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Conformal Invariant Quantum Field Theory and Composite Field Operators*

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The present status of conformal invariance in quantum field theory is reviewed from a non group theoretical point of view. Composite field operators dimensions are computed in some simple models and related to conformal symmetry.

Resenha-se a situação atual da invariância conforme em teoria de campos de um ponto de vista que não envolve teoria de grupos. Calculam-se as dimensões dos operadores de campos compostos em modelos simples relacionados à simetria conforme.

1. INTRODUCTION

As is well known, the conformal group, i.e., the Poincaré group, scaling (also called dilatation) and special conformal transformation of space--time, is the largest group which leaves invariant Maxwell's equations in vacuum. This fact was observed in the beginning of this century[1] and since that time many attempts were made[2] to incorporate the conformal symmetry into a quantum context. A complete review of the pioneering work can be found in Ref. 3.

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However, only in recent times the role played by conforma1 symnetry in quantum theory has been understood. This was achieved in two steps. First, the Weyl group (the subgroup which does not include the special conformal transformations) was studied, and after this, the group a as whole was taken into account. The first step was initiated by the works of Kastrup[4] and culminated in Mack's proposal[5] of a partially conserved dilatation current in analogy with some theories of internal symmetries which are verified approximately (PCAC). The need of a partia1 conservation in the case of dilatations is explained by the fact that exact scale invariance implies zero mass particles ora continuou~mass spectrum starting at the origin. Nevertheless, one expects the mass terms to be small at high energies and so one would have an asymptotic scale invariance. Bjorken[6] proposed certain asymptotic scale laws which motivated many attempts[7] to explain these laws as a manifestation of the asymptotic scale invariance.

In 1969, Wilson[e] introduced a fundamental concept for the understanding of scale symnetry: the anomalous dimension. Studying Johnson's solution[9] of the Thirring model[10], Wilson noted that the fermion has a definite scale factor under a space-time dilatation. It was surprising at that time that the scale factor does not coincide with the free field value, showing a dependence on the coupling constant. This means that, unlike the Poincaré symmetry, scale symmetry is linked to dynamics.

The systematic study of scale invariance was done by Callan[11] and Symanzik[12] by writing Ward identities for the dilatation current in terms of renormalized quantities. It was noticed that even theories with asymptotically vanishing mass terms could exhibit a scale invariance breaking in this limit. Of course, it was observed the possibility of asymptotic scale invariance but, surprisingly, it was noticed also the possibility of free field behavior for the asymptotic theory[13].

A motivation for the second step comes from the classical correspondente: from the classical point of view, for a large class of theories (including thosequantumversionswhich seemphysically relevant) scale invariance implies conformal invariance. Correspondingly, one expects this to be the case quantically or, if this is not the case, one would like to understand the nature of the quantum effects that are breaking the special conformal symmetry without breaking scale symnetry.

It is from the constructive point of view that one can see the usefulness of the special conformal symmetry. In the same way as the scale invariance fixes the two-point function of a quantum field theory, special conformal symmetry fixes the three-point function. Once the two and three-point functions are fixed, one can introduce them consistently into the Schwinger-Dyson equations (the nonhomogeneous term being absent because of conformal symrnetry). This results in numerical conditions for the field scale dimensions and coupling constants of the theory. One could thus construct the solution by inserting the self-energy and vertex parts into the skeleton expansion of any Green's function[**].

The main problem one has to face when dealing with theconformal group in quantum field theory comes from the peculiar feature of special conformal transformations being able to change space-like into time--like separations leading in this way to an apparent conflict with Einstein's causality. To see this explicitly, recall that a dilatations is a transformation of the form

$$x_\mu \to e^\rho x_\mu \qquad (\rho > 0)$$

and a special conformal transformation reads

$$x^\mu \to x^\mu_\tau = \frac{x^\mu - b^\mu x^2}{\sigma(x,b)} \quad ,$$

with $\sigma(x,b) = (1 - 2b.x + b^2x^2)$ and $b.x = -\vec{b}.\vec{x} + b_0 x_0$.

One can look at a special conformal transformation as a "local dilatation" since the line element "scales" as: $ds^2 \to [\sigma(x,b)]^{-2} ds^2$.

Let us see the apparent conflict with local commutativity (Einstein's causality) which asserts that the commutator of observables vanishes for space-like separations. One must observe that

a) a special conformal transformation can change the nature of a separation in Minkowski space:

$$(x-y)^2 = \sigma(x,b)\,(x_\tau - y_\tau)^2 \sigma(y,b) \;\; ;$$

b) scale invariance with anomalous dimensions forces the support of the vacuum expectation value of the commutator of observables to be concentrated in the interior of the light cone, this contrasting with the case of the free massless scalar field in four-dimensional space--time where the support is on the light cone.

Keeping in mind the above remarks and reasoning in analogy to known geometrical transformations (where the field is carried to the field in the transformed point multiplied by a certain factor), one recognizes a conflict with local commutativity.

Actually, the above considerations transcend the analogy because the existence of a unitary representation of the conformal group would imply[15] that (for example) the massless spin zero field would have a special conformal transformation of the form

$$\phi(x) \to |\sigma(x,b)|^{-d}\, \phi(x_\tau) \;\; ,$$

where d is the scale dimension of the field $\phi(x)$.

This conflict led Hortaçsu, Seiler and Schroer[16] to the concept of weak conformal invariance, formulated in terms of the invariance of the Euclidean Green's functions (where this conflict is absent).

However, the nature of the symmetry in Minkowski space would remain unsolved. The first progress in this direction was made by Swieca and Völkel[15] who, investigating the case of the free massless spin zero field in an arbitrary number of space-time dimensions, D, provedthat the special conformal transformation generators are essentially self-adjoint. This means that there is the unitary exponentiation of the special conformal symmetry. What does not exist is a unitary representation of the conformal group. Summarizing, these authors have shown how to go back from Euclidean to Minkowski space and this resulted in a nonlocal transformation for $D = 2n + 1$, n integer. The nonlocality means that the creation and annihilation parts of the free field for odd D (a situation in which the commutator has a support in the interior of the light cone) transform differently. This solves the apparent conflict. These authors conjectured the existence of a unitary representation of the universal covering group of the conformal group.

More recently[17,18], it was definitely established that in a local quantum field theory we are dealing with representations of the universal covering group of the conformal group. It was seen in a soluble model[17] that, analogously to the free field decomposition in creation and annihilation parts, the interacting field decomposes into "Fourier components" with respect to the center of the universal covering group of the conformal group which althoughnonlocal do transform in a simple way under the special conformal group. It can be shown that one does not have a true representation of the conformal group but, instead, the multiplication law of the elements of the representation is defined modulo a phase (ray representation). We have a true representation of the universal covering group of the conformal group. The same results were obtained later directly from the Wightman axioms[18].

The present paper is basically pedagogical in nature. We illustrate in a simple model the basic features of conformal symmetry as it is understood nowadays. We hope that an interested reader not familiar with group theoretical methods will profit from our selection of topics.

In Section 2, we review some early results restricting ourselves to the spin zero field case; we also obtain the most general classical conformally invariant theory of one spinless field. In Section 3, after a short review of the conformal algebra and infinitesimal transformations, we obtain the finite special conformal transformation for the spin zero field directly from the two-point function. This procedure allows us to easily obtain the finite transformations for the spin half field and the generalized free field. We show that in relativistic quantum field theory the relevant group is the universal covering of the conformal group[17]. As a preparation for the next Section, we also present in Section 3 an extensive discussion of the spin 0.5 field in two space-time dimensions and of the composite fields of this theory.

As we said earlier, we illustrate the basic features of conformal symmetry for interacting fields studying it in the generalized Schroer model in Section IV. We also study the composite fields of this model and relate its dimensions to the eigenvalues of the center operator of the universal covering of the conformal group. We will not, however, review conformal invariant operator product expansions, referring the interested reader to the original pioneer work of Ferrara, Gatto, Grillo and Parisi, Ref.32.

2. CONFORMAL SYMMETRY OF THE SPIN ZERO FIELD

2.1 Scale and Special Conforma1 Transformations for the Classical Spin Zero Field

First of all, we will introduce intuitively the scale and special umformal transformations in D space-time dimensions. To this end, we begin by considering the infinitesimal global dilatations

$$x'_\mu = (1 + \rho)x_\mu \quad . \tag{2.1}$$

From the condition that the field scales analogously to the space--time,

$$\phi'(x') = (1 - d\rho)\ \phi(x) \quad , \tag{2.2}$$

we obtain its infinitesimal variation $\delta_d\phi(x) = \phi'(x) - \phi(x)$,

$$\delta_d \phi(x) = -\rho(d + x^\mu \partial_\mu)\phi(x) \quad , \tag{2.3}$$

where this time we do not specify d.

If the Lagrangian scales as a field of the theory with $d = D$, the action, $I = \int d^D x \, L(x)$, will remain invariant;

$$\delta_D L = \rho(D + x^\mu \partial_\mu)L \quad . \tag{2.4}$$

This shows that we will have scale invariance in the case that the field scale dimension coincides with its mass dimension and dimensional constants are absent from the Lagrangian.

Recall that from Noether's theorem, one knows that for a symnetry transformation we have the conserved quantity

$$D^\mu_c(x) = \frac{\partial L}{\partial \phi_{,\mu}} \delta_d \phi + L \delta x^\mu \quad . \tag{2.5}$$

This means that if one takes the divergence of $D^\mu_c(x)$ and make use of (2.4) and the equation of motion, it follows that

$$\partial_\mu D^\mu(x) = \frac{\partial L}{\partial \phi} \delta_d \phi + \frac{\partial L}{\partial \phi_{,\mu}} \delta_d \partial_\mu \phi - \delta_d L = 0 \quad .$$

Now, if the action is not invariant,

$$\delta_d L = DL + x^\mu \partial_\mu L + \theta(x) \quad ,$$

and one gets

$$\partial_\mu D^\mu_c(x) = \theta(x) \quad .$$

Therefore, $\theta(x)$ expresses the breaking of scale symnetry in a local form.

In the following, we will introduce the special conforma1 transformations in analogy to the global dilatations and will obtain the necessary condition for special conforma1 invariance.

When the space-time is infinitesimally special conformal transformed,

$$x_\mu \to x'_\mu = x_\mu - b_\mu x^2 + 2b.x\, x_\mu \ , \tag{2.6}$$

the line element, ds^2, scales as

$$ds^2 \to (1 + 4b.x)ds^2 \ . \tag{2.7}$$

One can, therefore interpret a special conformal transformation as a local dilatation with a scale factor $(1 - 2b.x)$(note that (2.7) is quadratic in x) and one can write in analogy to (2.2):

$$\phi'(x') = (1 + 2b.x\ d)\ \phi(x) \ . \tag{2.8}$$

The infinitesimal variation $\delta_b\phi = \phi'(x) - \phi(x)$ will then be

$$\delta_b\phi(x) = b_\mu \delta^\mu_b \ \ \phi(x)$$

$$= (2db_\mu x^\mu + 2b_\mu x^\mu x^\nu \partial_\nu - x^2 b^\mu \partial_\mu)\ \phi(x) \ . \tag{2.9}$$

In the next Section we will obtain (2.9) by the method of induced representations[19] and the reader should regard the above procedure as a suggestion for the transformation of the Lorentz scalar field.

With (2.9), one obtains for the infinitesimal variation of the Lagrangian:

$$\delta^\mu L = (2x^\mu x^\nu - g^{\mu\nu}x^2)\partial_\nu L + 2Dx^\mu L \ +$$

$$2\ \frac{\partial L}{\partial\phi_{,\mu}}\ d\phi + 2x^\mu\left[\frac{\partial L}{\partial\phi}\ d\phi + \frac{\partial L}{\partial\phi_{,\nu}}\ \partial^\nu\phi + \frac{\partial L}{\partial\phi_{,\nu}}\ \partial^\nu\phi - DL\right] \ . \tag{2.10}$$

Note that we can combine the first two terms of (2.10) into a divergente $(\partial_\nu[(2x^\mu x^\nu - g^{\mu\nu}x^2)L])$. Moreover, in a scale invariant theory the term in brackets vanishes. Therefore, we will have special conformal invariance if we have scale invariance and the following condition is satisfied:

$$\frac{\partial L}{\partial \phi_{,\mu}} \, d\phi = 3\partial_\mu B(\phi) \ , \qquad (2.11)$$

where B is an arbitrary function of ϕ.

In this case, the variation of the Lagrangian reduces to a divergence. Recalling that (using the equations of motion) one can write

$$\delta^\mu L = \partial_\nu \left(\frac{\partial L}{\partial \phi_{,\nu}} \, \delta^\mu \phi \right) \qquad (2.12)$$

and subtracting (2.10) from (2.12), one gets

$$0 = \partial_\mu \left[\frac{\partial L}{\partial \phi_{,\nu}} \, \delta^\mu \phi - (2x^\mu x^\nu - g^{\mu\nu} x^2) L - g^{\mu\nu} 6B \right] \ . \qquad (2.13)$$

We have, then, the conserved currents

$$K^{\mu\nu}(x) = \frac{\partial L}{\partial \phi_{,\nu}} \left(2dx^\mu \phi + 2x^\mu x^\rho \partial_\rho \phi - x^2 \partial_\mu \phi\right) - (2x^\mu x^\nu - g^{\mu\nu} x^2) L - g^{\mu\nu} \, 6B \ . \qquad (2.14)$$

Let us write the most general Lagrangian that satisfies condition (2.11). Scale invariance imposes

$$L = f\left(\frac{\phi^m(x) \, (\partial_\mu \phi)^{2p}}{\phi^\ell(x)} \right) \phi^\ell(x) \, , \qquad (2.15)$$

with $md + 2p(d+1) = \ell d = D$, d being the dimension of the field ϕ, and D the space-time dimension. On the other hand, condition (2.1 1) imposes that

$$2pd\phi^{m+1} (\partial_\rho \phi)^{2p-2} (\partial_\mu \phi) \ f'\left(\frac{\phi^m (\partial_\mu \phi)^{2p}}{\phi^\ell} \right) = \partial_\mu 3B \ . \qquad (2.16)$$

But B does not depend on $(\partial_\mu \phi)$, therefore $(\partial_\rho \phi)^{2p-2} f'$ cannot depend on $(\partial_\mu \phi)$; anaiytically

$$(\partial_\rho \phi)^{2p-2} f' \left(\frac{\phi^m (\partial_\rho \phi)^{2p}}{\phi^\ell} \right) = h(\phi) \ .$$

One easily gets

$$L = g_1\phi^{n-2}(x)(\partial_\mu\phi)^2 + g_2\phi^\ell \; ; \tag{2.17}$$

g_1 and g_2 are dimensionless constants and the integers n and R are linked to the scale dimension d of the field $\phi(x)$ and to the space - time dimension D by

$$d = \frac{D-2}{n}, \quad \ell = \frac{nD}{D-2}, \quad n = 2,3,\ldots \; .$$

It is worthwhile to notice that only those theories with the usual kinetic energy term (though modulated by a "dielectric constant") exhibit conformal symmetry.

With the above machinery, one can easily show that (at least for a theory of one classical real scalar field) scale invariance implies special conformal invariance. In fact, one can, as in Ref.20, construct the energy-momentum tensor,

$$\theta_{\mu\nu}(x) = T_{\mu\nu}(x) + \frac{2dg_1}{3n}(\partial_\mu\partial_\nu - g_{\mu\nu}\Box)\phi^n(x), \tag{2.18}$$

generalizing the Belinfante construction to the conformal group ($T_{\mu\nu}$ is the canonical energy-momentum tensor). In terms of (2.18), the conformal currents, $K_{\mu\nu}$, read

$$K_{\mu\nu}(x) = 2x^\rho x_\nu \theta_{\mu\rho} - x^2\theta_{\mu\nu} \; . \tag{2.19}$$

We note from (2.19) that the conformal currents $K_{\mu\nu}$ will be conserved if and only if the trace of $\theta_{\mu\nu}$ vanishes. But from (2.18), one has $\theta_\mu{}^\mu = 0$ and this shows that one will have special conformal invariance if one has scale invariance.

2.2 Special Conformal Symmetry and the Vacuum Expectation Values[33]

In the following, we recall some constraints imposed by special conformal symmetry on the vacuum expectation values, and after we do briefly something analogous to the last subsection for the quantum case.

Consider first the two-point function

$$G(x) = <\phi_1(x)\phi_2(x)> . \tag{2.20}$$

Suppose a dilatation invariant vacuum and a unitary operator $U(\rho)$ implementing this dilatation:

$$U(\rho)|0> = |0> . \tag{2.21}$$

Then, introducing the operator $U^{-1}U$ in (2.20), one gets

$$G(x) = e^{(d_1+d_2)\rho} <\phi_1(e^{\rho}x)\phi_2(0)> , \tag{2.22}$$

where we have used

$$U(\rho)\phi_i(x)U^{-1}(\rho) = e^{d_i\rho}\phi_i(e^{\rho}x) , i = 1, 2 . \tag{2.23}$$

From (2.22), one concludes that

$$G(x) = c\left[- x_-^2\right]^{-(\frac{d_1+d_2}{2})} , \tag{2.24}$$

with $x_-^2 = \left[- \vec{x}^2 + (x_0 - i\varepsilon)^2\right]$.

Consider now an infinitesimal special conformal transformation. According to (2.8) , one has

$$G(x) = (1 + 2b.x\, d_1) <\phi_1(x_T)\, \phi(0)> . \tag{2.25}$$

Observing that $x_\tau^2 = x^2(1 + 2b.x)$ and recalling (2.22), it follows that

$$G(x) = c(1 + 2b.x\ d_1)\left[-x_-^2\right]^{-\frac{(d_1+d_2)}{2}}\left[1 - b.x(d_1 + d_2)\right] . \tag{2.26}$$

One then obtains the selection rule

$$d_1 = d_2 .$$

One therefore concludes that if we have a unitary operator implementing the special conformal symmetry and $U(b)|0\rangle=|0\rangle$, then the **unique** nonvanishing two-point function involves fields of the same dimension.

With an analogous calculation, one can show that the special conformal symmetry fixes the three-point function to be

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle \propto \left[(x_1-x_2)_-^2\right]^{\frac{1}{2}(d_3-d_1-d_2)}\left[(x_1-x_2)_-^2\right]^{\frac{1}{2}(d_2-d_1-d_3)} \times$$

$$\times \left[(x_2-x_3)_-^2\right]^{\frac{1}{2}(d_1-d_2-d_3)} ,$$

where d_i is the scale dimension of the field ϕ_i, $i = 1,2,3$.

In general, one can show that the special conformal symmetry restricts the number of independent variables of an n-point function to be (1/2) $n(n - 3)$.

Let us now recall that in terms of one-particle irreducible Green's functions $\Gamma^{(n)}(p_i,\mu,g)$ of a massive theory with coupling constant g, one has

$$\lim_{\lambda\to\infty} \Gamma^{(n)}(\lambda p_i,\mu,g) \to \lim_{\lambda\to\infty} \Gamma^{(n)}_{\text{pas}}(\lambda p_i,\mu,g) ,$$

where $\Gamma^{(n)}_{\text{pas}}$ satisfies the homogeneous Callan-Symanrik equation. But we also know that (for certain theories)

$$\lim_{\lambda\to\infty} \Gamma^{(n)}_{\text{pas}} (\lambda p_i,\mu,g) \to \lim_{\lambda\to\infty} K^n \Gamma^{(n)}_{\text{pas}} (\lambda p_i,\mu,\bar{g}(\infty)) ,$$

where K^n is some constant and $\Gamma^{(n)}_{\text{pas}} (\lambda p_i,\mu,\bar{g}(\infty))$ is exactly scale invariant.

Now, Schroer[21] showed by making use of the Zimmermann-Lowenstein "Normal Product" method that in the theory described by $\Gamma^{(n)}_{\text{pas}}(p_i,\mu,\bar{g}(\infty))$ the trace of the (renormalized) energy-momentum tensor vanishes. Therefore, analogously to the classical situation, one can construct the conserved spcial conformal currents and, since the function $\Gamma^{(n)}(p_i,\mu,g)$ approaches asymptotically $\Gamma^{(n)}_{\text{pas}}(p,\mu,\bar{g}(\infty))$, one concludes that one will have special conformal invariance if one has scale invariance.

3. THE CONFORMAL GROUP AND THE UNIVERSAL COVERING OF THE CONFORMAL GROUP

3.1 The Conformal Algebra and the Infinitesimal Conformal Transformations

In this Section, we obtain the infinitesimal conformal transformations using the method of induced representations. We will be brief and refer the reader to the References 19,22 for details. The Poincaré, $M_{\mu\nu}$ and P_μ, dilatation D and special conformal, K_μ, generators satisfy the following Lie algebra:

$$[M_{\mu\nu},M_{\rho\sigma}] = -i(g_{\mu\rho}M_{\nu\sigma} - g_{\nu\rho}M_{\mu\sigma} + g_{\mu\sigma}M_{\rho\nu} - g_{\nu\sigma}M_{\rho\mu}) , \tag{3.1a}$$

$$[M_{\mu\nu},P_\rho] = i(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu) , \tag{3.1b}$$

$$[P_\mu, P_\nu] = 0 , \tag{3.1c}$$

$$[M_{\mu\nu},D] = 0 , \tag{3.1d}$$

$$[P_\mu, D] = iP_\mu , \tag{3.1e}$$

$$[M_{\mu\nu},K_\rho] = -i(g_{\rho\mu}K_\nu - g_{\rho\nu}K_\mu) , \tag{3.1f}$$

$$[P_\mu, K_\nu] = 2i(g_{\mu\nu}D - M_{\mu\nu}) \quad , \tag{3.1g}$$

$$[K_\mu, K_\nu] = 0 \quad , \tag{3.1h}$$

$$[D, K_\mu] = iK_\mu \quad . \tag{3.1i}$$

Now one defines the infinitesimal generators $C_{\mu\nu}$, Δ and χ_μ by

$$[\phi(0), M_{\mu\nu}] = i\Sigma_{\mu\nu}\phi(0), \tag{3.2a}$$

$$[\phi(0), D] = i\Delta\phi(0), \tag{3.2b}$$

$$[\phi(0), K_\mu] = i\chi_\mu\phi(0) \quad . \tag{3.2c}$$

These matrices obey the Lie algebra **(3.1a,d,f,h,i)** with $M_{\mu\nu}$, D and K_μ substituted by $C_{\mu\nu}$, Δ and χ_μ, respectively.

In this paper, we will consider representations of this algebra **which involve** the number of fields required by the spin (then, $C_{\mu\nu}$ **is** the usual spin **matrix). It** follows, then, from the relation corresponding to **(2.1d)** and from Schur's **Lemma,** that Δ is a multiple of the identity; and from the relation analogous to (3.1 i) that $\chi_\mu=0$. This representation was called **I.*a*** in **Ref.19,** while the other possibilities $\chi_\mu \neq 0$ but finite and χ_μ infinite were denoted by **I.*b*** and **II,** respectively. **(In** the **following** Sections, we will analyze some soluble models and **we** will show some examples of **I.*b*** fields. For the **moment** we advance that the **I.*b*** fields can be obtained by differentiation of **I.*a*** fields present **in** the theory. We hope then that the fundamental fields of the theory are **I.*a*;** for example, they would form a "complete operator set" in thesensethat, acting on the vacuum with **all** operators of this set, one would **get** the whole Hilbert space).

We consider now a translation

$$[P_\mu, \phi(x)] = -i\partial_\mu\phi(x) \quad , \tag{3.3}$$

and compute the comnutator of the generators $Y (\equiv M_{\mu\nu}, D, K_\mu)$ with the field:

$$[\phi(x), Y] = [e^{iP.x}\,\phi(0)\,e^{-iP.x}, Y]$$
$$= e^{iP.x}[\phi(0), \tilde{Y}]e^{-iP.x} \quad , \qquad (3.4)$$

where

$$\tilde{Y} = e^{-iP.x}\,Y\,e^{iP.x} = \sum_n \frac{(-i)^n}{n!} x_{\mu_1}\ldots x_{\mu_n} [P^{\mu_1}, [P^{\mu_2}\ldots[P^{\mu_n}, Y]\ldots]].$$

With the comnutation relations (2.1), one can calculate the above sum, which reduces to a finite number of terms proportional to the generators of the conformal algebra.

'Then, with (3.2a,b,c) and (3.4), it follows that

$$[\phi(x), M_{\mu\nu}] = -i(x_\nu\partial_\mu - x_\mu\partial_\nu)\phi(x) + \Sigma_{\mu\nu}\phi(x) \quad , \qquad (3.5a)$$

$$[\phi(x), D] = i(\Delta + x^\mu\partial_\mu)\phi(x), \qquad (3.5b)$$

$$[\phi(x), K_\mu] = [2ix_\mu x^\nu\partial_\nu - ix^2\partial_\mu + 2x^\nu(g_{\mu\nu}i\Delta + \Sigma_{\mu\nu})]\phi(x). \qquad (3.5c)$$

Notice that these transformations agree with those we obtained for the spin zero field ($C_{\mu\nu} = 0$).

3.2 Finite Special Conformal Transformations for the Free Field

Our objective now is to determine the substitution law for the finite special conformal transformation of the quantized free field. To motivate our approach, we will recall the apparent conflict between a finite local dilatation and Einstein's causality.

We begin recalling that a special conformal transformation can change a space-like separatíon into a time-like one, leavíng the light cone invariant:

$$(x_2 - y_2)^2 = \frac{(x - y)}{\sigma(x,b)\ \sigma(y,b)} \tag{3.6}$$

For simplicity, we start by restricting ourselves to a theory withonly one neutra1 scalar field. From the hypothesis of scale invariance, one obtains for the vacuum expectation value of the comnutator:

$$<[\phi(x),\phi(y)]> = c\theta((x-y)^2)\varepsilon(x_0 - y_0)[(x-y)^2]^{-d} , \tag{3.7}$$

if d is non-integer. In the case d is an integer, the support of the comnutator will be on the light cone.

If one reasons in analogy to the known geometrical transformations where the field is carried from the original point to the transformed one, and multiplied by a factor, one should expect something like

$$\phi(x) \to \kappa(x,b)\phi(x_\tau) , \tag{3.8}$$

for a local dilatation. This relation being a symmetry transformation, the n-point function must remain invariant and, in particular, we should have for the vacuum expectation value of the commutator:

$$<[\phi(x),\phi(y)]> = \kappa(x,b)\kappa(y,b)<[\phi(x_\tau),\phi(y_\tau)]>. \tag{3.9}$$

The reader can verify (3.9) assuming the existence of an unitary operator that realizes (3.8) and leaves the vacuum invariant.

We notice now that, from the possibility of changing the nature of the separation, and from (3.7), this would be conflicting with local commutativity. We show in what follows that the conflict in Minkowski space is only apparent because it results from the hypothesis (3.8) which, in general, is incorrect. Our procedure consists in establishing the special conforma1 symmetry through the n-point function invariance.

In this Section, we will treat the free field case where it is necessary and sufficient to examine the two-point function invariance since the n-point functions reduce to products of two-point functions. For the free field, the vacuum expectation value of the commutator can be written as

$$<[\phi(x),\ \phi(y)]> = <\phi^-(x)\phi^+(y)> - <\phi^-(y)\phi^+(x)> \ . \quad (3.10)$$

Therefore, there won't be any conflict with Einstein's causality if the creation and the annihilation parts transform differently. We take this observation as our *Ansatz* verifying, of course, the two-point invariance:

$$<0|\phi^-(x)\phi^+(y)|0> = c[-(x-y)^2_-]^{-d} \ . \quad (3.11)$$

One can show that

$$[-(x-y)^2_-]^{-d} = [\sigma_-(x,b)]^{-d}[-(x_\tau-y_\tau)^2_{--}]^{-d}[\sigma_+(y,b)]^{-d}, \quad (3.12)$$

with

$$[\sigma_\pm(x,b)]^{-d} = [-b^2 \mp i\varepsilon b_0]^{-d}[-(x-\frac{b}{b^2})^2 \quad \mp i\varepsilon(x_0 - \frac{b_0}{b^2})]^{-d} \quad (3.13)$$

being the anaiytic continuation of the corresponding Euclidean expression from the respective positive and negative imaginary values of the b_0, x_0 variables. Then, we can write (3.11) as

$$<\phi^-(x)\phi^+(y)> = c[\sigma_-(x,b)]^{-d}[-(x_\tau - y_\tau)^2_{--}]^{-d}[\sigma_+(y,b)]^{-d} \ . \quad (3.14)$$

Consequently, there is a unitary operator $U(b)$ that realizes the transformation

$$U(b)\phi^\pm(x)U^{-1}(b) = [\sigma_\pm(x,b)]^{-d}\ \phi^\pm(x_\tau) \ . \quad (3.15)$$

We must observe that our method of obtaining the transformation leaves an indeterrnined phase. Nevertheless, as the reader can easily verify, we chose the phase in (3.15) in such a way that the field remains invariant at the origin according to (3.5c).

We conclude that there is no conflict with local commutativity because, for integer d (the support of the commutator being interior tothe light cone) the creation and annihilation parts transform differently, implying the commutator invariance in the following sense:

$$<\phi^-(x)\phi^+(y)> - <\phi^-(y)\phi^+(x)> = [\sigma_-(x,b)]^{-d}[\sigma_+(y,b)]^{-d}<\phi^-(x_\tau)\phi^+(y_\tau)>$$

$$- [\sigma_-(y,b)]^{-d}[\sigma_+(x,b)]^{-d}<\phi^-(y_\tau)\phi^+(x_\tau)>$$

$$= c\ \Theta((x-y)^2)\ \varepsilon(x_0 - y_0)[(x-y)^2]^{-d} \ . \quad (3.16)$$

The reader must note the peculiar nature of (3.15): we are dealing with a nonlocal transformation, contrary to what one would expect intuitively. The nonlocality means that the local field, $\phi(x)$, with non-integer dimension, is not taken from the original point to the transformed point, since the creation-annihilation decomposition is nonlocal.

The above procedure can be extended to the free field with spin different from zero and to the generalized free field.

The generalized free field (in the D-dimensional space-time) is defined by the two-point function (3.11) with all higher truncated n-point functions vanishing, for $d > (D-2)/2$. For certain values of d, we can offer an indirect but intuitive definition. For example, the generalized free field $\phi_{1/2}(x_0,x_1)$ of dimension 0.5, in the two dimensional space-time, can be viewed as

$$\phi_{1/2}(x_0,x_1) = \phi(x_0,x_1,0), \quad (3.17)$$

where $\phi(x_0,x_1,x_2)$ is a massless free field in three space-time dimensions:

$$\left[\frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right] \phi(x_0,x_1,x_2) = 0 \ . \quad (3.18)$$

Using (3.12), we conclude the special conformal invariance of the generalized free field of dimension d with the substitution rule

$$U(b)\phi_d^{\pm}(x)U^{-1}(b) = [\phi_{\pm}(x,b)]^{-d}\ \phi_d^{\pm}(x). \qquad (3.19)$$

In the same way, one can obtain the transformation for a field with spin 0.5. For example, the annihilation part transforms as

$$U(b)\psi^{(-)}(x)U^{-1}(b) = [1 - \gamma^{\mu}x_{\mu}\gamma^{\nu}b_{\nu}]\,[\sigma\_(x,b)]^{-d-\frac{1}{2}}\ \psi^{(-)}(x_{\tau}) \qquad (3.20)$$

and $\psi^{(+)}(x)$ has in general a different transformation law.

3.3 The Universal Covering of the Conformal Group and the Operator Z

We mentioned in the Introduction that the existence of a unitary representation of the conformal group would be, in general, incompatiblewith local commutativity and, therefore, with the substitution law (3.15). In fact, consider the transformation

$$\phi^{\pm\prime}(x) = e^{iK.b}\ \phi^{\pm}(x)\ e^{-iK.b} \qquad (3.21)$$

rewritten as

$$\phi^{\pm\prime}(x) = e^{iK.b}\ e^{iP.x}\ \phi^{\pm}(0)\ e^{-iP.x}\ e^{-iK.b} \qquad (3.22)$$

It was shown in Ref.16 that a generic element of the conformal group can be decomposed into a special conformal transformation, a Lorentz transformation, a dilatation and a translation; characterizing these transformations by the parameters c_{ν} $\omega_{\mu\nu}$, λ and a_{μ}, respectively, and applying it to our problem, we have

$$e^{iK_{\mu}b^{\mu}}\ e^{iP_{\mu}x^{\mu}} = e^{iP_{\mu}a^{\mu}}\ e^{iD\lambda}\ e^{iM^{\mu\nu}\omega_{\mu\nu}}\ e^{iK_{\mu}c^{\mu}}\ . \qquad (3.23)$$

Recalling now that, according to the Subsection 3.1 , K_{μ} and $M_{\mu\nu}$ act trivially at the origin, we have

$$\phi^{\pm}(x) = e^{-\lambda d}\ \phi^{\pm}(a)\ . \qquad (3.24)$$

One calculates h and a_μ by comparing the effect of the transformations represented in both sides of (3.23) at an arbitrary point; it results in

$$a_\mu = \frac{x_\mu - b_\mu x^2}{\sigma(x,b)} \equiv x_{\mu\tau} , \tag{3.25a}$$

$$\lambda = \log |\sigma(x,b)| . \tag{3.25b}$$

We would have, then,

$$e^{iK.b} \phi(x) e^{-iK.b} = |\sigma(x,b)|^{-d} \phi(x_\tau) . \tag{3.26}$$

The transformation law (3.26) is compatible with (3.15) only for D = $4\ell + 2$, R integer; in this case, we have representations of the conformal group. In the other cases, we are dealing with unitary representations of the universal covering of the conformal group[17]. To see this, let us recall the concept of projective multiplication law.

One says that one has a ray multiplication law in the case that the operators compose as

$$U(g_1)U(g_2) = e^{i\alpha(g_1,g_2)} U(g_1,g_2) ,$$

with a non-trivial phase $\alpha(g_1,g_2)$. Recall also that the rotation group is a familiar example of the following theorem due to Bargmann[23]: "The ray multiplication law for certain groups (including the conformal one) in a certain representation space implies (the usual) representations of their universal covering groups".

Consider now two general conformal transformations, C_1 and C_2. Let us look for the multiplication law of the operators $U(C_1)$ and $U(C_2)$. Observing that the transformations (3.15) and (3.20) differ only by a phase, one concludes that the effect of the operator $U^{-1}(C_2.C_1)U(C_2)U(C_1)$ is to multiply the field by a phase

$$U^{-1}(C_2 \cdot C_1)U(C_2)\,U(C_1)\,\phi^{\pm}(x)\quad U^{-1}(C_1)U^{-1}(C_2)U(C_2 \cdot C_1) =$$

$$= e^{\pm i\Theta(x,C_1,C_2)d}\,\phi^{\pm}(x)\ . \qquad (3.27)$$

One can simplify the calculation of $\Theta(x,C_1,C_2)$ observing that the phase is independent of x. In fact, applying the D'Alembertian to both sides of (3.27), one gets

$$[(\Box\ \exp\{-id\theta\})\ \phi^{\pm}(x)] + 2\partial^{\mu}\ \exp\{-id\theta\}\partial_{\mu}\phi^{\pm}(x) = 0\ .$$

Taking the matrix element of this equation between the vacuum and an one particle state with momentum p, and using the arbitrariness of p, one concludes that θ is independent of x.

One can therefore compute $\theta(C_1,C_2)$ by considering (3.27) with $x=0$; decomposing the transformation C_i canonically into a product of a dilatation e^{λ_i}. Lorentz transformation Λ_i, special conformal transformation b_i, and translation a_i, one has

$$U^{-1}(C_2,C_1)\ \frac{e^{(\lambda_1+\lambda_2)d}}{[\sigma_+(\Lambda_2 e^{\lambda_2}a_1 . b_2)]^d}\ \phi^{\pm}(\Lambda_2 e^{\lambda_2}a_1 + a_2)\ U(C_2 . C_1) = e^{\mp id\Theta}\ \phi^{\pm}(0)\ .$$

$$(3.28)$$

Operating with $U(C_2.C_1)\ldots U^{-1}(C_2.C_1)$ in (3.28), and observing that this transformation does not introduce any phase on the right hand side, it follows that

$$\theta d = \arg\ [\sigma_+(\Lambda_2\ e^{\lambda_2}a_1,b_2)]^d\ . \qquad (3.29)$$

Recalling that one is working near the real axis, it follows $d=\pm(\frac{D-2}{2})\pi$, $\pm(\frac{D-2}{2})\,2\pi$. One has also the possibility $\theta d = 0$, for C_1 and C_2 sufficiently close to the identity.

The nontrivial phase for $D \neq 4\ell + 2$ suggests the existence of a projective multiplication law. In fact, one can rewrite (3.28) as

$$\exp\{-id\theta N\}\ \phi^{\pm}(0)\ \exp\{id\theta N\} = e^{\mp id\theta}\ \phi^{\pm}(0), \qquad (3.30)$$

as is readily seen applying (3.30) to an eigenstate of the number operator N. Therefore, one can identify the operator $Z(C_2,C_1) = U^{-1}(C_2.C_1)$ $U(C_2)U(C_1)$ as

$$Z(C_2,C_1) = e^{-id\theta N} \quad . \qquad (3.31)$$

Therefore, for D $\neq 4\ell + 2$, we have a nontrivial operator Z and in its eigensectors one has a ray representation of the conformal group:

$$U(C_2)U(C_1) = e^{-id\theta n}\ U(C_2.C_1) \ , \qquad (3.32)$$

and following from Bargmann's theorem that we are dealing with (the usual) representations of the universal covering of the conformal group.

The operator Z plays a fundamental role since in order to know the conformal group multiplication law one must know its eigensectors. In the present example, Z is written in terms of the number operator, N, and since N is conserved only for the free field, this suggests that the Z operator is linked to dynamics. In fact, in the next Section we will consider an interacting field model and we will verify the existenceof a Z operator (different from (3.31)) linked to the dynamics.

On the other hand, it was shown generally that if one knows the eigenvalues of Z, one knows the dimensions of all composite operators occurring in the theory. Thus we can ask the inverse question: is the knowledge of the dimensions of all l.a composite fields equivalent to the knowledge of all eigenvalues of Z? The importance of this question lies in the fact that once one has a global operator expansion its answer is positive. In the following Section, we will answer affirmatively the above question in a soluble model (together with the inverse question raised above as an assertion).

The observations of the preceding paragraph allow us to conclude the dynamical nature of conforma1 symmetry. With the transformation law

(3.15), one can write the special conformal transformation of the local field $\phi(x)$ as:

$$U(b)\phi(x)U^{-1}(b) = \frac{1}{|\sigma(x,b)|^{\overline{d}}} e^{-id\theta N} \phi(x_\tau) e^{id\theta N} , \qquad (3.33a)$$

where 8 = arg $\sigma_+(x,b)$. Analogously, for a Lorentz scalar I.a interacting field, $A(x)$, one can expect to express the special conformal transformation as the effect of a Z operator (different from (3.31)):

$$U(b)A(x)U^{-1}(b) = |\sigma(x,b)|^{-d_A} Z(\theta)A(x_\tau)Z^{-1}(\theta) , \qquad (3.33b)$$

with 0 = arg $\sigma_+(x,b)$. Then from the fact that the dimnsions of al1 composite field are contained in Z, it follows that the transformation law of the field $A(x)$ involves the dimensions of all composite operators.

3.4 Free Fermion in D = 2: the Z Operator and the Conforma1 Composite Operators

In the next Section, we will deal with a soluble model of interacting fields in which the free fermion in D = 2 plays a central role. In particular, to get the dynamical dimensions of the composite fields, one must know the dimensions of the free fermion composite fields. For this reason, we study now the free fermion in D = 2.

We will use the basis

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \quad \gamma^1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} , \quad \gamma^5 = \gamma^0\gamma^1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} .$$

The two-point function for the free fermion $\psi_0(x)$ reads

$$<0|\psi_0(x)\psi_0^+(x)|0> = c \begin{bmatrix} (x^0 + x^1 - i\varepsilon)^{-1} & 0 \\ 0 & (x^0 - x^1 - i\varepsilon)^{-1} \end{bmatrix} . \qquad (3.34)$$

From (2.20), one has

$$U(b)\psi_0^{(-)}(x)U^{-1}(b) = [\sigma_-(x,b)]^{-1}\{1 - [\gamma_1 x_1 + \gamma_0(x_0 - i\varepsilon)] \times$$

$$\times [-\gamma_1 b_1 + \gamma_0(b_0 - i\tilde{\varepsilon})]\} \quad . \tag{3.35}$$

It is convenient to introduce the light-cone variables:

$$\begin{aligned} u &= x^0 + x^1 \quad , \\ v &= x^0 - x^1 \quad , \end{aligned} \tag{3.36}$$

and rewrite (3.35) in terms of these variables

$$U(b)\psi_0^{(-)}(u,v)U^{-1}(b) = \begin{bmatrix} [1-(b_v - i\varepsilon)(u-i\varepsilon)]^{-1} & 0 \\ 0 & [1-(b_u - i\varepsilon)(v-i\varepsilon)]^{-1} \end{bmatrix} \psi_0^{(-)}(u_\tau, v_\tau) \quad , \tag{3.37}$$

where $b_u = b^0 + b^1$, $b_v = b^0 - b^1$, and

$$u_\tau = \frac{u}{1 - b_v u} \quad , \qquad v_\tau = \frac{v}{1 - b_u v} \quad ;$$

one also has

$$<0|\psi_0^+(x)\psi_0(0)|0> = c\begin{bmatrix} (x^0+x^1-i\varepsilon)^{-1} & 0 \\ 0 & (x^0-x^1-i\varepsilon)^{-1} \end{bmatrix} . \tag{3.38}$$

Therefore, we can proceed as in Subsection 2.2, obtaining for $\psi_0^{+(+)}(x)$ the same factor in the transformation as for $\psi_0^{(-)}(x)$. Since $\psi_0^{(+)} = [\psi_0^{+(-)}]^+$, it follows that

$$U(b)\psi_0^{(+)}(u,v)U^{-1}(b) = \begin{bmatrix} [1-(b_v+i\varepsilon)(u+i\varepsilon)]^{-1} & 0 \\ 0 & [1-(b_u+i\varepsilon)(v+i\varepsilon)]^{-1} \end{bmatrix} \psi_0^{(+)}(u_\tau,v_\tau) . \tag{3.39}$$

Notice that in two space-time dimensions, the creation and annihilation parts of the free fermion transform in the same way, but this does not conflict with Einstein's causality because the anticommutator of $\psi_0(x)$ has a support on the light cone and this implies a support on the light cone for the commutator of the observables.

To get the operator analogous to Z (which we will denote by Z_3), we will proceed as in the last Subsection. As in (3.23-26), it is readily shown that the existence of a unitary representation of the conformal group would imply the special conformal transformation law:

$$U(b)\psi_0(u,v)U^{-1}(b) = \begin{bmatrix} \frac{1}{|1-b_v u|} & 0 \\ 0 & \frac{1}{|1-b_u v|} \end{bmatrix} \psi_0(u_\tau,v_\tau) . \tag{3.40}$$

In the same way as it has been obtained (3.27), one can get the effect of the operator $Z_3 = U^{-1}(C_2 . C_1)U(C_2)U(C_1)$ (C_1 and C_2 being two general conformal transformations):

$$Z_3\ \psi_0(u,v)\ Z_3^{-1} = \exp\{-i\begin{pmatrix}\theta_1 & 0\\ 0 & \theta_2\end{pmatrix}\}\ \psi_0(u,v) . \tag{3.41}$$

With the same arguments of the preceding Subsection, it is readily shown that θ_1 and θ_2 are independent of x and determined by 0, $\pm\pi$. Then,

$$Z_3 = \exp\{-i[(\frac{Q+\tilde{Q}}{2})\ \theta_2 + (\frac{Q-\tilde{Q}}{2})\ \theta_1]\} , \tag{3.42}$$

where Q and $\tilde{Q}$ are, respectively, the charge and the pseudo-charge.

One reads the multiplication law in an eigensector of Z_3 as

$$U(C_2)U(C_1) = \exp\{-i[(\frac{q+\tilde{q}}{2})\theta_2 + (\frac{q-\tilde{q}}{2})\theta_1]\}U(C_2.C_1) \quad . \tag{3.43}$$

Again, it follows from Bargmann's theorem that we are dealing with (usual) representations of the universal covering of the conformal group.

We will try to get the dimension for the composite fields starting constructively from $\psi_0(x)$ and $\psi_0^{+}(x)$ and their derivatives. Two facts should be noted: in terms of light cone variables the equation of motion is

$$\psi_{01} = \psi_{01}(u), \quad \psi_{02} = \psi_{02}(u) \quad . \tag{3.44}$$

Therefore, we can separate the construction of composite fields into fields that depend only on u, and fields that dependent only on v. Another fact comes from statistics:

$$\psi_{01}^2(u) = 0 = \psi_{02}^2(v) \quad , \quad \{\psi_{01}(u),\psi_{02}(v)\} = 0. \tag{3.45}$$

This means that the normal product of a field such as

$$\ldots \psi_{01}(u)\psi_{02}(v) \quad \frac{\partial}{\partial n} \quad \psi_{01}(u)\psi_{01}^{+}(u)\psi_{02}(v) \ \ldots$$

vanishes. So, we have to introduce a sufficient number of derivatives in the construction of composite fields to prevent them from trivially vanishing.

The conformal fields whose dimensions we are trying to determine belong to the conformal algebra representation in which the infinitesimal generator vanishes. So, we will exclude fields that transform as $\frac{\partial}{\partial u}\psi_0(u)$:

$$U(b) \frac{\partial}{\partial u} \psi_{01}(u)U^{-1}(b) = b_v(1 - b_v u)^{-2} \psi_{01}(u_\tau)$$
$$+ (1 - b_v u)^{-3} \frac{\partial}{\partial u_\tau} \psi_{01}(u_\tau) \quad , \tag{3.46}$$

and we will call "spurious", terms like $b_v(1-b_v u)^{-2}\,\psi_{01}(u_\tau)$ above. According with subsection 3.2, we will denote by I.a , fields that transform with $\chi_\mu = 0$ and by I.b , fields that transform with $\chi_\mu \neq 0$ (as for example $\frac{\partial}{\partial u}\psi_{01}(u)$).

We will illustrate the construction process by forming the fields starting from $\psi_{01}(u)$. We can get the transformation law far any composite field since we know how to transforrn ψ_0. Hence, it is a simple task to determine the general infinitesimal special conforma1 transformation law which we will need :

$$U(b):\partial^{m_1}\psi_{01}(u)\partial^{m_2}\psi_{01}(u)\ldots\partial^{m_k}\psi_{01}(u):U^{-1}(b) =$$

$$= \sum_{j=1}^{k} m_j^2 b_v:\partial_\tau^{m_1}\psi_{01}(u_\tau)\ldots\partial^{m_j}\psi_{01}(u_\tau)\ \ldots:$$

$$+ \left[1 + (2\sum_{j=1}^{k} m_j + K)b_v u\right]:\partial_\tau^{m_1}\psi_{01}(u_\tau)\ldots\partial_\tau^{m_k}\psi_{01}(u_\tau): \quad , \tag{3.47}$$

where

$$\partial^{m_j} = \frac{\partial^{m_j}}{\partial u^{m_j}}\ , \quad \partial_\tau^{m_j} = \frac{m_j}{\partial u_\tau^{m_j}}\ .$$

We see that the lowest dimensional field which we can build from q fields $\psi_{01}(u)$ is

$$U^0(q) = :\partial^{q-1}\psi_{01}(u)\partial^{q-2}\psi_{01}(u)\ \ldots\ \partial\psi_{01}(u)\psi_{01}(u): \ , \tag{3.48}$$

whose dimension is $q^2/2$. From (3.45) and (3.47), we conclude that the field $U^0(q)$ is I. a. To construct composite fields from q fields $\psi_{01}(u)$ of higher dimension we must take derivatives of $U^0(q)$. As it happened with $U^0(1)$, it may occur that the resultant field is not I.a. Nevertheless, we can combine I.b fields such that the result is a I.a field.

First, let us notice a general fact from the example:

$$U^1(2) = :\partial^1\psi_{01}(u)\psi_{01}(u): \ . \qquad (3.49)$$

There is no I.a field in charge sector with dimension $2^2/2 + 1$. This follows from (3.47) applied to $U^1(2)$: the spurious term $:\partial\psi_{01}(u)\psi_{01}(u):$ could only be compensated by $:\partial\psi_{01}(u)\partial\psi_{01}(u): = 0$. For the same reason, it is not possible to build an I.a field of dimension $q^2/2 + 1$.

Let us go on in the charge 2 sector, trying to determine which are the allowed dimensions. It is simple to verify that the field

$$U^2(2) = \frac{1}{3^2} :\partial^3\psi_{01}(u)\psi_{01}(u) - \partial^2\psi_{01}(u)\partial\psi_{01}(u): \qquad (3.50)$$

is I.a and that the field

$$U^3(2) = \frac{1}{4^2} :\partial^4\psi_{01}(u)\psi_{01}(u) - \partial^3\psi_{01}(u)\partial\psi_{01}(u): \qquad (3.51)$$

is I.b. The reason for not being possible to construct a I.a field with the dimension of $U^3(2)$ is the following: in the process of subtracting the spurious terms we notice that we have to add $:a^3\$_{01}(u)\partial\psi_{01}(u):$ to $(1/4^2):\partial^4\psi_{01}(u)\psi_{01}(u):$ in order to cancel the spurious term coming from the last one. But the field $:\partial^3\psi_{01}(u)\partial\psi_{01}(u):$ will generate the spurious term $:\partial^2\psi_{01}(u)\partial\psi_{01}(u):$ which could only be cancelled by $:\partial^2\psi_{01}(u)\partial^2\psi_{01}(u): = 0$.

In general, we see that we must introduce the field

$$:\partial^{n+1}\psi_{01}(u)\,\partial^{n-1}\psi_{01}(u): \qquad (3.52)$$

to build $U^{2n-1}(2)$. But the field *(3.52)* will generate the spurious term $:a^n\$_{01}(u)\partial^{n-1}\psi_{01}(u):$ which cannot be cancelled.

On the other hand, the construction of a I.a field of dimension $q^2/2 + 2n$ is possible because the process closes itself with the field

$$:\partial^{n+1}\psi_{01}(u)\partial^{n}\psi_{01}(u): \qquad (3.53)$$

which does not introduce spurious terms due to *(3.45)*. We conclude that in charge 2 sector there are l.a fields, $U^{2n}(2)$, with dimension:

$$\dim U^{2n}(2) = \frac{q^2}{2} + 2n \quad . \tag{3.54}$$

we can explicitly write

$$U^{2n}(2) = \frac{1}{(2n+1)}\, \partial^{2n+1}\psi_{01}(u)\psi_{01}(u) - \partial^{2n}\psi_{01}(u)\partial\psi_{01}(u) \ \ldots$$

$$\ldots \frac{[(2n)^2(2n-1)^2\ldots(j+1)^2]}{[(2n+1-j)^2(2n-j)^2\ldots 2^2]}\, \partial^{j}\psi_{01}(u)\partial^{2n+1-j}\psi_{01}(u)\ldots$$

$$\ldots \frac{[(2n)^2\ldots(n+2)^2]}{[n^2(n-1)^2\ldots 2^2\ ]}\, \partial^{n+1}\psi_{01}(u)\partial^{n}\psi_{01}(u) \quad . \tag{3.55}$$

Unfortunately, in sectors with charge larger than two the situation is not so simple. We will write below the initial l.a fields in the charge *3* sector:

$$U^2(3) = \frac{1}{4^2} :\partial^4\psi_{01}(u)\partial\psi_{01}(u)\psi_{01}(u) - \frac{1}{2^2}\,\partial^3\psi_{01}(u)\partial^2\psi_{01}(u)\psi_{01}(u):$$

$$U^3(3) = \frac{1}{5^2}:\partial^5\psi_{01}(u)\partial\psi_{01}(u)\psi_{01}(u) - \frac{1}{2^2}\,\partial^4\psi_{01}(u)\partial\psi_{01}(u)\psi_{01}(u):$$

$$U^4(3) = \frac{1}{6^2} :\partial^6\psi_{01}(u)\partial\psi_{01}(u)\psi_{01}(u) - \frac{1}{2^2}\,\partial^5\psi_{01}(u)\partial^2\psi_{01}(u)\psi_{01}(u)$$

$$+ \frac{5^2}{3^2.2^2}\,\partial^4\psi_{01}(u)\partial^3\psi_{01}(u)\psi_{01}(u): \quad .$$

Notice that the even-odd rule for the charge *2* sector does not apply to charge 3 sector. We can understand this fact observing that when we introduce $2n$ derivatives in $U^0(2)$ we form $n+1$ I.b fields (notice that the number of formed fields is equal to the number of partitions[24] of the integer n in two parts). On the other hand, with 2n–1 derivatives we form n I.b fields.

When we combine the n+l fields resulting from the introduction of $2n$ derivatives in $U^0(2)$ weget a composite field that will generate n spurious terms (because the spurious terms have one derivative less). Therefore, we can use the arbitrariness of the n+l coefficients in the linear combination of the n+l I.b fieldsto cancel the n spurious fields resulting in a system of n homogeneous equations and n+l unknowns, whose solution was already written (3.55). Naturally, in cornbining n composite fields formed by the insertion of 2n-1 derivatives in $U^0(2)$, we get a field that will generate spurious terms; we will have in this case a system of n homogeneous equations in n unknowns and this is the reason why we cannot build I.a fields introducing an odd number of derivatives in $U^0(2)$.

However, in the charge 3 sector for a number of derivatives larger than one, the number of unknowns is sufficient. Again, this is seen by computing the number of I.b fields to be combined against the numberof spurious terms generated by them (remember: the spurious terms have one less derivative than the field they form). Denoting by N_n^3 the number of I.b fields formed by introducing n derivatives in $U^0(3)$, there are six possibilities:

$$N^3_{3m+3} - N^3_{3m+2} = \begin{cases} \dfrac{m+2}{2} & \text{even } m\,, \\[2ex] \dfrac{m+3}{2} & \text{odd } m\,, \end{cases}$$

$$N^3_{3m+2} - N^3_{3m+1} = \begin{cases} \dfrac{m+2}{2} & \text{even } m\,, \\[2ex] \dfrac{m+1}{2} & \text{odd } m\,, \end{cases}$$

$$N^3_{3m+1} - N^3_{3m} = \begin{cases} \frac{m}{2} & \text{even } m \\ \frac{m+1}{2} & \text{odd } m;\ 0,1,2,\ \ldots \end{cases}$$

Therefore, the number of equations (with exception of $n=1$) is less than the number of unknowns.

It can be readily shown that $N^m_{n+1} > N^m_n$. We concl de that when we introduce n (n > 1) derivatives in $U^0(m)$, we obtain a number of fields larger than the number of spurious terms generated by them. So unless there is a numerical accident (for it may happen that the system of equations has no solution), the allowed dimensions will be $q^2/2 + n$, with n an integer larger than'l for $q \geq 3$, and an even integer for $q = 2$.

In the same way, we determine the dimensions for the fields composed by $\psi^+_{01}(u)$, $\psi_{02}(v)$ and $\psi^+_{02}(v)$, namely, $U^n_{-q_2}(u)$, $V^\ell_{q_3}(v)$ and $V^k_{-q_4}(v)$. It would remain the task of composing those fields among themselves and with $U^n_q(u)$, including new derivatives. But, we will see that for our purposes it is enough to know that a general field will have dimension

$$\frac{q^2}{2} + \frac{q_2^2}{2} + \frac{q_3^2}{2} + \frac{q_4^2}{2} + \text{integer}\ . \tag{3.56}$$

We are able to show that the knowledge of Z is equivelent to know the dimension of the composite fields. Since we have the decomposition in the u and v variables, we can decompose the operator Z in Z_+, acting on the fields that depend only on u,

$$Z_+ = e^{-i\theta_1\left(\frac{Q-\tilde{Q}}{2}\right)}\ , \tag{3.57a}$$

and Z_- acting on the fields that depend only on v:

$$Z_- = e^{-i\theta_2\left(\frac{Q-\tilde{Q}}{2}\right)}\ , \tag{3.57b}$$

with

$$Z = Z_+ Z_- \ .$$

For any field, $O(u)$, constructed with q_1 fields $\psi_{01}(u)$ and q, fields $\psi^+_{01}(u)$, we have

$$Z_+ \, O(u) \, Z_+^{-1} = \exp\{-i\theta_1 \, 2d_u\} \, O(u), \tag{3.58}$$

with $d_u = q_1^2/2 + q_2^2/2 + n$, the integer n being subject to the restrictions of the preceding paragraphs.

On the other hand, we know from (3.57a) that the eigenvalues of Z_+ are

$$\text{eigenvalues } (Z_+) = e^{-i\theta_1 \left(\frac{q-\tilde{q}}{2}\right)} \tag{3.59}$$

But $q = q_1 - q_2$ and $\tilde{q} = -(q_1 - q_2)$, so

$$\text{eigenvalues } (Z_+) = e^{-i\theta_1 (q_1 - q_2)} \ , \tag{3.60}$$

and because $\theta_1 = 0, \pm\pi$, we have

$$\text{eigenvalues } (Z_+) = e^{-i\theta_1 (q_1^2 + q_2^2 + n)} \ . \tag{3.61}$$

By an analogous procedure, one can show that for fields composed by q_3 fields $\psi_{02}(v)$ and q_4 fields $\psi^+_{02}(v)$, one has

$$\text{eigenvalues } (Z_-) = e^{-i\theta_2 (q_3^2 + q_4^2 + m)} = e^{i\theta_2 d_v} \tag{3.62}$$

Therefore, all eigenvalues of the operator Z are written in the form $\exp\{-i[\theta_1 2d_u + \theta_2 2d_v]\}$, d_u and d_v being the dimensions of the composite fields in a free fermion theory. Those results will be useful in the next Section.

4. CONFORMAL SYMMETRY OF INTERACTING FIELDS

4.1. Schroer's Generalized Model[25]

The classical version of the model that we will use as a laboratory for the conformal symmetry study is defined by the Lagrangian

$$L = \frac{1}{2}\left[(\partial_\mu \phi)^2 + (\partial_\mu \tilde{\phi})^2\right] + \bar{\psi}\gamma^\mu \partial_\mu \psi(x) + ig\bar{\psi}\gamma^\mu \psi \partial_\mu \phi + i\tilde{g}\bar{\psi}\gamma^\mu \gamma^5 \psi \partial_\mu \tilde{\phi}(x) \qquad (4.1)$$

in two space-time dimensions. See[26] for a similar study in the Thirring model.

The scalar field, $\phi(x)$, and the pseudoscalar field, $\tilde{\phi}(x)$, are free fields, i.e.

$$\Box \phi = 0 \ ,$$

$$\Box \tilde{\phi} = 0 \ ,$$

and the equation of motion of the fermion is

$$\gamma^\mu \partial_\mu \psi(x) = ig\gamma^\mu \psi(x) \partial_\mu \phi(x) + i\tilde{g}\gamma^\mu \gamma^5 \psi(x) \partial_\mu \tilde{\phi}(x) \ . \qquad (4.2)$$

We get the solution constructing the fields $\phi(x)$ and $\tilde{\phi}(x)$ in independent Hilbert spaces (this construction requires certain care due to infrared divergences - see Refs. 27 and 28, for example). The proof of the positivity in the model is given in Ref.27, and we mention only that the basic prerequisite is charge conservation. With the help of the free fermion $\psi_0(x)$, previously defined, we see that the solution of (4.2) is

$$\psi(x) = \exp\{ig\phi(x)\} \exp\{i\tilde{g}\gamma^5 \tilde{\phi}(x)\}\psi_0(x) \ . \qquad (4.3)$$

Therefore, the $\psi(x)$ field is defined in $H_\phi \otimes H_{\tilde{\phi}} \otimes H_{\psi_0}$ and the exponentials are properly regularized:

$$\exp\{ig\phi(x)\} = \exp\{ig\phi^+(x)\} \exp\{ig\phi^-(x)\} \ , \qquad (4.4)$$

$$\exp\{i\tilde{g}\gamma^5 \tilde{\phi}(x)\} = \exp\{i\tilde{g}\gamma^5 \tilde{\phi}^+(x)\} \exp\{i\tilde{g}\gamma^5 \tilde{\phi}^-(x)\} \ . \qquad (4.5)$$

It should be mentioned that the $\psi(x)$ field has the same Lorentz spin as $\psi_0(x)$.

It will be necessary to work with the n-point functions for our present purposes. To write them, it is convenient to introduce the following notation

$$\psi_{[g_i, \tilde{g}_i, \gamma^5_i]}(x_i) =$$

$$\exp\{ig_i\phi^+(x_i)\}\exp\{ig_i\phi^-(x_i)\}\exp\{i\tilde{g}_i\gamma^5_i\tilde{\phi}^+(x_i)\} \cdot$$

$$\cdot \exp\{i\tilde{g}_i\gamma^5_i\tilde{\phi}^-(x_i)\}\ \psi_{0\,[g_i, \tilde{g}_i, \gamma^5_i]}\ , \qquad (4.6)$$

where $g_i = \pm g$, $\tilde{g}_i = \pm\tilde{g}$ and $\gamma^5_i = \pm 1$ according to

$$\psi_{01}(x_i) = \psi_{0\,[g, \tilde{g}, -1]}(x_i)\ ,$$

$$\psi_{02}(x_i) = \psi_{0\,[g, \tilde{g}, 1]}(x_i)\ ,$$

$$\psi^+_{01}(x_i) = \psi_{0\,[-g, -\tilde{g}, -1]}(x_i)\ ,$$

$$\psi^+_{02}(x_i) = \psi_{0\,[-g, -\tilde{g}, -1]}(x_i)\ .$$

The vacuum being $|0\rangle = |0\rangle_\phi \times |0\rangle_{\tilde{\phi}} \times |0\rangle_{\psi_0}$, one gets for the $2n$-point functions:

$$\langle 0|\prod_i \psi_{[g_i, \tilde{g}_i, \gamma^5_i]}(x_i)|0\rangle = {}_\phi\langle 0|\prod_i \exp\{ig_i\phi^+(x_i)\}\exp\{ig_i\phi^-(x_i)\}|0\rangle_\phi \cdot$$

$$\cdot\ {}_{\tilde{\phi}}\langle 0|\prod_i \exp\{ig_i\gamma^5\tilde{\phi}^+(x_i)\}\exp\{i\tilde{g}_i\gamma^5_i\phi^-(x_i)\}|0\rangle_\phi \cdot$$

$$\cdot\ {}_{\psi_0}\langle 0|\prod_i \psi_{0\,[g_i, \tilde{g}_i, \gamma^5_i]}(x_i)|0\rangle_{\psi_0}\ , \qquad (4.7)$$

and explicitly

$$<0|\prod_i \psi_{[g_i,\tilde{g}_i,\gamma_i^5]}(x_i)|0> = \prod_i \prod_{j>i} [-(x_i - x_j)_-^2]^{g_i g_j} .$$

$$\cdot \prod_i \prod_{j>i} [- x_i - x_j)_-^2]^{\tilde{g}_i \gamma_i^5 \tilde{g}_j \gamma_j^5} \quad {}_{\psi_0}<0| \prod_i \psi_{0\,[g_i,\tilde{g}_i,\gamma_i^5]} |0>_{\psi_0} , \qquad (4.8)$$

where $j > i$ reflects the fact that the annihilation part of the fieldwith argument x_i has **been commuted** through **all** the creation parts on its right.

We are now ready to study conforma1 **symmetry.** For interacting fields the dynamics of the theory is not entirely contained in the **two-point** **function** as is the case for the free **field**[29]**.** Therefore we must examine the covariance of the **$2n$-point** functions, **Ref.17.** Using the factorized form of **(4.8),** we will simplify the calculations examining each factor **separately.** **It** will be enough to study the first factor, for the second is **formally** obtained from the first one by the **substitution** $g_i \to \tilde{g}_i\gamma_i^5$, and free **fermion** covariance for D = 2 was established in the last Section.

Let us see, then, how the 2n-point function of the **"scalar exponential"** transforms as one special conforma1 transforms space-time. Using the formula **(3.12),**

$$[-(x_i - x_j)_-^2]^{g_i g_j} =$$

$$[\sigma_-(x_i)]^{g_i g_j} [-(x_{i\tau} - x_{j\tau})^2]^{g_i g_j} [\sigma_+(x_j)]^{g_i g_j} ,$$

we get

$$\prod_i \prod_{j>i} [-(x_i - x_j)^2]^{g_i g_j} =$$

$$\prod_i \prod_{j>i} [\sigma_-(x_i)]^{g_i g_j} [-(x_{i\tau} - x_{j\tau})_-^2]^{g_i g_j} [\sigma_+(x_j)]^{g_i g_j} . \qquad (4.9)$$

But since

$$\prod_i \prod_{j>i} \left[\sigma_+(x_j)\right]^{g_i g_j} = \prod_j \prod_{i<j} \left[\sigma_+(x_j)\right]^{g_i g_j} = \prod_i \prod_{j<i} \left[\sigma_+(x_i)\right]^{g_j g_i} , \qquad (4.10)$$

we get

$$\prod_i \prod_{j>i} \left[-(x_i-x_j)^2_-\right]^{g_i g_j} =$$

$$\prod_i \left[\sigma_-(x_i)\right]^{\sum\limits_{j>i} g_i g_j} \prod_i \prod_{j>i} \left[-(x_{i\tau}-x_{j\tau})^2\right]^{g_i g_j} \prod_i \left[\sigma_+(x_i)\right]^{\sum\limits_{j<i} g_i g_j} . \qquad (4.11)$$

And noticing that charge conservation implies C $g_i = 0$:

$$\prod_i \prod_{j>i} \left[-(x_i-x_j)^2_-\right]^{g_i g_j} = \prod \left[\sigma_-(x_i)\right]^{\sum\limits_{j>i} g_i g_j} \prod_i \prod_{j>i} \left[-(x_{i\tau}-x_{i\tau})^2_-\right]^{g_i g_j} .$$

$$. \prod_i \left[\sigma_+(x_i)\right]^{-g_i^2 - \sum\limits_{j>i} g_i g_j} . \qquad (4.12)$$

Analogously,

$$\prod_i \prod_{j>i} \left[-(x_i-x_j)^2_-\right]^{\tilde g_i \gamma^5_i \tilde g_j \gamma^5_j} = \prod_i \left[\sigma_-(x_i)\right]^{\sum\limits_{j>i} g_i \gamma^5_i \tilde g_j \gamma^5_j} .$$

$$. \prod_i \prod_{j>i} \left[-(x_{i\tau}-x_{j\tau})^2_-\right]^{\tilde g_i \gamma^5_i \tilde g_j \gamma^5_j} \prod_i \left[\sigma_+(x_i)\right]^{-\tilde g_i^2 - \sum\limits_{j>i} \tilde g_i \gamma^5_i \tilde g_j \gamma^5_j} \qquad (4.13)$$

From (4.12-13), we see that the field $\psi_{[g_i, \tilde g_i, \gamma^5_i]}(x_i)$ does not have a definite special conformal transformation law because the transformation law of "scalar" and "pseudo-scalar exponential" factors depends on the number of free fermion fields, $\psi_0{}_{[g_j, \tilde g_j, \gamma^5_j]}(x_j)$, which occur on the right of $\psi_0{}_{[g_i, \tilde g_i, \gamma^5_i]}(x_i)$ in (4.8) . In formulae,

$$<0|\prod_i \psi_{[g_i,\tilde{g}_i,\gamma_i^5]}(x_i)|0> =$$

$$_\phi<\prod_i [\sigma_-(x_i)]^{\sum_{j>i} g_i g_j} [\sigma_+(x_i)]^{-g_i^2 - \sum_{j>i} g_i g_j} e^{ig_i\phi^+(x_{i\tau})} e^{ig_i\phi^-(x_{i\tau})}>_\phi \cdot$$

$$\cdot {}_{\tilde{\phi}}<\prod_i [\sigma_-(x_i)]^{\sum_{j>i} \tilde{g}_i\gamma_i^5\tilde{g}_j\gamma_j^5} [\sigma_+(x_i)]^{-\tilde{g}_i - \sum_{j>i} \tilde{g}_i\gamma_i^5\tilde{g}_j\gamma_j^5}$$

$$\cdot\, e^{i\tilde{g}_i\gamma_i^5\tilde{\phi}^+(x_{i\tau})} e^{i\tilde{g}_i\gamma_i^5\tilde{\phi}(x_{i\tau})}>_{\tilde{\phi}} \quad {}_{\psi_0}<\prod_i \psi_{0\,[g_i,\tilde{g}_i,\gamma_i^5]}(x_i)>_{\psi_0} \cdot \qquad (4.14)$$

But we notice that the number of free fermions $\psi_{0\,[g_j,\tilde{g}_j,\gamma_j^5]}(x_i)$ placed to the right of $\psi_{0\,[g_i,\tilde{g}_i,\gamma_i^5]}(x_i)$ defines the sector with charge $qg = \sum_{j>i} g_j$ and pseudo-charge $\tilde{g}q = \sum_{j>i} \tilde{g}_j\gamma_j^5$. Therefore, the projection of the field operator $\psi_{[g_i,\tilde{g}_i,\gamma_i^5]}(x_i)$, on a sector with charge q and pseudo charge $\tilde{q}$, has a define transformation law. Taking the projection $\psi^{q\tilde{q}}_{[g_i,\tilde{g}_i,\gamma_i^5]}(x_i) = \psi_{[g_i,\tilde{g}_i,\gamma_i^5]}(x_i)\, P(q)P(\tilde{q})$ (with $\sum_q P(q) = 1 = \sum_q P(\tilde{q})$ and remernbering the transformation formula (3.40), we get

$$U(b)\ \psi^{q\tilde{q}}(x)\ U^{-1}(b) =$$

$$[\sigma_-(x)]^{g^2q+\tilde{g}\gamma^5\tilde{q}}\ [\sigma_+(x)]^{-[g^2+\tilde{g}^2+g^2q+\tilde{g}^2\gamma^5q]} \cdot$$

$$\cdot\begin{bmatrix} [1-(b^0-b^1)(x^0+x^1)]^{-1} & 0 \\ 0 & [1-(b^0+b^1)(x^0-x^1)]^{-1} \end{bmatrix} \psi^{q\tilde{q}}(x_\tau)\ . \qquad (4.15)$$

The nonlocal componentç, $\psi^{q\,\tilde{q}}(x)$, of the interacting field are analogous to the creation and annihilation components of the free field.

It ca: be seen tha:, as in subsection 3.3, the existence of a unitary representation of the conformal group would imply the same transformation law for all components $\psi^{q\tilde{q}}(x)$. We can examine the effect of the operator $Z = U^{-1}(C_2.C_1)U(C_2)U(C_1)$ (with C_1 and C_2 being two general conformal transformations) on $\psi^{q\tilde{q}}(x)$. For that purpose, it is convenient to rewrite (3.15) as

$$U(b)\ \psi^{q\tilde{q}}(x)\ U^{-1}(b) = |\sigma(x)|^{-(g^2+\tilde{g}^2)} \exp\{-(g^2+\tilde{g}^2+2gq+2\tilde{g}\gamma^5\tilde{q}) \cdot$$

$$\log \frac{\sigma_+(x)}{|\sigma(x)|}\} \cdot \begin{bmatrix} [1-(b^0-b^1)(x^0+x^1)]^{-1} & \\ 0 & [1-(b^0-b^1)(x^0-x^1)] \end{bmatrix} \psi^{q\tilde{q}}(x_\tau) \qquad (4.16)$$

because $\sigma_- = [\sigma_+]^*$. Then,

$$Z\ \psi^{q\tilde{q}}(x)\ Z^{-1} =$$

$$\exp\{-i(g^2+\tilde{g}^2+2gq+2\tilde{g}\gamma^5\tilde{q})\ \theta(C_2,C_1) - i\begin{pmatrix}\theta_1 & 0\\ 0 & \theta_2\end{pmatrix}\}\ \psi^{q\cdot\tilde{q}}(x) \ . \qquad (4.17)$$

The phase $\theta(C_1,C_2)$ can take the values 0, $\pm\pi$ and is independent of x (this can be seen by noticing that the phase that occurs in (3.16) is a multiple of the dimension 0.5 generalized free field phase; so that θ is the same for both fields. Because the dimension 0.5 generalize free field can be seen as a restriction of the form

$$\phi_{0.5}(x) = \phi(x_0,x_1,0)$$

of the free field $\phi(x_0,x_1,x_2)$ in 3-dimensional space-time, and since the two-dimensional conformal group is a subgroup of thetridimensionai conforma! group which leaves the x_2 coordinate invariant and, finally, recalling that the phase does not depend on x for the free field case, we conclude that θ is independent of x. To write Z explicitly, we put

$$Z = Z_1 Z_3 \ , \qquad (4.18)$$

where Z_3 is given by (3.42) and Z_1 is easily seen to be

$$Z_1 = e^{-ig^2Q^2\theta}\, e^{-i\tilde{g}^2\tilde{Q}^2\theta} \quad . \tag{4.19}$$

Putting (3.19) in (3.17), applying the result on a charged state $|q,\tilde{q}>$, and recalling that

$$[Q,\psi] = \psi(x) \quad ,$$
$$[\tilde{Q},\psi] = \gamma^5\psi(x) \quad ,$$

we see that, in fact, we reproduce the righthand side of (3.17)

Notice that the charge Q and the pseudo-charge $\tilde{Q}$ are the same as in the free theory for

$$:\bar{\psi}(x)\gamma^\mu\psi(x): \; = \; :\bar{\psi}_0(x)\gamma^\mu\psi_0(x):$$

with

$$Q = \int j^0(x)dx = \int :\psi_0^+(x)\;\psi_0(x):dx$$

and

$$\tilde{Q} = \int \tilde{j}^0(x)dx = \int :\psi_0^+(x)\gamma^5\psi_0(x):dx = \int j^1(x)dx \;;$$

Z becomes

$$Z = e^{-ig^2Q^2\theta}\, e^{-i\tilde{g}^2Q^2\theta}\, e^{-i\left[\left(\frac{Q+\tilde{Q}}{2}\right)\theta_2 + \left(\frac{Q-\tilde{Q}}{2}\right)\theta_1\right]} \quad . \tag{4.20}$$

Therefore, in a sector of charge q and pseudocharge $\tilde{q}$, the multiplication law for the conformal group is

$$U(C_2)U(C_1) = e^{-ig^2q^2\theta}\, e^{-i\tilde{g}^2\tilde{q}^2\theta}\, e^{-i\left[\left(\frac{q+\tilde{q}}{2}\right)\theta_2 + \left(\frac{q-\tilde{q}}{2}\right)\theta_1\right]}\, U(C_2.C_1) \quad . \tag{4.21}$$

Again we see, recalling Bargmann's theorem, that it follows from the ray multiplication law (4.21) the existence of a (usual) representation of the universal covering of the conformal group.

The structure of the operator Z shows its connection to the dynamical details of the theory. That is, the composition law of the conformal group besides being projective depends on the theory under consideration, one not being able to determine it from Group Theory alone. As we had promised, we will see in the following the relation between the eigenvalues of Z and the dimensions of the composite fields.

Any field, $O(x)$, built (with appropriate limits) from the product of q_1 fields $\psi_1(x)$, q_2 fields $\psi_1^+(x)$, q_3 fields $\psi_2(x)$ and q_4 fields $\psi_2^+(x)$, will have an exponential part with the structure

$$\exp\{ig(q_1-q_2+q_3-q_4)\ \phi^+(x)\}\ \ \exp\{ig(q_1-q_2+q_3-q_4)\tilde{\phi}^-(x)\}\ .$$

$$\exp\{ig(-q_1+q_2-q_3+q_4)\ \phi^+(x)\}\ \ \exp\{i\tilde{g}(-q_1+q_2-q_3+q_4)\ \tilde{\phi}^-(x)\}\ .$$

Its dimension is

$$d_0 = (q_1-q_2+q_3-q_4)^2g^2 + (-q_1+q_2-q_3+q_4)^2\tilde{g}^2$$

$$+\frac{1}{2}\sum_1^4 q_i^2 + n\ . \tag{4.22}$$

In subsection 2.5, we have shown that the knowledge of the dimensions $\frac{1}{2}\sum_1^4 q_i + n$ is equivalent to knowing the eigenvalues of the operator Z (and vice-versa, modulo the addition of an integer) for the free fermion and, as the Z operator has eigenvalues, z,

$$z = e^{-ig^2\theta q^2}\ e^{-i\tilde{g}^2\theta\tilde{q}^2}\ z_3\ , \tag{4.23}$$

we see that the knowledge of the dimensions of all the composite fields exhausts all eigenvalues of Z since $q = (q_1-q_2+q_3-q_4)$ and $\tilde{q} = -q$. Naturally, our calculation shows that the knowledge of all eigenvalues of

Z is equivalent to the knowledge of the dimensions of all composed fi-elds (modulo the addition of an integer).

Observing that the Lorentz spins, of $O(x)$, is

$$\left[-\frac{q_1^2}{2}-\frac{q_2^2}{2}+\frac{q_3^2}{2}+\frac{q_4^2}{2}+\text{integer}\right]$$

and that $\theta = \theta_1 + \theta_2$, we can rewrite (3.23) as

$$z = e^{-i\left[\theta_1(d_0 - s) + \theta_2(d_0 - s)\right]} \quad . \qquad (4.24)$$

As we mentioned in subsection 3.3, it was shown in all generality[17,18] that all dimensions present in a conformally invariant theory are contained in the Z operator and the inverse question motivated our calculation. In Ref.17, it was proved that the fact all eigenvalues of Z can be expressed in terms of the composite field dimensions is a necessary condition for the existence of a complete basis of composite fi-elds, and although we are not able to prove it, it is conjectured that it is also a sufficient condition. Our calculations consist in a constructive demonstration of this condition in the present model. Concerning the existence of operator product expansions in the Thirring model, see Ref.30.

As we anticipated in subsection 3.3, conformal symmetry possesses a dynamical nature in the sense that a special conformal transformation of a local field involves the dimensions of all composite fields. This is seen recalling the transformation formula (4.16) and the decomposition $\psi(x) = \sum_{q\tilde{q}} \psi^{q\tilde{q}}(x)$:

$$U(b)\ \psi(u,v)\ U^{-1}(b) = |\sigma(u,v,b)|^{-g^2-\tilde{g}^2} \begin{bmatrix} |1-b_v u|^{-1} & 0 \\ 0 & |1-b_u v|^{-1} \end{bmatrix} \times$$

$$\times\ Z(\theta_1,\theta_2)\ \psi(u,v)\ Z^{-1}(\theta_1,\theta_2) \quad , \qquad (4.25)$$

where $\theta_1 = \arg(1 - b_v u)$ and $\theta_2 = \arg(1 - b_u v)$.

One should notice that, both in the free field case and in our present mdel, the $Z(C_1,C_2)$ operator commutes with any operator that represents an element of the universal covering of the conformal group, Z being, therefore, a central element of this group (it can be shown independently from the model considered that $Z(C_1,C_2)$ is a central element).

By successive Z products, we can form new central elements. In this way we see that the infinitely sheeted nature of the covering reflects itself by the fact that $[\exp\{ig^2\pi\}]^n \neq 1$, n = 1,2,3..., for non-integer g^2.

4.2 The Center Z

We saw in the preceding Sections that the conformal symmetry is directly linked to the dynamics and we realize that contrary to our intuition when we make a special conformal transforrnation a local field is not taken to the field in the transformed point. Naturally, it will be interesting to interpret "geometrically" the non-local transformation of the field. Let us write, then, a generalization of formulae (3.33a) and (3.25), but for the sake of sirnplicity let us consider an interacting scalar field, $A(x)$:

$$U(b)\ A(x)\ U^{-1}(b) = |\sigma(x,b)|^{-d_A}\ Z(\theta)\ \ A(x_\tau)\ Z^{-1}(\theta)\ , \qquad (4.26)$$

where $\theta = \arg\ \sigma_+(x,b)$ $(=0,\ \pm\pi,\ \pm 2\pi)$. We cannot interpret transformation (4.26) geometrically; nevertheless we can define the field $A(x,n)$ by

$$A(x,n) = Z^n(\theta{=}\pi)\ A(x)\ Z^{-n}(\theta{=}\pi) \qquad (4.27)$$

which, due to (4.26), transforms as

$$U(b)\ A(x,n)\ U^{-1}(b) = |\sigma(x,b)|^{-d_A}\ A(x_\tau,n_\tau)\ , \qquad (4.28)$$

where $n_\tau = n,\ n\pm 1,\ n\pm 2$ if $\arg\sigma_+ = 0,\ \pm\pi,\ \pm 2\pi$, respectively. We can interpret the pair (x,n) as a point in a new space where the field

$A(x,n)$ has a conventional transformation law. Naturally, if we make an infinitesimal transformation (arg $\sigma_+ = 0$) we have $n_\tau = n$. Let us make successive infinitesimal transformations adding up to a finite one. Recalling that

$$x^\mu_\tau = \frac{x^\mu - b^\mu x^2}{\sigma(x,b)} \quad , \quad \sigma(x,b) = (1 - 2b.x + b^2x^2) \ ,$$

we see that we can have $n_\tau = n \pm 1$ only if $x^\mu_\tau \to 0$ (i.e., if $\sigma(x,b)$ has gone through a zero).

Those ideas can be expressed in a schematic diagram (see Fig.1) where the horizontal lines represent the Minkowski space. This way, when we change point x into $x_\tau \to \infty$, point P = (x,n) can be transformed into $(x_\tau, n \pm 1)$ or $(x_\tau, n \pm 2)$, and when we come back we will have points of a space characterized by P = $(x, n \pm 1)$ or P = $(x, n \pm 2)$. In Ref. 17, it was shown that the non local fields $A^\xi(x)$, analogous to $\psi^{q\tilde{q}}(x)$ and $\phi^\pm(x)$, can be defined through the Fourier transform of the $A(x,n)$ field. So we can understand geometrically the non local nature of $A^\xi(x)$.

5. CONCLUSION

We wish to emphasize once more the possibility of consistently formulating conformal symmetry in the context of quantum field theory: the relevant group is the universal covering group of the conformal group and the local fields do not possess a conventional local transformation law. This results from the dynamical nature of conformal symmetry entailing the diagonalization of the operator Z and the consequent transformation law defined for the projections of a local field in eigensectors of Z.

Our calculations support the conjecture that there is a complete basis of composite fields. We hope, nevertheless, that in the near future the existence of this basis can be proved since it would be of great use for the constructive side[30,31] of the problem.

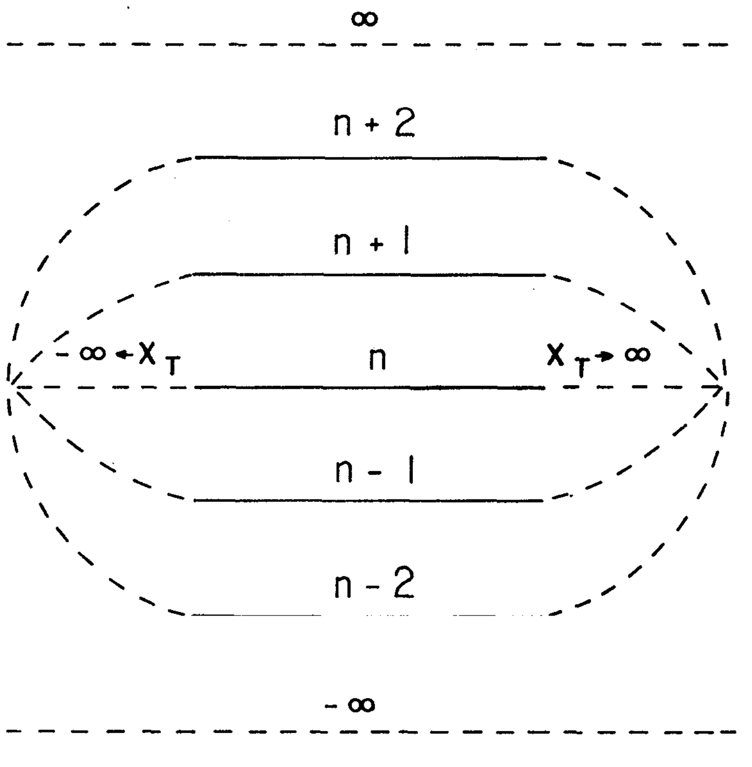

Fig.1

Although conformal symmetry as a whole should not be looked at as a symmetry in the active sense, we wish to express our hope that from the point of view of experimental facts the ideal conformal invariant theory presented here will be a first approximation to a realistic theory which would have to take into account the corrections due to mass terms.

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