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A Quantum Phenomenology of Viscosity*

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A quantum formalism, including phenomenological dissipation, is developed from an extension of the classical canonical formalism proposed by Morse and Feshbach and applied to the time evolution of a Gaussian wave-packet.

Desenvolvemos neste trabalho um formalismo quântico que inclui fenomenologicamente efeitos de dissipação. Esse formalismo é obtido a partir de uma extensão do formalismo canônico clássico proposto por Morse e Feshbach e aplicado à evolução temporal de um pacote de ondas gaussiano.

1. INTRODUCTION

The observation of strongly inelastic collisions between heavy ions at energies well above the Coulomb barrier has kindled interest once again on the description of dissipative processes in quanta1 systems[1]. The high level densities of the colliding nuclei at the relevant excitation energies suggest moreover that the transfer of energy from

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the relative motion to other modes may tolerate a phenomenological description in terms of some quantum extension of the classical concept of viscosity.

A classical deterrninistic treatment of viscous effects fits naturally in the scherne of Newtonian mechanics, by means of the inéroduction of forces that cannot be derived from a potential function. This very fact, however, tends to make it alien to quantum mechanics, at least in so far as it is constructed upon the formal structure of the classical canonical formalism. This formalism is particularly ill adapted to deal with nonconservative phenomena. Its essential coherence is irrevocably broken by the introduction of *ad–hoc* devices such as the Rayleigh dissipation function. Explicitly time-dependent Lagrangians, on the other hand, are known to lead, upon quantization, to difficulties related to the resulting explicit time-deperidence of the canonical variables[2].

In this paper we want to explore within a quantum context, a possible gap in the imperviousness of the canonical formalism to phenomenological dissipation. It has been pointed out by Morse and Feshbach[3] that a bilinear Lagrangian in the two degrees of freedom, x and y,

$$L_B = m\dot{x}\dot{y} + \frac{R}{2}(\dot{y}x - \dot{x}y) - g_1 x - g_2 y - kxy$$

leads, by means of the least action principle, to two independent equations for $x(t)$ and $y(t)$ which contain velocity-dependent dissipative forces:

$$m\ddot{x} + R\dot{x} + g_2 + kx = 0 ,$$

$$m\ddot{y} - R\dot{y} + g_1 + ky = 0 ,$$

R standing for the friction coefficient.

Since L_B is a time-independent Lagrangian, one might use the standard canonical quantization procedure to obtain a theory that would, at least, contain in its classical limit the behavior of a linearly damped system under linear conservative forces.

2. THE ACTION FOR THE BILINEAR LAGRANGIAN

It turns out, however, that such a straightforward procedure meets with several problems. In order to see what they are, we may take the simplified Lagrangian

$$L_B = m\dot{x}\dot{y} + \frac{R}{2}(\dot{y}x - \dot{x}y) ,$$

from which the linear conservative forces have been omitted, and study the behavior of the corresponding action functional

$$S_B[x(t), y(t)] = \int_{t_1}^{t_2} L_B \, dt$$

for suitably parametrized families of space-time paths $x(t)$, $y(t)$. In particular, taking a family of fixed end-points, uniformly accelerated motions with accelerations γ_x and γ_y respectively, one gets, in the limiting case when $R=0$,

$$S_B[\gamma_x, \gamma_y] = \text{Const}_B \, \gamma_x \gamma_y ,$$

while the usual quadratic Lagrangian for two free particles

$$L_Q = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

gives

$$S_Q[\gamma_x, \gamma_y] = \text{Const}_Q (\gamma_x^2 + \gamma_y^2) .$$

This illustrates the fact that, while being both stationary at the uniform motion trajectories with $\gamma_x = \gamma_Y = 0$, these two action functionals have different properties for paths differing from the classical space-time path. Since quantum propagation can be described in terms of the behavior of such action functionals for all space-time paths between the chosen end-points, and given that S_Q leads to the usual quantum mechanical description of two noninteracting particles, it follows that one should not expect to obtain a proper quantum me-

chanical description by quantizing L_B, even in the frictionless (i.e., $R=0$) limit, without any further considerations.

It should be borne in mind, however, that the y degree of freedom introduced in the bilinear Lagrangian L_B is not meant to be physically meaningful in the same sense as x. In fact, the classical equation of motion gives unphysical exponentially growing solutions for this variable when R is not zero. In a purely classical context, this is of little consequence as the Euler-Lagrange equations for the two degrees of freedom are completely independent. Quantum-mechanically, however, this separation argument is no longer applicable (as can be seen e.g. from the non-separability of the Lagrangian in thevariables x and y) and one has to properly specify the role of y.

In order to do that, we recall that what we would like to build cut of L_B is a quantum formalism for a single degree of freedom that would include the effects of a linear viscous force. In addition to having the correct classical limit (by which we mean that the solution can be described in terms of that of the classical Euler-Lagrange or Newton equations with a linear damping force when the action involved are large in units of $\hbar$), we require the theory to reproduce the usual free particle quantum mechanics in the limit of zero friction. We will show next that this additional correspondence requirement specifies to a large extent the role played by y.

3. THE BILINEAR PROPAGATOR: CORRESPONDENCE WITH STANDARD QUANTUM MECHANICS OF A FREE PARTICLE

For reasons to be made clear below, we choose to describe the x-system in terms of a density matrix $\rho_x(x, x';t)$. The time evolution of this density matrix in the frictionless limit is more simply expressed in momentum space by using the standard quantum mechanical propagator for a free particle:

$$\tilde{\rho}_x(k_x k_x';\, t) = e^{\frac{i\hbar t}{2m}(k_x'^2 - k_x^2)}\, \tilde{\rho}_x(k_x k_x';0) \ ,$$

where now $\tilde{\rho}$ is just the double Fourier transform of ρ. If one wants to use the bilinear Lagrangian, in the frictionless limit, one gets, for a two particle density matrix $\tilde{\rho}(k_x k'_x, k_y k'_y; t)$,

$$\tilde{\rho}(k_x k'_x, k_Y k'_Y; t) = e^{\frac{i\hbar t}{m}(k'_x k'_y - k_x k_y)} \tilde{\rho}(k_x k'_x, k_Y k'_y; 0) .$$

Since, however, all we want to describe is system x, we may require in addition that

$$\mathrm{tr}_y \tilde{\rho}(k_x k'_x, k_y k'_y; t) = \int dk_y \, \tilde{\rho}(k_x k'_x, k_y k'_y; t) = \tilde{\rho}_x(k_x k'_x; t)$$

provided that the initial density has been also so chosen that

$$\mathrm{tr}_y \tilde{\rho}(k_x k'_x, k_y k'_y; 0) = \tilde{\rho}_x(k_x k'_x; 0) .$$

Use of the explicit form of the two propagators and of the symmetries of $\tilde{\rho}$ leads at once to the condition that the density in the two degrees of freedom be of the form

$$\tilde{\rho}(k_x k'_x, k_y k'_y; 0) = \tilde{\rho}_x(k_x k'_x; 0) \delta\left(\frac{k_y + k'_y}{2} - \frac{k_x + k'_x}{2} \right) G(k_y - k'_y) ,$$

where $G(z)$ is a properly normalized, arbitrary, even function of its argument. By Fourier transforming back into configuration space we get, correponding to the initial density for x, $\rho_x(xx'; 0)$, a density for x and y which is of the form

$$\rho(xx', yy'; 0) = \rho_x\left(x + \frac{y-y'}{2}, x' - \frac{y-y'}{2} ; 0 \right) g\left(\frac{y+y'}{2} \right) ,$$

where again $g(z)$ is any properly normalized even function of its argument. This function (or equivalently, the function $G(z)$) are enti-

rely redundant in the frictionless limit we just considered. However, as we will show below, it plays an important role when frictional effects are included.

We see, thus, that it is possible to do ordinary one particle quantum mechanics by using the bilinear Lagrangian L_B (with R = 0) and considering densities in two degrees of freedom of the particular form given above. What we attempt to do next is to include frictional effects by merely turning on the R-term of L_B in the bilinear propagator. Because of the special form of L_B, this may be done by directly evaluating the corresponding path integral

$$K(xx',yy';t) = \int D[x(t)] \int D[y(t)] \exp \frac{i}{\hbar} \int_0^t L_B \, dt$$

which is of the Gaussian type and can thus be worked out explicitly[4]. Propagation including friction would then appear as

$$\rho_x(xx';t) = \int dy \iiiint dudvdu'dv' \; K(u'x',v'y';t) \times$$

$$\times \rho_x \left(u + \frac{v-v'}{2}, u' - \frac{v-v'}{2} ; 0 \right) g \left(\frac{v+v'}{2} \right) K^*(ux,vy;t) \equiv$$

$$\equiv \iint dudv \; K_{eff}(ux,vx';t) \rho_x(uv;0) \; ,$$

where the effective propagator K_{eff} can be explicitly written as

$$K_{eff} = \frac{R}{2\pi\hbar\beta_-} \exp\{ - \frac{\alpha}{2} \left[(u-v) - \frac{\gamma}{\beta_-} (x-x') \right] \left[\beta_+ (x+x') + \gamma(u+v) \right] \} \times \Gamma_t(x-x')$$

where

$$\alpha = \frac{R}{2\hbar} \frac{1}{1-\cosh \frac{Rt}{m}} \; ; \; \beta_\pm = 1 - e^{\pm \frac{Rt}{m}} \text{ and } \gamma = \sinh \frac{Rt}{m} \; .$$

The last factor, $\Gamma_t(x-x')$, originates from the function g and is given by

$$\Gamma_t(x-x') = \int dz \exp\left[\frac{iR\beta}{4\hbar}(x'-x)z\right] g(z) \ .$$

It is constant for the diagonal part of the density matrix (probabiiities), but effects the time propagation of theoff diagonal part (correlation). In particular, if $g(z)$ is a Gaussian, Γ_t will attenuate and eventually destroy all correlations in configuration space.

This feature of the theory actually requires that the description of the x-system be in terms of a density matrix. It is an extremely reasonable feature on physical grounds. In fact, any viscous effects must be seen ultimately as resulting from the coupling of the explicitely retained degree of freedom to other degrees of freedom that remain unobserved. This situation will lead in general tothe increasing occurrence of correlations between observed and unobserved modes, which implies in the loss of initial quantum coherence in the observed degree of freedom alone. This point of view hints, moreover, to the possible interpretation of the function $g(z)$ as a phenomenological function describing the effectiveness of the unobserved modes (here mocked up by the single degree of freedom y) in destroying the quantum coherence of the initial state in the x-variable.

Wemay also mentionother easily verified features of the theory. First, the bi1inear Hami1tonian, associated with L_B, being the generator of time translations and being itself independent of time, is a constant of motion. It must not, however, be interpreted as the energy of the system. The latter may be defined by correspondence with the classical energy, i.e.,

$$E_x = \frac{1}{2} m \dot{x}^2$$

and is <u>not</u> a constant of motion unless $R=0$. In general, E_x decays exponentially with time. The Hamiltonians is, on the other hand, a Hermitian operator, leading to the conservation of the integrated pro-

babilities. A more delicate point concerns the position-velocity uncertainty relations for the bilinear theory. The canonical momenta are, in fact, in the frictionless limit, $p_x = m\dot{y}$ and $p_y = m\dot{x}$, so that the canonical comnutation relations (which are of course consistent with the path-integral quantization) give, in general,

$$[x,\dot{x}] = [y,\dot{y}] = 0 \; ; \; [x,\dot{y}] = [y,\dot{x}] = \frac{i\hbar}{m} \; .$$

This is at variance with what one should expect for the proper quantum description of two degrees of freedom. It can be verified, however, that these comnutation relations, used in connection with density matrices of the special form considered here, give results that agree with those obtained from the usual commutator $[x,\dot{x}] = i\hbar/m$ and single degree of freedom densities $\rho_x(x,x')$.

4. GAUSSIAN WAVE-PACKET WITH FRICTION

We finish by briefly mentioning the result of the application of the above theory to the propagation of a Gaussian wave-packet given initially by the density

$$\rho_x(xx';0) = N\, e^{ik_0(x-x')} \; e^{-\frac{x^2+x'^2}{2b_0^2}} \; .$$

The probability distribution $\rho_x(xx;t)$, at time t, computed with the effective propagator K_{eff} evolves in time retaining its Gaussianshape and in such a way that its center of gravity obeys Newton's equation for a particle under the effects of a linear viscous force. The spreading of the wave packet is quenched by the frictional effects[4] in such a way that its width b(t) tends to a constant for large times. As shown in the Fig.1, this constant may actually be less than the initial width $b(0) = b_0$ if the friction coeficient R is largeenough. For one special value R_c of R, in particular, one has $b(\infty) = b_0$: the packet will therefore stop and freeze under the effects of friction. The time dependence of correlations involves a more detailed study of

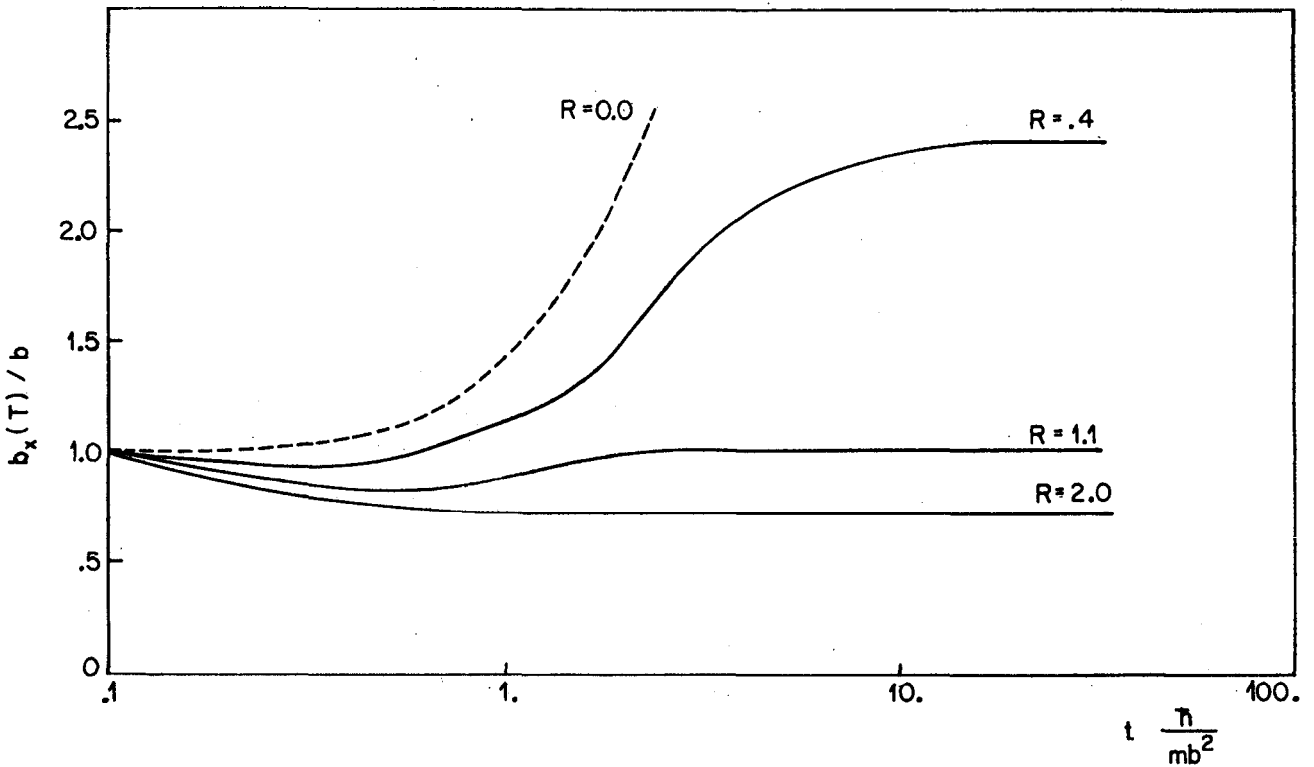

Fig.1. Width of the Gaussian probability distribution as a function of time for various values of the friction coefficient R; R is given in units of $\hbar/b^2$.

the physical content of the function g and will not be discussed here.

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