\partial t} = 0$, and equation (79) becomes

$$\frac{\partial P}{\partial t} + \gamma_0 P = D\,\frac{\partial^2 P}{\partial \theta^2}\ . \tag{80}$$

Looking for solution of the form

$$P = e^{-\gamma' z}\ F(\theta)\ , \tag{81}$$

we obtain on substitution into equation (80)

$$(\gamma_0 - \gamma')F(\theta) = D\,\frac{d^2 F}{d\theta^2}\ . \tag{82}$$

Treating non-trapped rays as having infinite loss, F must satisfy the boundary condition

$$F(\theta_c) = 0\ . \tag{83}$$

With this boundary condition, equation (82) has two sets of solutions:

$$F(\theta) = \cos\left[(2n+1)\frac{\pi}{2}\,\theta/\theta_c\right], \quad n = 0,1,2,\ldots$$

and (84)

$$F(\theta) = \sin\left[n\pi\,\theta/\theta_c\right] \qquad n = 1,2,\ldots \quad .$$

In the first case, the loss is given by

$$\gamma' = \gamma_0 + \frac{(2n+1)^2\,\pi^2}{4\theta_c^2} D \ , \tag{85}$$

and in the latter by

$$\gamma' = \gamma_0 + \frac{n^2\pi^2}{\theta_c^2} D \tag{86}$$

The lowest loss is therefore

$$\gamma_0' = \gamma_0 + \frac{\pi^2}{4\theta_c^2} D \quad ,$$

or using $\theta_c^2 = 2\Delta$

$$\gamma_0' = \gamma_0 + \frac{\pi^2 D}{8\Delta} \quad . \tag{87}$$

Equation (87) is the loss constant belonging to the eigenfunction with least loss. It represents therefore the ultimate loss of a fiber in the limit of long fiber length.

We now give a rough explanation why in the limit of long fiber length, the width of the impulse response increases by the square root of the fiber length instead of in direct proportion. Suppose the length L is divided into N portions of length Δl each. Imagine that a "photon" traversing the fiber length L has at each interval Δl the option of choosing any of the available trapped modes in which to propagate. The total delay time, for N intervals, is then the sum of the individual delay times τ_n

$$T = \sum_{n=1} \tau_n \ . \tag{88}$$

Assuming complete independence of all τ_n, application of the elementary theory of random walk yields

$$\begin{aligned} \langle \Delta T^2 \rangle &= N \langle \Delta\tau^2 \rangle \\ &= L \frac{\langle \Delta\tau^2 \rangle}{\Delta l} \end{aligned} \tag{89}$$

which is the desired result.

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10. Strictly speaking, equation (10) is correct only for the slab guide, but we chose this form because of its simplicity, while the differences in results are of little importance in this Section.