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New Solutions of a Nonlinear Classical Field Theory: a "Charge Barrier" Mechanism

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We have obtained new solutions of a relativistic, classical, Field Theoretical Model having logarithmic nonlinearities. Some of these solutions correspond to field not bounded in time but having finite Energy and Charge. There are no bounded solutions (bound states and resonances in particular) if the charge exceeds a certain value. This effect is due to the existance of a "charge barrier" in this field theoretical model. All calculations are performed in any number of spatial dimensions.

Obtêm-se novas soluções de um modelo relativístico em Teoria Clássica de Campos que apresenta não-linearidades logarítmicas. Algumas dessas soluções não são limitadas no tempo mas têm carga e energia finitas. Não existem soluções limitadas (estados ligados ou ressonâncias) se a carga excede um certo valor crítico. Esse efeito é devido a um mecanismo de "barreira de carga". Todos os cálculos são feitos em um número qualquer de dimensões espaciais.

1. INTRODUCTION

Hadron-like properties exhibited by some solutions of nonlinear Field Theoretical Models have motivated an increasing search for classical

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solutions of relativistic nonlinear Field Theories. Up to now, however, explicit solutions having dynamical time dependence and finite energy have been obtained mainly in the realm of 1+1 dimensional models.

Concerning this last remark, the model proposed by Birula and Mycielski[1,2] constitutes an exception. In this paper, we will enlarge the class of solutions of the relativistic version of the Birula-Mycielski model.

Based upon an *Ansatz* solution, the problem of getting solutions of this nonlinear Field Theoretical Model reduces to that of finding solutions of a classical mechanical system having only two degrees of freedom. Due to charge conservation, this system is integrable.

Another interesting feature which can be abstracted from 'óur analysis is that, in this classical mechanical analog model, the charge is represented by the angular momentum. Then, there exists a contribution to the energy due to a "charge barrier" (analog to the centrifugal barrier in classical mechanics). A consequence of the existence of the "charge barrier" is that there are no well behaved solutions (bound states[3] or resonances[4] within the usual interpretation) with too large a charge. We have well behaved solutions as long as the charge does not exceed a critical value.

Besides energy and charge, classical solutions differ one from another in that some of them are bounded in time while others are not.

We discuss the stability of some of these solutions in the light of two criteria of stability. From our study, we conclude that these criteria are not equívalent.

The plan of this paper is as follows: in section II we present Birula-Mycielski model as well as the simplifying *Ansatz* solutions. In Section III, we discuss the solutions of this model, exhibiting explicitly some "almost" periodic solutions. Questions related to the stability of classical solutions are discussed in Section IV. Conclusions are left to Section V.

2. THE BIRULA-MYCIELSKI MODEL

Motivated by a formulation of a nonlinear Quantum Mechanics which maintains the factorization property of wave functions for composed systems, Birula and Mycielski[1] were led to a model whose relativistic extension is described by the Lagrangian density:

$$L(\vec{x},t)= \partial_\mu\phi^*\partial^\mu\phi - (\lambda^{-2}+\ell^{-2})\phi^*\phi + \ell^{-2}\phi^*\phi\ln(\phi^*\phi a^{n-1}), \qquad (2.1)$$

where $\phi(\vec{x},t)$ is a scalar complex field; ℓ, λ and a are dimensional parameters, and n stands for the number of spatial dimensions.

The Euler-Lagrange equation derived from (2.1) is

$$\left[\Box + \lambda^{-2} - \ell^{-2}\ \ln(|\phi|^2 a^{n-1})\right]\phi(\vec{x},t)= 0 \ . \qquad (2.2)$$

The conserved quantities, Energy and Charge, are obtained by integrating their respective densities:

$$E = \int d^n\vec{x}\ \ \{2\,\frac{\partial}{\partial t}\,\phi^*\,\frac{\partial}{\partial t}\,\phi - L\,\}\ \ , \qquad (2.3)$$

$$Q = \int d^n\vec{x}\ i\ \{-\ \phi^*(\frac{\partial}{\partial t}\phi) + (\frac{\partial}{\partial t}\,\phi^*)\phi\}\ . \qquad (2.4)$$

The striking feature of this model is that if one looks for an *Ansatz* solution of the form

$$\phi(\vec{x},t) = f(t)\ \exp\ (-\ \vec{x}^2/2\ell^2)\ , \qquad (2.5)$$

one can reduce the study of certain solutions of a system with an infinite number of degrees of freedom to the study of solutions of a system with two degrees of freedom. Charge conservation makes that system integrable. In order to show this, we shall write first the equation satisfied to by $f(t)$. From *(2.2)* and from our *Ansatz* (2.5), we shall get

$$\left[\frac{d^2}{dt^2} + \omega_0^2 - \ell^{-2} \ln |f(t)|^2\right] f(t) = 0 \,, \tag{2.6}$$

where ω_0^2 is

$$\lambda^{-2} + \ell^{-2}(n - \ln a^{n-1}) . \tag{2.7}$$

By using (2.3), we can find the classical energy associated to solution (2.5). It is given by

$$E = c_n \{ \left|\frac{df}{dt}\right|^2 + \omega_0^2 - \ell^{-2} \ln|f(t)|^2 \cdot |f(t)|^2 \} \,, \tag{2.8}$$

while its charge is (see equation *(2.4)*)

$$Q = i\, c_n \left[(\frac{d}{dt} f^*) f - f^* (\frac{d}{dt} f) \right] , \tag{2.9}$$

where c_n is given by

$$\int d^n \vec{x} \, \exp(-\vec{x}^2/\ell^2) = (\sqrt{\pi}\, \ell)^n . \tag{2.10}$$

Equation (2.6) can be integrated if we make use of energy and charge conservation. This is a very common procedure in classical mechanics where one exploits energy and angular momentum conservation in order to integrate the equations of motion of a particle under the action of a conservative central force. That can be achieved more easily by introducing new real variables $\rho(t)$ and $\theta(t)$, defined by

$$f(t) = \rho(t) \exp[i\theta(t)] . \tag{2.11}$$

In terms of these new variables, we shall get, after substituting *(2.11)* into (2.9),

$$\frac{Q}{2c_n} = \rho^2(t)\, \dot{\theta}(t) = L \,, \tag{2.12}$$

and, for the Energy,

$$\frac{E}{c_n} = \left(\frac{d}{dt}\rho\right)^2 + \frac{L^2}{\rho^2} + (\omega_0^2 + \ell^{-2})\rho^2 - \ell^{-2}\rho^2 \ln \rho^2 . \quad (2.13)$$

Equations (2.12) and (2.13) justify our earlier statements on obtaining a very large class of solutions. As we shall see shortly, solutions obtained earlier[1,2] are vecy special cases of this general class.

Looking to (2.12) and (2.13), we can see that for a fixed value of E and Q this system of equations is integrable. Some explicit solutions, as well as discussions concerning their general features, will be presented in the next Section.

3. CLASSICAL SOLUTIONS

What we have achieved in the last Section was, ultimately, to reduce the problem of obtaining some classical solutions of a relativistic Field Theoretical Model to that of obtaining solutions of a classical mechanical system, namely, the motion of a particle whose mass is 2 in a central field whose "effective" potential is

$$V_{eff}(\rho) = \frac{L^2}{\rho^2} + (\omega_0^2 + \ell^{-2})\rho^2 - \ell^{-2}\rho^2 \ln \rho^2 . \quad (3.1)$$

In Fig.(1), we plot V_{eff} vs. ρ. From this figure, one can infer most of the general features of the solutions.

The solutions for (2.6) can be classified into two categories: the bounded (in time) solutions and the unbounded ones. The bounded solutions are those that, for a fixed $\vec{x}$, and for any time, satisfy

$$|\phi(\vec{x},t)| < \text{const.} \quad (3.2)$$

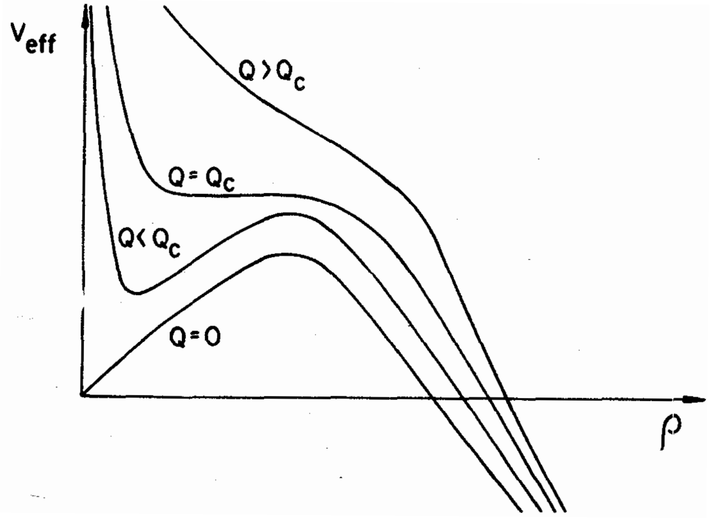

Fig.1 - Effective potential for different values of the charge.

From (2.12) and (2.13), we can realize the existence of the "charge barrier". In Fig.1, we plot $V_{eff}(\rho)$ for various values of the charge. As the charge increases, the depth of the "effective well" diminishes. As a consequence, there are no bound solutions if the charge exceeds a critical value Q_c .

Periodic solutions are examples of bounded solutions. One can always find, for the potential (3.1), simple periodic solutions corresponding to the "circular motion". These solutions are obtained by imposing that

$$\rho(t) = R = \text{constant} \; . \tag{3.3a}$$

From (2.12) and (3.3a) it follows that

$$\dot{\theta} = \omega = \frac{L}{R^2} \; . \tag{3.3b}$$

The constant R, in Eq.(3.3a), can be expressed in terms of the constants of the theory and the "angular momentum" L. In the points of extremum of the effective potential, we have

$$\frac{L^2}{R^4} = \omega_0^2 - \lambda^{-2} \ln 2 = \omega^2 . \tag{3.4}$$

From (2.5), (2.10), (3.3a) and (3.3b), we shall get

$$\phi(\vec{x},t) = R(\omega) \exp(i\omega t - \vec{x}^2/2\ell^2) \; . \tag{3.5}$$

These periodic solutions correspond to the ones obtained by the authors of Refs.(1) and (2). In a recent paper, we studied[4] the stability of these solutions. Here, we shall tackle the stability of (3.5) by using a procedure which exploits the study of the stability of classical mechanical solutions.

This question of mechanical stability is related to the problem of obtaining explicit (but approximated) solutions of (2.6), or equivalently (2.2), by considering small oscillations around a circular orbit Keeping the charge (angular momentum) fixed, and defining

$$\omega_r = \sqrt{4(\ell^2/R^4) - 2\ell^{-2}} = \sqrt{4\omega^2 - 2\ell^{-2}} \quad , \tag{3.6}$$

these new solutions can be written as

$$\rho(t) \simeq R + \varepsilon \cos\omega_r t \quad , \tag{3.7}$$

and (assuming $\varepsilon/R << 1$)

$$\dot{\theta}(t) = \omega[1 - 2(\varepsilon/R) \cos \omega_r t] \ . \tag{3.8}$$

From equations (3.7) and (3.8), we get

$$\phi(\vec{x},t) \simeq \exp(i\omega t - \vec{x}^2/2\ell^2)R\left[1+ \frac{\varepsilon}{R}(\cos\omega_r t + i\,\frac{\omega}{\omega_r}\sin\omega_r t)\right] \ . \tag{3.9}$$

Since in general $\frac{\omega_r}{\omega} \neq \frac{k}{m}$ (k and m being integers), solutions (3.9) are not periodic. Based upon them, we shall study the stability of solutions (3.5). We shall postpone this discussion until next Section.

Up *to* now, we have given explicit examples of periodic arid nonperiodic bounded solutions of (2.6). An example of an unbounded çolution, with zero energy and charge, was presented by Birula and Mycielski[1] and was discussed in Ref. (2). In our scheme, this solution if of the form

$$\rho(t) = A \exp(t^2/2\ell^2) \ . \tag{3.10}$$

The constant A can be fixed by substituting (3.10) into (2.13) and remembering that it corresponds to L=0=E.

Although all unbounded solutions, and in particular (3.10), have a peculiar behavior in time, we have not found any reason (on physical grounds) for ruling them out in the context of Classical Field Theory.

In Fig.(2), we plot E vs. Q. In this figure, we have called the attention to the curves corresponding to "circular motions". The dashed curve shows the "circular motions" with higher energy resonances[4], whereas the continuous curve shows "circular motions" having lower energy bound states. All bounded classical solutions, in the sense of (3.2), have energies lying within the region limited by the resonance and bound states curves. The remaining part of the E-Q plane is occupied by the unbounded solutions.

Finally, integrating explicitly the system of equations (2.12) and (2.13), we obtain

$$t - t_0 = \pm \int_{\rho_0}^{\rho} \frac{d\rho'}{\sqrt{\left(\frac{E}{c_n}\right)^2 - \left(\frac{Q}{2c_n}\frac{1}{\rho'}\right)^2 - (\omega_0^2 + \ell^{-2})\rho'^2 + \ell^{-2}\rho'^2 \ln \rho'^2}} \qquad (3.11a)$$

and

$$\theta - \theta_0 = \frac{Q}{2c_n} \int_{t_0} \frac{dt'}{\rho^2(t')} \qquad (3.11b)$$

4. STABILITY OF CLASSICAL SOLUTIONS

If one wants to associate classical solutions of field theories to extended Hadrons, the question of stability of these solutions is the one which must be investigated first. Concerning to the stability of solutions of classical field theoretical models, there are, in the literature, two widely used criteria of stability.

The first one was proposed by Poincaré[5], about seventy years ago. According to Poincaré, a given classical solution is stable if the as-

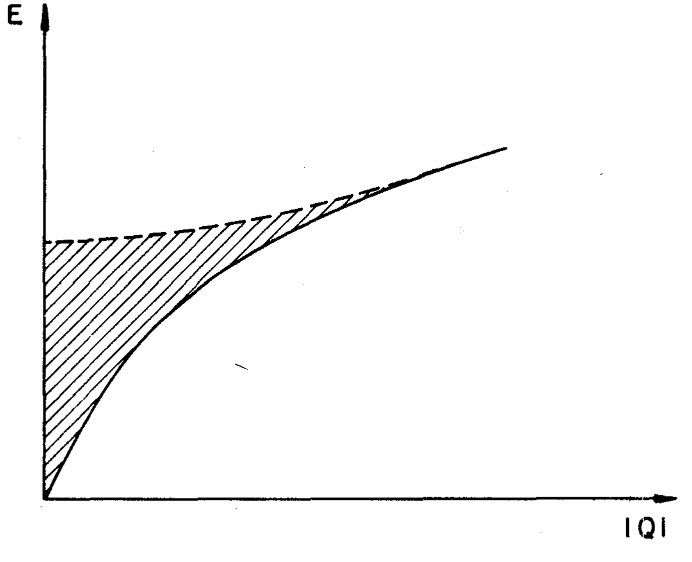

Fig.2 - E–Q plane: the shaded region shows the habitat of the bounded solutions.

sociated canonical Energy-Momentum Stress tensor has vanishing components corresponding to self-stresses, namely,

$$T^{c}_{ij} = 0 \ . \qquad (4.1)$$

The canonical Energy-Momentum Stress tensor associated which the Lagrangian (2.1) is

$$T^{c}_{\mu\nu} = \partial_\mu\phi^*\partial_\nu\phi + \partial_\mu\phi\partial_\nu\phi^* - g_{\mu\nu}L \ . \qquad (4.2)$$

Solutions (3.5), and those corresponding to (3.10), are not stable in the sense of (4.1). However, Birula and Mycielski discovered an improved Energy-Momentum Stress tensor given by

$$T^{BM}_{\mu\nu} = T^{c}_{\mu\nu} + \frac{1}{2}\left[\partial_\mu\partial_\nu\phi^*\phi - g_{\mu\nu}\Box\phi^*\phi\right] , \qquad (4.3)$$

such that

$$T^{BM}_{ij} = 0 \ , \qquad (4.4)$$

implying that, at least in the sense of (4.4), solutions (3.5) and (3.10) are stable.

Another method of studying stability is that of the "infinitesimal stability"[6]. It relies upon the study of the behavior of a small fluctuation about a classical solution. In this way, if $\phi_c(\vec{x},t)$ is a classical solution, and $\phi(\vec{x},t)$ is another solution close to $\phi_c(\vec{x},t)$, we can write

$$\phi(\vec{x},t) = \phi_c(\vec{x},t) + \eta(\vec{x},t) \ . \qquad (4.5)$$

Now, if we insert (4.5) into (2.2), and linearize the resulting equation for $\phi(\vec{x},t)$, we say that $\phi_c(\vec{x},t)$ is stable if, for a fixed $\vec{x}$, $\eta(\vec{x},t)$ is bounded for all t.

The procedure just sketched leads, for the solution (3.10), to the following equation for the fluctuation:

$$\left[\Box - \frac{(t^2-\vec{x}^2)}{\ell^4} + \lambda^{-2} - \ell^{-2} - \ell^{-2}\ln(a^{n-1}A^2)\right]\eta = \ell^2\eta^* \ . \qquad (4.6)$$

From (4.6), it is not difficult to see that there exist fluctuations $\eta(\vec{x},t)$ not bounded in time. Then, under the latter criterion, the solution (3.10) is unstable. This result is essentially the one which we have reached by using the criterion (4.1). We mention that if instead of (3.10) we had looked for another unbounded solution, the conclusions, concerning stability, would be the same.

The stability of solution (2.5), in the sense of (4.5), has been discussed in Ref.(4). Here, one observes that a mechanical study[7] of the stability of the "circular motions" (3.3a) and (3.3b) happens to exhibit the main features of our analysis of Ref. (4).

Pursuing the analogy with the two dimensional system a little further, let us study the stability of the "circular motions", solutions (3.3a) and (3.3b). As can be seen from (3.6) and (3.7), the radially perturbed motion is bounded if ω_r is real. This implies that the "circular motion" is stable if

$$4\omega^2 - 2\ell^{-2} > 0 \ . \qquad (4.7)$$

The main consequence of the conditional stability (4.7) is that, for a given charge (not greater than the critical value Q_c) there are two classical states differing in its energies. The one with the greatest energy is unstable (which we have interpreted as a resonance[4]) while the lowest is stable. In Ref. (4), we reached the same conclusions by using the "infinitesimal stability".

Equation (3.6) indicates that solutions with too large periods are unstable, under the second criterion. On the other hand, criteria (4.1) and (4.3) do not impose any condition on the periods. In this way, from the examples we studied we can infer that the criterion of Poincaré and the one employing small fluctuations are not equivalent. We have essentially two distinct criteria. This situation is common in classical mechanics, where a given solution can be stable under the "Orbital Criteria" of stability, and be unstable under "Lyapunov's Criteria"[8].

5. CONCLUSIONS

We have obtained a very large class of solutions of a nonlinear relativistic Field Theoretical Model, in any number of dimensions. Some of these solutions have been obtained previously. The most interesting features which emerged from our analysis are given below.

There exist classical solutions not bounded in time whose associated energy and charge are finite. Among the bounded solutions, there are periodic ones that can be associated to bound states, and others which, being unstable and having higher energies than the stable ones, we have associated to resonances.

We have found a "charge barrier" which forbids the occurrence of bounded solutions whose charge exceeds a critical value. If one associates these bounded solutions with extended hadrons, the existence of a "charge barrier" has far reach consequences since we cannot build hadrons with charge greater than a critical one. This critical value is[4]

$$\pi^{3/2}\,(\ell/a)^2\,\exp\left[\frac{5}{2}+\left(\frac{\ell}{\lambda}\right)^2\right], \qquad (5.1)$$

where we have taken $n=3$ (the physical world).

The conclusion is that exotic hadrons (particles having very large charge) are forbidden. This is in agreement with experimental facts. We could speculate that this "charge barrier mechanism" can also operate within more realistic theories leading to the non existence of exotic particles.

The study of the stability of classical solutions has been carried out by using two stability criteria. We have pointed out that these criteria are not equivaient.

These achievements were possible because the interaction Lagrangian of the model is such that, if one seeks for a solution like (2.5), one can reduce the search of classical field theoretical solutions to that of a classical mechanical system having two degrees of freedom.

Some difficulties which we found in the quantization of the model (2. 1) - even at a semiclassical level (WKB approximation) - have been pointed out in Ref.(4).

It is well known[3,6] that, in the process of semi-classical quantization, the relevant solutions are the periodic ones. This might imply that our unbounded solutions have no counterpart within the quantized theory.

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