Revista Brasileira de Física, Vol. 7, N 3, 1977

Causality and Relativity

MARIO SCHONBERG
Sociedade Brasileira de Física, C. P. 20553. São Paulo SP

Recebido em 7 de Dezembro de 1976

The causal relation between a pair of physical events is taken as the physical foundation of the Riemannian geometry of the Space-Time and of the arrow of time, the basic differentiable manifold of the physical continuum being associated to a pre-causal level of the physical theory. The Einstein partial differential equations of General Relativity are associated to a physical field of Causality. The four-dimensionality is related to electromagnetic properties-at the pre-causal level, especially to the field of displacement and the electric current. Gravitation is obtained from the physical field of Causality, which may be used also to develop a form of already unified theory. Causality and Determinism are related by a variational principle of the Hilbert type for the field of Causality and matter.

Toma-se a relação causal entre um par de eventos físicos como fundamento físico da geometria riemanniana & espaço-tempo e da direçã - temporal, a variedade diferencial básica do *continuum* físico sendo associado a um nível pre-causal da teoria física. Associam-se as equações diferenciais parciais, de Einstein, da relatividade geral, a um campo físico de causalidade. Relaciona-se o carater quadridimensional com as propriedades eletromagnéticas, a um nivel pre-causal, e especialmente com o campo de deslocamentos e com a corrente elétrica. Obtem-se a gravitação do campo físico de causalidade, que pode ser tambem utilizado para se desenvolver uma forma de teoria já unificada. Relacionam-se Causalidade e Determinismo através de um princípio variacional, do tipo de Hilbert, para o campo de Causalidade e matéria.

1. PRIMARY PHYSICAL CONTINUUM AND SPACE-TIME

In Relativistic Physics, the physical continuum is taken usually as a four dimensional Riemannian manifold endowed with a normal hyperbolic metric defined by a symmetric tensor with determinant $g(x)\neq 0$ whose quadratic differential form $g_{ij}(x)\, dx^i\, dx^j$ has signature -2. That signature gives a Space-Time structure with three space dimensions and one time dimension. *The existence of the normal hyperbolic Riemannian metric of the physical continuum is generally taken as the foundation of the relativistic theory of Causality.*

In the mathematical theory of an N-dimensional Riemannian manifold, the Riemannian metric is introduced in a basic differentiable manifold D_N by means of a symmetric tensor field $g_{ij}(x)$, with $g(x)\neq 0$. In Relativity, little attention is generally given to the basic differentiable manifold. *We shall emphasize the role of the basic differentiable manifold as a fundamental physical structure, distinguishing it carefully from the physical structure given by the normal hyperbolic Riemannian Space-Time.*

In the present paper, we shall take Causality as the physical foundation of the Space-Time structure by introducing the metric $g_{ij}(x)$ as a tensor associated to the theory of Causality. The signature of $g_{ij}(x)\, dx^i\, dx^j$ must obviously have absolute value 2, in order to give satisfactory relations of Causality between physical events, as shown by Special Relativity in a flat Space-Time.

The concept of a basic differentiable manifold can be introduced independently of Causality. It will be related to a pre-causal level of the theory of the physical events in Section 2. The normal hiperbolic Riemannian metric of the physical continuum will be introduced in connection with the causal relation between physical events in Section 3.

The distinction between the basic differentiable manifold and the Space-Time gives the possibility of introducing disconnected domains of Causality in D_4 each of them corresponding to a different Space-Time, and thereby to a different universe.

The $g_{ij}(x)$ of Causality can be taken as variables of a physical field of Causality. This leads to a reinterpretation of the Einstein field of Gravitation of General Relativity, in which the Einstein partial differential equations appear as the expression of a physical law relating the distribution of energy-momentum-stress of matter to the physical field of Causality. *Thus, the whole General Relativity becomes included in the theory of Causality.*

2. PHYSICAL CONTINUUM AND PHYSICAL EVENTS

We shall now use the concept of physical event for the discussion of the primary physical continuum, by means of the following physical law:

Law of the Physical Events: A physical event is associated to a point of the basic differentiable manifold D_4 of the physical continuum.

The point $x(E)$, associated to the physical event E, will be called the position of E in the D_4 of the physical continuum. *D_4 is the set of the possibte positions of events, and endowed with a differentiable structure of the class C^r, with $r > 1$, taken sufficiently large to allow the construction of the physical theory.*

The differentiable structure of D_4 allows for the existence of relative tensors of all orders, with all the values of the weight, as well as that of the relative pseudo-tensors. *The condition $r > 1$ allows also the existence of affinities.*

The level of the theory of the physical events based only on the above law will be called pre-causal. At this level, there is not yet the dis-tínguishable system of lines of the physical continuum allowing for the introduction of Causality relations between events at different points.

We have the following theorem:

Theorem of the Directions: At the pre-causal level of the theory of the physical events there is still no distinction of different kinds of linear directions at any point x of the physical continuum.

At the points x, of an N-dimensional differentiable manifold D_N, with $N > 1$, the linear directions constitute an irnportant projective space $P_{N-1}(X)$, whose "points" have as homogeneous coordinates the X^i components of the contravariant vectors $X \neq 0$ at x. The straight lines of $P_{N-1}(X)$ describe the planar directions at the point x of D_N. We have the following theorem:

Theorem of $P_3(x)$: The dimensionality $N=4$, of the physical continuum, is distinguished by special properties of the planar directions at its points x, corresponding to those of the straight lines of $P_3(x)$, given by the self-duality of the straight line in the thre-dimensional projective geometry.

The special properties of the planar directions at the points of D_4 are related to the existence of the Levi Civita basic relative tensor ε_{ijhk}. The homogeneous coordinates $X^{ij} = -X^{ji}$ of a planar direction at x satisfy the quadratic condition

$$\varepsilon_{ijhk} X^{ij} X^{hk} = 0 , \tag{2-1}$$

which is the Plücker condition for the homogeneous Plücker coordinates X^{ij} of the straight lines of $P_3(x)$.

In the pre-causal level of the theory of the physical events of the present Section, the *a priori* assumption of the four-dimensionality of the physical continuum is made, since it is based on the law of the physical events. *Thereby, in the above pre-causal level, the four-dimensionality directly as an experimental fact is assumed, but not the Space-Time structure.*

3. THE CAUSAL RELATION OF PHYSICAL EVENTS

We shall now introduce the concept of a causal relation between two physical events, as a nonsymmetric type of relation, in which one of the events plays the role of cause, and the other event that of effect. *The asymmetry of the causal relation will be assumed as the foundation of the arrow of time.*

We shall also introduce the concept of the possibilitv of a causal relation between two events, associated to their positions in the physical continuum, without further specification of their special nature. In a similar way, we shall also introduce the concept of the impossibility of a causal relation between two events specified only by their positions. *In this approach to the possibility (impossibility) of a causal relation between two events, we shall not distinguish the "cause" and the "effect", unless it becomes necessary in connection with the arrow of time.*

In both the Special and the General theories of Relativity, Causality is based on the previous introduction of a normal hyperbolic Riemannian metric in the physical continuum. The Riemannian metric, $g_{ij}(x)$, allows us to formulate purely positional conditions for the possibility of causal relations between physical events. *In our approach, we shall introduce the tensor $g_{ij}(x)$ as a symmetric tensor of Causality, instead of taking it as a geometric tensor, by means of a physical law giving the positional conditions for the possibility of causa2 relations between physical events.*

The quadratic differential form, $g_{ij}(x)\, dx^i\, dx^j$, of Causality will be taken with the signature (-2), as the quadratic differential form of the Riemannian metric in General Relativity. *The signature (-2) is related to the role played by that quadratic differential form in the definition of the lines of Causality, the lines in which $g_{ij}(x)$ & dx^j > 0 for any infinitesimal displacement dx on them, at any of their points.*

We shall now introduce the following physical law:

Law of Causal Possibility: There is the possibi1ity of a causal relation between two physical events at different points of D if and only if they can be connected by a line of Causality of the tensor field $g_{ij}(x)$ of Causality, having $g_{ij}(x)\, dx^i\, dx^j$ with signature (-2).

The lines of Causality, with $g_{ij}(x)\, dx^i\, dx^j > 0$ for any $dx \neq 0$ in them, are particularly important because of the following physical law:

Law of the Observers: The lines of Causality with $g_{ij}(x)\, dx^i\, dx^j > 0$, for any $dx \neq 0$ in them, are the possible lines of the observers. The variation ds of the proper-time of the observer for the displacement dx is given by the equation

$$ds^2 = g_{ij}(x)\, dx^i\, dx^j \ . \tag{3-1}$$

It follows from the law of the observers the

Theorem of the Riemannian Metric: The tensor $g_{ij}(x)$ of Causality is the metric tensor of the physical continuum, as a consequence of the law of the observers, since it underlies the chronometry of the physical observers.

4. THE CONFORMAL METRIC

The conformal metric $C_{ij}(x)$ of the physical continuum, corresponding to the Riemannian metric $g_{ij}(x)$ given by Causality, is

$$C_{ij}(x) = |g(x)|^{-1/4} g_{ij}(x) \ , \tag{4-1}$$

$g(x)$ denoting the determinant of the $g_{ij}(x)$. Since $g(x)$ is negative, the determinant $C(x)$ of the $C_{ij}(x)$ is

$$C(x) = -1 \ . \tag{4-2}$$

The conformal metric $C_{ij}(x)$ is a relative tensor of weight (-1/2), thereby the quadratic differential form $C_{ij}(x) dx^i\, dx^j$ is not a scalar, but its sign is invariant when one changes the coordinate systems in D_4. Thus, we get the following theorem:

Theorem of the Directions: The distinction of time-like, space-like and null directions at a point x of the physical continuum, defined by the dx at x, depends only on its conformal metric $C_{ij}(x)$. The equation $C_{ij}(x)\, dx^i\, dx^j = 0$ defines the cone of the null directions at x. The time-like directions correspond to $C_{ij}(x)\, dx^i\, dx^j$ positive, and the space-like directions to $C_{ij}(x) dx^i dx^j$ negative. The lines of Causality can be defined by means of the conformal metric $C_{ij}(x)$.

In the causal approach to Geometry, the time-like and null directions at a point x of the physical continuum appear as the two kinds of directions of Causality at x, whereas the space-like directions at x may be called directions of Acausality. *The passage from the conformal to the metric geometry of the physical continuum allows us to associate to each direction at* x *a geodesic of the* $g_{ij}(x)$ *metric containing* x, *and only one,* for *sufficiently smooth functions* $g_{ij}(x)$. *The geodesics, defined by* the *time-like, null and space-like directions at a point* x, *have tangent directions of the same kind at all their points. They will be called time-like, null* and *space-like geodesics of the physical continuum, respectively.*

The above discussion shows that the construction of the causal levelof the theory of the physical events can also be done in the follwing two steps:

(a) The development of an almost-causal level by the introduction of the conformal metric $C_{ij}(x)$;

(b) The passage from the almost-causal level to the causal one by the introduction of $|g(x)|$, and the definition of $g_{ij}(x)$ in terms of $C_{ij}(x)$, and $|g(x)|$ by the equation

$$g_{ij}(x) = |g(x)|^{1/4} C_{ij}(x) \,. \qquad (4\text{-}3)$$

The step (a) is associated to the introduction of an Eucliadean measure of the angles of the space-like directions orthogonal to a given time-like direction, at a point x of the physical continuum, which gives the angular measure of the space-directions of the observers at x. The step (b) introduces a measure of infinitesimal hypervolumes at x by means of $|g(x)|^{1/2}$.

The quasi-causal level of the theory of the physical events is closely related to the Maxwell equations of General Relativity, which are conformally invariant, as is well known. This is due to the possibility of defining the dual $*A(x)$ of a covariant bi-vector $A(x)$ by means of $C_{ij}(x)$, as a consequence of the four-dimensionality of the physical continuum,

$$*A_{ij}(x) = -\frac{1}{2} C_{ia}(x) \ C_{jb}(x) \ \varepsilon^{abcd} \ A_{cd}(x) \ , \qquad (4\text{-}4)$$

since the electromagnetic displacement $D(x)$ and the electromagnetic field $F(x)$ are related by Duality:

$$D(x) = *F(x) \text{ and } F(x) = - *D(x) \ . \qquad (4\text{-}5)$$

5. ELECTROMAGNETISM AND FOUR-DIMENSIONALITY

The dimensionality $N=4$ of the physical continuum is a property of its primary differentiable manifold D_N. We shall now see that there is a remarkably simple and natural electromagnetic foundation of the dimensionality $N=4$ of the D_N of the physical continuum based on the relation between the field of displacement $D(x)$ and the electric current given by the first Maxwell equation

$$(1/2) \ \delta^{abc}_{ijk} \ \partial_a \ D_{bc}(x) = J_{ijk}(x) \ , \qquad (5\text{-}1)$$

$J(x)$ denoting the oriented covariant trivector of electric current.

We shall introduce the following primary law of Electromagnetism:

Law of the Electric Current: The electric current can be described by an oriented contravariant vector-density, $J(x)$, at the D_N level of the geometry of the physical continuum.

It follows from the above law and the Maxwell equation (5-1) that the oriented trivector $J_{ijk}(x)$ of the electric current must be equivalent to the oriented contravariant vector-density $J^a(x)$ of electric current at the D_N level of the geometry of the physical continuum. This requires the existence of a contravariant Levi Civita epsilon with four indices ε^{abcd}, so that we must have $N=4$. Thus, we have the

Theorem of the Four-Dimensionality: The electromagnetic displacement $D(x)$ gives a physical foundation of the **four-dimensionality** of **the primary** differentiable manifold of the physical continuum, as a **consequence** of the primary form (5-1) of the first Maxwell equation and the law of the electric current, which does not require the introductionof the electromagnetic field $F(x)$ and the second Maxwell equation.

The first Maxwell equation may be seen as the expression of the electric current as a current of polarization, by assuming $D(x)$ to describe an electric polarization of the vacuum as a physical **medium.** This assumption requires $D(x)$ to be equivalent to an oriented contravariant bivector-density $\mathcal{D}(x)$ at the D_N level of the geometry of the physical continuum. This would require the existence of a contravariant Levi Civita epsilon with four indices, and thereby the **dimensionality** $N=4$. Thus, we get the

Theorem of $D(x)$**:** The **equivalence** of $D(x)$ to an oriented contravariant bivector-density $\mathcal{D}(x)$, describing a polarization of the vacuum, is a sufficient condition for the dimensionality $N=4$ of the D_N of the **physical** continuum, independently of the first Maxwell equation (5-1).

It is possible to introduce $F(x)$ as independent of $D(x)$ at the D_4 level of the geometry of the physical continuum, and to write the second Maxwell equation

$$(1/2)\,\delta^{abc}_{ijk}\,\partial_a\,F_{bc}(x) = 0 \; . \qquad (5\text{-}2)$$

The eventual existence **of** magnetic **currents** would require a **generalization** of the second Maxwell equation, with the introduction of a covariant trivector $M_{ijk}(x)$ **in** the right hand **side** of **(5-2).** This would allow **us** to use the generalízed second Maxwell equation to get a **theorem** analogous to the above theorem of the four-dimensionality.

The'equivalence of $F(x)$ and a contravariant bivector-density $\mathcal{F}(x)$, at the D_N level of the geometry of the physical continuum, is a sufficient condition for the four-dimensionality, since it requires the existence of a contravariant Levi Civita epsilon with four indices.

By means of $D(x)$ and $F(x)$, we can build the oriented tensor,

$$T_{ab,cd}(x) = D_{ab}(x)\ F_{cd}(x) - D_{cd}(x)\ F_{ab}(x), \qquad (5\text{-}3)$$

introduced (Schönberg 1971) in connection with the electromagnetic energy. We shall discuss it in later Sections.

6. THE ARROW OF TIME

The determination of the arrow of time by means of the asymmetry of the causal relation between two physical events is associated to a physical distinction of the two half-cones of the past and the future at a point x of the physical continuum given by the tensor $g_{ij}(x)$ of Causality. *From a geometrical point of view, those two half-cones have similar roles, but this is not true with respect to the cause - effect structure of the causal relation between physical events.*

For the sake of mathematical simplicity, we shall discuss firstly the relation between the arrow of time and Causality in the case of a flat Space-Time, similar to that of Special Relativity. Now, the light cone at a point x is generated by the straight lines with null directions containing x. It is constituted by two half-cones having in common only the point x.

We shall assume as a physical law of Special Relativity that a physical event at x can be a cause only of events at points lying either inside or on a distinguished half-cone of the Zight-cone at x, which will be called the half-cone of the future at x, the half-cone of the future varying continuously for a displacement of x.

The existence of the half-cone of the future allows us to introduce an orientation of the non space-like straight lines containing x, corresponding to the displacement from the half-cone of the past into that of the future. *The orientation of those straight lines inside the light-cone, at x, allows us to choose a sense of the proper-time on such a line, with $ds>0$ for a dx along it in the sense of the line. Thus we get an arrow of time based on the distinction of cause from effect in the causal relation between physical events.*

In the general case of a curved Space-Time, there is an open neighbourhood $N(x)$ of the point x in which are defined the systems of normal coordinates of the metric $g_{ij}(x)$ with origin at x, characterized by giving zero values to the first order derivatives of the $g_{ij}(x)$. All the points $\bar{x}$ of $N(x)$ can be connected to x by segments of geodesics of the $g_{ij}(x)$ metric lying inside $N(x)$, there being only one such segment for every $\bar{x}$. We shall define the light-cone $K(x)$ at x as the set of the points of $N(x)$ lying in the null geodesics of the $g_{ij}(x)$ metric containing x. $K(x)$ is constituted by two half-cones having in common only the point x.

In order to extend, to the general case, the above orientation of the non space-like geodesics of the $g_{ij}(x)$ metric, by means of the asymmetry of the causal relation between physical events, we shall introduce the following physical law:

Law of the Half-Cone $K_+(x)$: A physical event at x cannot be a cause of any physical events at the points $\bar{x}$ of $N(x)$ lying out-side a half-cone $K_+(x)$ of the light cone $K(x)$ at x. The half-cone of the future, $K_+(x)$, varies continuously with the displacement of its vertex x.

The half-cone $K_-(x)$ of $K(x)$, complementary to $K_+(x)$, will be called the half-cone of the past at x. The non space-like geodesics of the $g_{ij}(x)$ metric containing x can be oriented by the choice of the sense from $K_-(x)$ into $K_+(x)$. *This orientation allows us to choose the sense of the proper time variable s, in a time-like geodesic containing the point x, by the condition that s increases in displacements in the sense given by that orientation.*

Any line of Causality containing a point x must have a segment lying on $K(x)$ or inside it, which can be oriented by choosing in it the sense from $K_-(x)$ into $K_+(x)$. Thus, we get a generalization of the orientation of the non space-like geodesics containing x, allowing a choice of the sense of the proper time variable s in the lines of the physical observers, as in the case of a time-like geodesic.

The above results show that the existence of the arrow of time is based both on the normal hyperbolic Riemannian type of the geometry of the physical continuum, given by the $g_{ij}(x)$ of Causality, and on the asymmetry of the causal relation between the physical events, since the signature of $g_{ij}(x)\ dx^i\ dx^j$ determines the special structure of the light-cone $K(x)$.

7. SPACE AND SIMULTANEITY

We can associate a "space" to any point x of a line of an observer, constituted by the points $\bar{x}$ of $N(x)$ lying on the space-like geodesics containing x, whose tangent directions at x are orthogonal to the time-like tangent direction of the line of the observer at the point x. Since $N(x)$ is the domain of existence of the systems of normal coordinates of the $g_{ij}(x)$ metric with origin at x, the above definition, of the "space" of an observer at x in its line allows us to obtain a natural parameterization of its points $\bar{x}$ by means of the three normal coordinates with origin at x, whose corresponding coordinate lines lie in that "space".

Special Relativity showed that the concepts of space and simultaneity for an observer are closely related, the events at the points of the space of an observer at a point x being simultaneous for that observer at x. We shall now generalize that result by the introduction of the following physical law:

Law of Simultaneity: For an observer at the point x of its line, the events at the points of its "space" are simultaneous.

It *follows from* the *above Zaw* that *Simultaneity is related* to *Causality*, *since* the *"space"* of an *observer* at the *point* x *of its line is built with* the *space-like geodesics* of the $g_{ij}(x)$ *metric*.

We shall now consider a tubular domain T(L) containing the line L of an observer, and such that each point $\hat{x}$ of T(L) lies in the "space" of the observer at a point $x(s)$, but in no other such "space" corresponding to a proper time $s' \neq s$. We can now determine the order of succession of the events at the points inside T(L) for the observer, by means of the following physical law:

Law of Time: For an observer with line L, the time of an event at the point $\hat{x}$ of its tubular region T(L) is given by the value of s corresponding to its "space" containing $\hat{x}$. The order of succession of those events for the observer is that of the increasing values of their time values s.

The systems of Fermi coordinates of the physical continuum based on the line L of an observer are particularly adequate for the parameterization of the tubular region T(L), because one of the Fermi coordinates is the time variable s given by the proper time of the observer, and the other three of them give a parameterization of the "space" of the observer corresponding to the value s of the time, in a way similar to that of orthogonal Cartesian coordinates in its "space", with the three coordinate lines of a specially chosen orthogonal system of normal coordinates lying in the "space" of the observer at $x(s)$ playing the role of the Cartesian axes, the fourth coordinate line being the geodesic tangent at the point $x(s)$ to the line L of the observer. The tetrads of the orthogonal normal systems ofcoordinates at the points of L are related by the Fermi-Walker transport associated with L, which allows to have one of the vectors of the tetrad always tangent to L.

8. THE RELATION OF ACAUSALITY

We based our discussion of Causality on the positional condition for the possibility of a causal relation between two physical events. We *shall now stute that there is a relation of Acausality between two events E' and E'' when there is a positional impossibility for any of them to be a cause of the other.*

In the pseudo-Euclidean flat Space-Time of Special Relativity, the positional condition for the impossibility of any of the two events E' at x', and E'' at x'', to be a cause of the other is that the interval of the points x' and x'' be an imaginary number. Thereby in this case Acausality is related to the pseudo-Euclidean metric of the Space-Time.

The above definition of the relation of Acausality shows that it is symmetrical with respect to the two events E' and E'', not depending on the asymmetry of the causal relation with respect to cause and effect, associated to the arrow of túne.

We shall now assume that there is a symmetric tensor $g_{ij}(x)$ associated to the relatíon of Acausality of physical events, whose quadratic differential form $g_{ij}(x)dx^i\, dx^j$ has the signature (-2). The direct association of $g_{ij}(x)$ to Acausality looks satisfactory, because the $g_{ij}(x)$ do not determine the arrow of time.

By means of the tensor $g_{ij}(x)$ we can define a normal hyperbolic Riemannian metric of the physical continuum and its Space-Time structure, according to the following physical law:

Law of the Space-Time: The Riemannian metric of the Space-Time is given by the symmetric tensor $g_{ij}(x)$ associated to the relation of Acausality of physical events.

The null geodesics of the Riemannian metric given by $g_{ij}(x)$ allow us to define at a point x the light-cone $K(x)$, in the domain $N(x)$ of existente of its systems of normal coordinates with origin at x. We have the following physical law:

Law of Acausality: There is a relation of Acausality between any event at a point x and any event at a point $\bar{x}$ of $N(x)$ lying outside the light--cone $K(x)$, for any choice of $\bar{x}$.

In the case of the flat Space-Time of Special Relativity, $N(x)$ is the whole manifold, and the points outside $K(x)$ are those $\bar{x}$ for which the intervals to x are imaginary numbers. Thereby the Law of Acausality is a generalization of the condition for the positional impossibility of a causal relation between events at x and $\bar{x}$, for any order of causation, well known from Special Relativity.

The passage of an observer at a point x of its line is a physicalevent in a relation of Acausality with any event at a point $\bar{x}$ of its "space" at x, since $\bar{x}$ is a point of $N(x)$ lying outside the light-cone $K(x)$. *Thereby, Acausality appears as the natural foundation of Space, and also of Simultaneity.*

9. THE PHYSICAL FIELD OF CAUSALITY

We shall now go over to a higher level of the theory of Causality by assuming the existence of a structure of physical field having the $g_{ij}(x)$ as its variables. *This introduction of the physical field of Causality is analogous* to *that of* a *physical field described by the components of the Riemannian metric in General Relativity.*

The higher level of the theory of Causality is characterized by the central role played by the Riernann-Christoffel tensor associatedto the Riemannian metric defined by the tensor $g_{..}(x)$ of Causality, which will be denoted by $R^{a}{}_{bcd}(x)$. By means of the Ricci tensor, $R_{ab}(x)$, and the curvature scalar $R(x)$, we can build the symmetric Einstein tensor $G_{ab}(x)$:

$$G_{ab}(x) = R_{ab}(x) - (1/2)g_{ab}(x)\ R(x)\ . \qquad (9\text{-}1)$$

The theory of the physical field of Causality requires the introduction of a dimensional constant K in order to define its action integral I_C by means of $g(x)$ and $R(x)$:

$$I_C = (2K)^{-1} \int R(x)\ |g(x)|^{1/2}\ d_4x\ . \qquad (9\text{-}2)$$

We shall now introduce the following physical law:

Law of Causality and Energy: The physical field of Causality determines the energy tensor $T_{ij}(x)$ of matter and ordinary fieids by means of the Einstein tensor $G_{ij}(x)$,

$$K\, T_{ij}(x) = G_{ij}(x) \ . \qquad (9\text{-}3)$$

The equations (9-3) are a system of partial differential equations for the $g_{ij}(x)$, with the same form as the Einstein partial differential equations of General Relativity. Thereby we have the following theorem:

Theorem of Causality and Gravitation: The whole theory of Space-Time and Gravitation of General Relativity can be naturally obtained from the higher level of the theory of Causality. In particular, the components of the gravitational field of General Relativity are given by the Christoffel symbols of the tensor $g_{ij}(x)$ of Causality.

The equations (9-3) can be obtained from a variational principle

$$\delta I_{\text{tot}} = 0 \ , \ I_{\text{tot}} = I_C + I \ , \qquad (9\text{-}4)$$

with I denoting the action of matter and ordinary physical fields, by assuming the following physical law:

Law of Action and Energy: The energy tensor $T_{ij}(x)$ of matter and the ordinary physical fields is given by the equation

$$T_{ij}(x) = - \delta I / \delta g^{ij}(x) \ ,$$

with $\delta I/\delta g^{ij}(x)$ denoting the functional derivative of I with respect to the variable $g^{ij}(x)$ of the physical field of Causality.

The equations (9-3) expressing the law of Causality and Energy are those given by the variational principle (9-4), corresponding to the variation of the $g^{ij}(x)$ as variables of the physical field of Causality. We have the following theorem:

Theorem of *Determinism and Causality*: The variational principle (9-4) expresses the law of Determinism of the Classical Physics. It shows that the Classical Determinism is related to Causality by the dependente of the action I of matter and the ordinary physical fields on the tensor $g^{ij}(x)$ of Causality.

10. THE ELECTROMAGNETIC ENERGY TENSOR

We shall now discuss the oriented tensor $T_{ab,cd}(x)$ defined by (5-3), in order to get a clearer understanding of the relations between Causality and Energy in Electromagnetism, since that tensor can already be defined at the pre-causal level of electromagnetic theory.

The definition (5-3) can be applied for any value $N \geq 3$ of the dimensionality, because $T_{ab,cd}(x)$ is antisymmetric with respect to the two indices of each pair, and satisfies also the condition

$$T_{ab,cd}(x) = - T_{cd,ab}(x) \quad . \tag{10-1}$$

$N = 4$ *is thereby* the *lowest even value* of N for *which oriented tensors with* the *symmetries* of $T_{ab,cd}(x)$ may *exist*.

For $N = 4$, the oriented tensor $T_{ab,cd}(x)$ is equivalent to an oriented affinor-density $T^{i}_{\dot{j}}(x)$, with trace $T^{i}_{\dot{i}}(x)=0$, given by

$$T^{j}_{\dot{i}}(x) = (1/4)\varepsilon^{jbcd}\, T_{ib,cd}(x) \quad . \tag{10-2}$$

We have the following theoreni:

Theorem of $T_{ab,cd}(x)$: The dimensionality $N = 4$ of the D_N of the physical continuum is determined by the equivalence of $T_{ab,cd}(x)$ to an oriented affinor-density with zero trace, at the D_N level of the geometry of the physical continuum.

The equation (10-2) does not depend on the special algebraic structure of $T_{ab,cd}(x)$ given by (5-3), similar to that of a Grassmann outer product of two six-dimensional vectors, which leads to the equation

$$T^{a}_{\cdot i}(x)\; T^{j}_{\cdot a}(x) = \delta^{j}_{i}\; Q(F,D), \qquad (10\text{-}3)$$

with

$$Q(A,B) = P(A,B)^2 - P(A)\; P(B), \qquad (10\text{-}4)$$

$$P(A,B) = 1/8\; \varepsilon^{abcd}\; A_{ab}(x) A_{cd}(x) \text{ and } P(A) = P(A,A)\;, \qquad (10\text{-}5)$$

$A(x)$ and $B(x)$ denoting covariant bivectors at x.

$T^{j}_{\cdot i}(x)$ becomes the oriented affinor-density of electromagnetic energy when $D(x)$ is taken as the dual $*F(x)$, with respect to the $g_{ij}(x)$ metric of Causality. It follows from (5-3) and (4-4) that

$$T^{j}_{\cdot i}(x) = |g(x)|^{1/2}\; T_{ia}(x)\; g^{ja}(x)\;, \qquad (10\text{-}6)$$

$T_{ij}(x)$ denoting the usual electromagnetic energy tensor, expressed in terms of the components of $F(x)$ and the $g_{ab}(x)$ by

$$T_{ij}(x) = F_{ia}(x)\; F^{a}_{\;\,j}(x) + (1/4)\; g_{ij}(x)\; F_{ab}(x)\; F^{ab}(x) \qquad (10\text{-}7)$$

which satisfies the three well known Rainich algebraic conditions.

It is interesting to note that the two Rainich conditions notdepending on the signature of $g_{ij}(x)\; dx^{i}\; dx^{j}$ are related to the pre-causal properties of $T^{j}_{\cdot i}(x)$ given by the second equation (10-2) and the equation (10-3). We may therefore see $T^{j}_{\cdot i}(x)$ as a kind of pre-causal oriented affinor-density releted to electromagnetic energy-momentum-stress. The *symmetric energy tensor* $T_{ij}(x)$ *of Electromagnetism can only* be *obtained* by the *introduction of* the *tensor* $g_{ab}(x)$ *of Causality.*

The introduction of the electromagnetic action integral **I**.

$$I = \int P(*F,F)\; d_4 x \quad , \qquad (10\text{-}8)$$

and the equation (9-5), applied to the case in which there is only the electromagnetic field in interaction with the field of Causality, leads to the expression (10-7) of $T_{ij}(x)$. In the present case, the equations (9-3) become simply

$$K\, T_{ij}(x) = R_{ij}(x) \quad . \tag{10-9}$$

It is interesting to note that the expression (10-8) of the electromagnetic action can be obtained from its pre-causal primary form $\int P(D,F)\, d_4x$ by taking $D(x) = {*F}(x)$, with ${*F}(x)$ denoting the dual of $F(x)$ corresponding to the Riemannian metric $g_{ij}(x)$ of Causality.

The existence of an oriented scalar-density built only with $D(x)$ and $F(x)$ is possible only for N = 4. *Thereby the existence of a primary form of the electromagnetic action I gives a remarkably simple physical foundation of the four-dimensionality.*

11. CAUSAUTY AND ELECTROMAGNETIC PERMEABILITY

We introduce in our paper (Schönberg 1971) an oriented tensor $L_{ab}^{cd}(x)$, antisymmetric with respect to the two indices of each pair, such that for any bivector $A(x)$,

$$*A_{ab}(x) = (1/2) L_{ab}^{cd}(x)\, A_{cd}(x) \;; \tag{11-1}$$

$L_{ab}^{cd}(x)$ will be called the oriented tensor of electromagnetic permeability of the vacuum because

$$F_{ab}(x) = (1/2) L_{ab}^{cd}(x)\, F_{cd}(x) \quad . \tag{11-2}$$

It follows from (4-4) that

$$L_{ab}^{cd}(x) = -\, C_{ap}(x)\, C_{bq}(x)\, \varepsilon^{pqcd} \quad . \tag{11-3}$$

Thus, we have the following theorem:

Theorem of Causality and Permeability: The electromagnetic permeability of the vacuum is determined by the field of Causality, by means of its quantities $C_{ab}(x)$.

Equation (10-6) shows that the electromagnetic energy tensor $T_{ij}(x)$ cannot be defined only in terms of $F(x)$ and the components $C_{ab}(x)$ of the conformal metric, but requires also the determinant $g(x)$. The definition of $T_{ij}(x)$ by means of the equation (9-5) applied to the field $F(x)$ requires that $P(D,F)$ be expressed in terms of $F(x)$ and the tensor of Causality, although $P(D,F)$ can be written in terms of $C_{ab}(x)$ and the field $F(x)$, because it involves the functional derivative $\delta I/\delta g^{ij}(x)$. Thus we get the following theorem:

Theorem of $T_{ij}(x)$: The energy tensor $T_{ij}(x)$ of Electromagnetism is associated to a higher level of the geometry of the physical continuum than the properties of electromagnetic permeability of the vacuum described by $L^{cd}_{ab}(x)$.

It is important to note that there are important relations between $D(x)$ and $F(x)$ at the D_N level of the geometry of the physical continuum, given by the following law:

Law of $D(x)$ and $F(x)$: The ranks of the bivector $F(x)$ and the oriented bivector $D(x)$ are the same at any point x, and their pfaffians satisfy the condition

$$P(D) = - P(F) \ . \qquad (11\text{-}4)$$

It is interesting to note that the above law requires only that the dimensionality N be even, in order to allow for the existence of the pfaffians of the bivectors.

We have the following theorem:

Theorem of $P(D)$: The dimensionality $N = 4$ of the physical continuum is determined by the condition of the existence of a scalar-density built with the components of $D(x)$ at the D_N level of the geometry of the physical continuum, which is $P(D)$ up to a numerical factor.

It follows from (10-4) and (11-4) that

$$Q(D,F)^2 = P(D,F)^2 + P(D)^2 \ . \qquad (11\text{-}5)$$

$Q(D,F)$ is therefore always non negative as a consequence of the law of $D(x)$ and $F(x)$. When $Q(D,F) \neq 0$, the matrix of the $T^j_{\cdot i}(x)$ hasdeterminant $\neq 0$. Since $P(A)$ is quadratic in the $A_{ab}(x)$ for $N = 4$, the sign minus in (11-4) requires the linear independence of $D(x)$ and $F(x)$. *Thereby $T_{ab,cd}(x)$ is a nonzero oriented tensor for $F(x) \neq 0$, as a consequence of the law of $D(x)$ and $F(x)$.*

It follows from (4-4) and the relation $D(x) = {}^*F(x)$ that

$$P(D) = C(x)\ P(F) \ , \qquad (11\text{-}6)$$

$C(x)$ denoting the determinant of the $C_{ab}(x)$. We get from (11-6) and (11-4) that $C(x) = -1$, so that the absolute value of the signature of $C_{ij}(x)\, dx^i\, dx^j$ is 2 as a consequence of the minus sign in the right hand side of (11-4). *Thus we see that the law of $D(x)$ and $F(x)$ is compatible only with a normal hyperbotic Riemannian metric of the physical continuum.*

12. CAUSALITY AND THE "ALREADY UNIFIED FIELD

The equation (10-9) with $T_{ij}(x)$ taken as the energy tensor of an electromagnetic field without sources can be used to relate the higher level of the theory of Causality to that of the "already unified field " of Gravitation and Electromagnetism of Rainich, Misner and Wheeler (Rainich 1925, Misner and Wheeler 1957), by the introduction of the Rainich algebraic conditions and the differential condition of Rainich, Misner and Wheeler.

We shall discuss this matter in more detail elsewhere. Now we shall only note that, in the relation of the higher level of the theory of Causality to the "already unified field", it is convenient to take the field $D(x)$ of displacement as the basic field of Electromagnetism, rather than $F(x)$, with $F(x)$ defined in terms of $D(x)$ and the tensor $g_{ij}(x)$ of Causality. *Thus, we get an "already unified field" of Causality and displacement. The dimensionality $N=4$ can be now naturally determined by imposing the condition of equivalence of $D(x)$ and an oriented bivector-density $\mathfrak{D}(x)$ of polarization of the vacuum as in the Theorem of $D(x)$ of Section 5.*

The theory of the already unified field of Causality and Displacement requires also the topological refinements of the theory of Misner and Wheeler, instead of the simpler topology of the primary differentiable manifold D_4.

REFER ———

1. C.W. Hisner and J.A. Wheeler, Ann. Phys. 2, 525, 1957.
2. G.Y. Rainich, Trans. Am. Math. Soc. 27, 106, 1925.
3. M. Schönberg, Rev. Brasil. de Fis. 1, 91, 1971.