Revista Brasileira de Física, Vol. 7, Nº 2, 1977

A Basis Expansion for the Hyperspherical Approach in a Simple Triton Model*

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Recebido em 24 de Novembro de 1976

Generalized harmonic oscillator functions (GHOF) are shown to be a very convenient (complete, orthonormal) set of basis functions in which to expand the radial functions in the K-harmonic approach to the three body problem. This is illustrated for a simple ^3H model. The binding energy, root mean square radius and form factor for elastic electron scattering are shown to be very stable against variation in the parameter of the GHOF. The Morpurgo and Feshbach-Rubinow methods are also compared with the exact K-harmonic approach. It is shown that the former is the K=O K-harmonics solution and that the latter constitutes a very accurate approximation when purely attractive potentials are used.

Tornando-se como exemplo um modelo simplificado do núcleo de ^3H, mostra-se que as funções do oscilador harmônico generalizado (FOHG) constituem um conjunto muito conveniente de funções de base para se expandir as funções de onda radiais do problema de três corpos, no método dos K-harmônicos. A energia de ligação, o raio quadrático médio, e o fator de forma elástico para o espalhamento de elétrons, resultam bas-

* Supported in part by CNPq (Brazil) and National Research Council (Canada).

tante estáveis frente à variação do parâmetro das FOHG. Comparam-se os métodos de Morpurgo e Feshbach-Rubinow com o método dos K-harmônicos exato, mostrando-se que o de Morpurgo corresponde ao método dos K-harmônicos com $K=0$, enquanto que o de F-R constitue-se numa aproximação deveras precisa para potenciais puramente atrativos.

1. INTRODUCTION

The analysis of the few body systems took a leap forward in the last decade with the advent of so-called exact methods, such as Faddeev's equations (Faddeev, 1961) and the K-harmonics (also called hyperspherical) approach (Simonov, 1966). The former method reduces to a set of integral equations whose kernels depend on the two-body scattering amplitudes on and off energy-shell. The K-harmonic method, on the other hand, is a generalization of the spherical harmonics used in the $N=2$ body system to cases $N\geq 3$ (hence the hyperspherical label) and is meant to be used for bound state problems. In this way, the N-body wave f'unction (depending on $3N-3$ coordinates in the C.M. frame) is expanded in terms of the hyperspherical harmonics, which form a complete orthonornial set spanning the $3N-4$ space of hyperangles. The problem is thus reduced to solving an infinite set of coupled differential equations for the hyperradial function, which in turn depends on the knowledge of the interaction between the particles.

Few papers have been published using the K-harmonics approach. V.D. Efros (Efros, 1971; Efros, 1972) has derived, and used in the case of H^3 and He^4, generalized hyperspherical functions for systems with total angular momentum $L\neq 0$. We use on the other hand the L=O hyperspherical functions designed by Yu. Simonov (Simonov, 1966). G. Erens et *al.* (Erens, 1971) have used a matrix diagonalization approach but with a particular set of hyperspherical harmonics and a set of hyperradial functions. His approach is discussed i'n Section 2. M. Beiner and M. Fabre de la Ripelle (Beiner, 1971) and J.L. Ballot et *al.* (Ballot, 1972) looked at the convergence of the hyperspherical expansion itself, solving numerically the

system of coupled differential equations for the radial functions. They introduce the notion of optimal subset, which is a recombination of the original hyperspherical functions, to reduce the number of functions to be included in the expansion.

The aim of this paper is to show that the solution of the system of coupled differential equations is more efficiently obtained if one chooses to further expand the hyperradial function in a complete set of basis functions, enabling one to use standard matrix diagonalization techniques in the solution. We Illustrate our point by calculating observables (binding energy, root mean square radius and form factor for elastic scattering) for a 'model triton', treating only the symmetric L=O part of the wave-function. In this paper, we do not use the technique of optimal subset, because in the case of a spinless boson system we need to include only a reasonable number of functions because of the selection rule (symmetry). Further, we do not look at the convergence of the hyperspherical expansion **per** *se* , but only at the convergence of the expansion of the radial functions. After explaining the method in Section 2, we give numerical results for the model triton in Section 3, using various two-body potentials, emphasizing the convergence rate of the solution in each case.

It is also of great interest to compare the three-body solution obtained using the K-harmonic method with solutions resulting of simpler (but approximate) methods proposed earlier. We do this in Section 4, where we show that the Morpurgo (1952) method is identical to the K=O K-harmonic formalism and that the Feshbach-Rubinow (1955) method yields a very accurate solution when purely attractive potentials are used.

2. THE METHOD

The three-body wave-function depends on six relative coordinates, once the C.M. motion is separated out. If we use the Jacobi coordinates, na-

mely,

$$\vec{\xi} = \sqrt{2/3}\ \left(\frac{\vec{r}_1+\vec{r}_2}{2} - \vec{r}_3\right),$$

$$\vec{\eta} = 1/\sqrt{2}\ (\vec{r}_1-\vec{r}_2), \qquad (2.1)$$

$$\vec{R} = 1/\sqrt{3}\ (\vec{r}_1+\vec{r}_2+\vec{r}_3),$$

the Schrödinger equation becomes

$$\left[-\frac{h^2}{2m}\left(\nabla_{\vec{\xi}}^2 + \nabla_{\vec{\xi}}^2\right) + V_{123}\right]\psi(\vec{\xi},\vec{\eta}) = E\psi(\vec{\xi},\vec{\eta}), \qquad (2.2)$$

where, for a pairwise interaction,

$$V_{123} = V_{12}\ (\sqrt{2}\ \vec{\eta}) + V_{13}\left(\left|\sqrt{3/2}\ \vec{\xi} + \frac{1}{\sqrt{2}}\ \vec{\eta}\right|\right) \qquad (2.3)$$

$$+ V_{23}\left(\left|\sqrt{3/2}\ \vec{\xi} - \frac{1}{\sqrt{2}}\ \vec{\eta}\right|\right).$$

In the K-harmonics approach, we expand the relative wave function $\psi(\vec{\xi},\vec{\eta})$ in terms of products of radial functions $\chi_{K\nu}(\rho)$ and angular functions $u_{K\nu}(\{\theta_i\})$:

$$\psi = \sum_{K\nu} \chi_{K\nu}(\rho) u_{K\nu}(\{\theta_i\}), \qquad (2.4)$$

where the global length is defined as $\rho = \{\vec{\eta}^2 + \vec{\xi}^2\}^{1/2}$ and $\{\theta_i\}$ stands for five angles specified by $\vec{\eta}$, $\vec{\xi}$ and 8 through $\eta = \rho\cos\theta$, $5 = \rho\sin\theta$, $0 \leq \theta \leq \pi/2$. A complete set of such angular functions $u_{K\nu}(\{\theta_i\})$ has been derived for the case where the total angular momentum L is zero ($L = 0$) by Simonov (1966). In this case, the $u_{K\nu}$ are the angular parts of the homogeneous polynomials of degree K which satisfy the Laplace equation. K is referred to as the global angular momentum and characterizes the convergence of the wave function expansion. In this case (L=0), only one extra quantum number (Simonov, 1966), denoted ν($\nu = -K/2, -K/2+2, \ldots K/2$), is necessary to specify the states.

It is known that the symmetric $L=0$ wave function accounts for 90% of the ground state wave functions of the physical triton (Strayer, 1974). For the purpose of testing our approach, we consequently use the 'model triton', treating only the $L=0$ symmetric part of the wave function. These functions, in the hyperspherical method, have been given explicitly by Simonov (1966) and are denoted $v_{K\nu}$ ($\nu = K/2, K/2 - 2, \ldots \geq 0$). If we substitute the expansion (2.4) into (2.2), we can see that $\chi_{K\nu}(\rho)$ will have to satisfy a system of coupled differential equations (Simonov, 1966). V_{123} will enter the equations through an effective radial potential term, defined as the matrix elements of V_{123} in the hyperspherical angular part.

This system of coupled differential equations is usually solved numerically when V_{123} and appropriate boundary conditions are specified. It was the purpose of our earlier work (Vallières et *al.*, 1976) to warn that a more convenient way of obtaining the solution of the system of coupled differential equations is to expand the radial function in some complete set of basis functions, thus enabling one to use standard matrix diagonalization techniques. Further, we proposed the use of the generalized harmonic oscillator functions (GHOF) as a convenient set in which to perform the expansion. In this paper, we want to analyze, by explicit numerical calculation, the convergence properties of the expansion. Before we proceed to do so in the next Section, for completeness we first sketch the approach.

The GHOF are the solutions of the hyperspherical radial equation for a diagonal effective potentiai proportional to ρ^2. They are denoted $R_{nK}(\mu\rho)$, n being the 'main quantum number' and μ playing the role of the characteristic length of the 3-D harmonic oscillator functions. Pertinent formulas concerning these functions are collected in Appendix A. The radial functions are expanded in terms of these GHOF,

$$\chi_{K\nu}(\rho) = \sum_{n=0}^{\infty} C_n^{(K\nu)} R_{nK}(\mu\rho). \qquad (2.5)$$

In practical calculations, this sum is truncated at $n = \bar{n}$ and, in this situation, the parameter μ can be treated as a variational parameter .

Certainly, if $\bar{n} \to \infty$, the solution will be independent of μ. If the functions R_{nK} are reasonably compact, the expansion should converge rapidly. The $C_n^{(K\nu)}$ are determined by solving the eigenvalue equation

$$\sum_{n'} T_{nn'}^{(K)} C_{n'}^{(K\nu)} + \sum_{K'\nu' n'} v_{nn'}^{(K\nu,K'\nu')} C_{n'}^{(K'\nu')} = -\frac{2m}{\hbar^2} B\, C_n^{(K\nu)} \qquad (2.6)$$

where

$$T_{nn'}^{(K)} = \langle R_{nK} | \frac{1}{\rho^5} \frac{d}{d\rho} \rho^5 \frac{d}{d\rho} - \frac{K(K+4)}{\rho^2} | R_{n'K} \rangle , \qquad (2.7)$$

and

$$v_{nn'}^{(K\nu,K'\nu')} = \langle R_{nK} | V^{\text{eff}}_{K\nu,K'\nu'}(\rho) | R_{n'K'} \rangle \qquad (2.8)$$

$V^{\text{eff}}_{K\nu,K'\nu'}$ being the effective radial potential (Vallières *et al.*, 1976).

This approach presents many advantages: among others, that the boundary conditions are automatically taken into account and that excited states as well as the ground state are solved for. One possible drawback, however, is that the R_{nK} functions have a wrong asymptotic behavior. There exists a way to systematically correct the expansion in the asymptotic region (Vallières *et al.*, 1976). However, we will show in the next Section that in practical situations this correction is not necessary if $\bar{n}$ is taken reasonably large.

Erens *et al.* (1971) have proposed another set of functions,

$$R_n(\rho) = \alpha^3 \left[\frac{n!}{(n+5)!}\right]^{1/2} L_n^5(\alpha\rho) \exp(-\tfrac{1}{2}\alpha\rho),$$

(α is a scaling parameter) in which to expand the radial function. The claim for using this set is that is has the right asymptotic behavior and presents fast convergence; however, it does not correspond to any physical global potential. As we will show in the next Section, we find comparable or better convergence using the GHOF basis as Arens *et al.* did and we do not see any necessity to apply an asymptotic correction in practical solutions. Besides the above hyperradial set of functions

used in Erens paper, a particular set of hyperspherical harmonics which was developed by Fabre de la Ripelle was also used. We do not use this 'optimal subset' in our calculation simply because it would not save us a great amount of computing. This is because, as will be explained in the next Section, we use the K-harmonics with $K \leq 16$, which is enough to illustrate the convergence of the radial expansion. In this case, the optimal subset would save us only 2 out of the 10 hyperspherical functions used.

3. NUMERICAL RESULTS

In this paper, we are considering the 'model triton', treating the three nucleons as bosons, as a test case to illustrate the method. We are also using nucleon-nucleon potentials for which results can be compared with previous authors. For the purpose stated, we need to consider only two extreme classes of potentials (Erens, 1971):

a) One with a singular behaviour at the origin, which we represent by a Yukawa potential (Y) (Bell, 1970)

$$V(r) = - \left(\frac{\hbar^2}{M}\right) b^2 \frac{e^{-r/a}}{(r/a)} ,$$

where $(\hbar^2/M)$ = 41.468 MeV.fm^2, b^2 = 0.7 fm^{-2}, a = 1.58 fm.

b) Another, well-behaved at the origin, which we represent by the Volkov potential (V) (Volkov, 1974)

$$V(r) = \frac{\hbar^2}{M} \left(V_r\, e^{-\mu_r^2 r^2} - V_a\, e^{-\mu_a^2 r^2}\right),$$

where $(\hbar^2/M)$ = 41.4686 MeV.fm^2, V_r = 3.4932455 fm^{-2}, V_a = 2.0097133 fm^{-2}, μ_r^{-1} = 0.82005790 fm and μ_a^{-1} = 1.600512 fm.

For completeness, in a few instances, results using these potentials will be contrasted with those of the purely attractive and smooth Baker po-

tential (B) (Baker, 1962), of the form

$$V(r) = -\frac{\hbar^2}{M} b^2 \exp(-r/a)^2$$

where $(-\hbar^2/M) = 41.496$ MeV fm^2, $b^2 = 1.2418736$ fm^{-2}, and a = 1.6 fm. All of the above forms are taken as the average over the singlet and triplet channels; this is in accordance with our restriction to the symmetric L-O solution.

An important aspect of the numerical solution of the problem, within the K-harmonics framework, concerns the evaluation of the matrix element of the potential. The effective radial potential can be written as (Simonov, 1966)

$$V^{\text{eff}}_{K\nu,K'\nu'}(\rho) = -\left(\frac{2m}{\hbar^2}\right)24\pi^2 \int_0^{\pi/2} d\theta \cos^2\theta \sin^2\theta \int_0^1 d(\cos\phi) v_{K\nu} V_{12}(\sqrt{2}\rho\cos\theta) v_{K'\nu'} \qquad (3.1)$$

where $\cos\phi = \vec{\xi}.\vec{\eta}/\xi\eta$. Since V_{12} does not depend on ϕ, the integral over ϕ can be performed analytically. We are left only with the integral over 8, which is done numerically in two intervals, using 16 points Gaussian quadrature in each interval. Next, the integral over $x = \mu\rho$ in Eq. (2.8) is done numerically in three intervals $(0,a)$, (a,b), (b,x). In general, we find that 20 points Gaussian quadrature, used in each of the three intervals specified by a = 2.5, b = 6.0 and x = 10.0, gives a good accuracy.

Another very important aspect of the numerical work concerns the choice of the diagonalization rnethod. We are using the Lanczos method of 'minimized iterations' (Ibarra, 1973). The algorithm starts from an initial function or eigenvector, and converts the original eigenvalue probleminto one of finding the eigensolution of a tridiagonal matrix. If the initial vector has been chosen properly, the few low-lying eigenvectors of the original eigenvalue problem can be solved accurately by diagonalizing small matrices. This is the real significance of this procedure. Even

TABLE 1

Choice of an optimum μ by looking at the convergence behaviour of the binding energy (Volkov potential). Experimental values for H^3 binding energy and rms radius are -8.48 MeV and 1.70±0.05, respectively.

$K = 10$, $\bar{n} = 8$

$\mu(fm^{-1})$	0.4	0.6	0.7	1.1
B (MeV)	-8.2108	-8.3761	-8.363	-7.992
$\sqrt{\langle r^2 \rangle}$ (fm)	1.927	1.881	1.858	1.764

$K=10$, $\bar{n}=14$

$\mu(fm^{-1})$	0.5	0.6	0.7	0.9	1.1
B (MeV)	-8.435	-8.4396	-8.4368	-8.418	-8.378
$\sqrt{\langle r^2 \rangle}$ (fm)	1.907	1.903	1.898	1.882	1.860

$K=10$, $\bar{n}=19$

μ (fm^{-1})	0.4	0.6	0.9
B (MeV)	-8.4376	-8.4429	-8.4388
$\sqrt{\langle r^2 \rangle}$ (fm)	1.908	1.907	1.899

$K=10$, $\bar{n}=24$

μ (fm^{-1})	0.4	0.6	0.9
B (MeV)	-8.4421	-8.4432	-8.4413
$\sqrt{\langle r^2 \rangle}$ (fm)	1.908	1.908	1.904

$K=12$, $\bar{n}=24$

μ (fm^{-1})	0.4	0.6	0.9
B (MeV)	-8.4686	-8.4697	-8.4678
$\sqrt{\langle r^2 \rangle}$ (fm)	1.909	1.909	1.905

TABLE 2

Choice of an optimum μ by looking at the convergence behaviour of the binging energy (Yukawa potential). Experimental values *for* H^3 binding energy and rms radius are *-8.48* MeV and 1.70±0.05 respectively.

K=10, $\bar{n}$=8

μ (fm^{-1})	0.4	0.6	0.8	1.1
B (MeV)	-3.1351	-3.2524	-3.1402	-2.7150
$\sqrt{\langle r^2 \rangle}$ (fm)	2.132	1.985	1.875	1.747

K=10, $\bar{n}$=14

μ (fm^{-1})	0.4	0.6	0.7	0.8	0.9
B (MeV)	-3.4693	-3.5709	-3.5790	-3.5694	-3.543
$\sqrt{\langle r^2 \rangle}$ (fm)	2.158	2.087	2.054	2.021	1.989

K=10, $\bar{n}$=19

μ (fm^{-1})	0.4	0.6	0.7	0.8	0.9
B (MeV)	-3.566	-3.6416	-3.6535	-3.6551	-3.615
$\sqrt{\langle r^2 \rangle}$ (fm)	2.162	2.121	2.100	2.079	2.009

K=10, $\bar{n}$=24

μ (fm^{-1})	0.7	0.8	0.9
B (MeV)	-3.6748	-3.6777	-3.6757
$\sqrt{\langle r^2 \rangle}$ (fm)	2.112	2.098	2.082

K=12, $\bar{n}$=24

μ (fm^{-1})	0.7	0.8	0.9
B (MeV)	-3.7347	-3.7382	-3.7362
$\sqrt{\langle r^2 \rangle}$ (fm)	2.113	2.097	2.080

though the average size of the matrices involved in this work (number of $(K\nu)$ values x $\bar{n} \lesssim 240$) is small as compared with the matrices encountered in most shell model calculations, the use of the Lanczos procedure makes it particularly rapid and efficient to use the expansion in terms of GHOF.

Tables 1 and 2 show the procedure to obtain an optimum value of the parameter U, when $\bar{n}$ and the largest K values are supplied. This is shown for the Volkov and Yukawa potentials. Calculations are done for the ground state binding energies and rms radius of ^{3}H. The rms radius of the three-boson system is corrected bytaking into account the rms radius of the nucleon (Jansen et *al*. 1966). It is significant that for $\bar{n} \geq 19$ the results for both potentials are stable against variation in μ. Table 3 shows optimum binding energies for calculations with various K and $\bar{n}$ for the three different potentials.

Ultimately, we want to show that the GHOF constitute a' good set of basis functions. We already have seen the stability of both the binding ener-

TABLE 3

Binding energies, $|B|$ (MeV), for different potentials and different K, $\bar{n}$ and μ (fm^{-1}) (optimum ones) values. Complete convergence is achieved only for the Baker potential.

K	$\bar{n}$	Volkov		Yukawa		Baker	
		$\|B\|$	μ	$\|B\|$	μ	$\|B\|$	μ
0	24	7.7352	0.6	2.5221	0.6	9.2076	0.6
10	8	8.3761'	0.6	3.252	0.6	-	
	14	8.4396	0.6	3.5790	0.7	-	
	19	8.4428	0.6	3.6551	0.8	-	
	24	8.4432	0.6	3.6778	0.8	-	
12	24	8.4697	0.6	3.7382	0.8	9.7792	0.6
16	24	8.4847	0.6	3.7782	0.8	9.7792	0.6

gy and rms radius against variation in μ. Another way to confirm this fact is by looking at the radial function for large ρ. Figs.1 and 2 show the K=0 radial function for the Volkov and Yukawa potentials. We see that for $\bar{n} \simeq 19$ the asymptotic behavior of the function has the proper $\exp(-\alpha\rho)$ form for $\rho \lesssim 10$ fm.

Table 4 shows the importance of each K, ν partial wave by calculating $\sum_{n=0}^{\bar{n}} C_n^{(K\nu)^2}$, for fixed values of μ and $\bar{n}$. One baric feature that can be noticed from the Table is the importance of the K=0 contribution in comparison with the higher hyperspherical functions.

TABLE 4

Contribution of each partial wave (with $\bar{n}$=24)

Volkov (μ = 0.6 fm^{-1})			Yukawa (μ = 0.8 fm^{-1})		
K	ν	$\sum_{n=0}^{\bar{n}} C_n^{(K\nu)^2}$	K	ν	$\sum_{n=0}^{\bar{n}} C_n^{(K\nu)^2}$
0	0	0.993	0	0	0.984
4	0	0.494×10^{-2}	4	0	0.101×10^{-1}
6	3	0.157×10^{-2}	6	3	0.411×10^{-2}
8	0	0.193×10^{-3}	8	0	0.608×10^{-3}
10	3	0.131×10^{-3}	10	3	0.409×10^{-3}
12	0	0.257×10^{-4}	12	0	0.793×10^{-4}
12	6	0.516×10^{-4}	12	6	0.155×10^{-3}
14	3	0.224×10^{-4}	14	3	0.668×10^{-4}
16	0	0.509×10^{-5}	16	0	0.155×10^{-4}
16	6	0.103×10^{-4}	16	6	0.309×10^{-4}

From what was seen before one can also conclude that convergence is much dependent on the behavior of the potential near the origin. This agrees with Erens *et al.* (1971) conclusions.

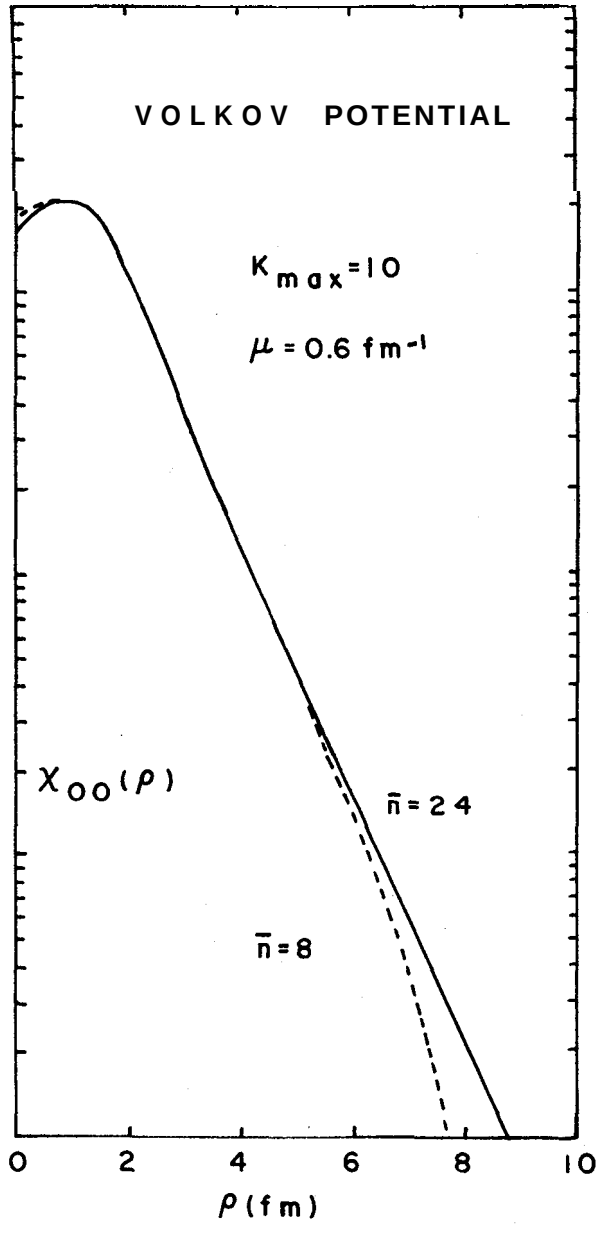

FIG.1. Plot of $\chi_{00}(\rho)$ versus ρ for $\bar{n}$=8 and 24. Correct asymptotic behavior of $\chi_{00}(\rho)$ is achieved for $\bar{n}\sim 20$. The Volkov potential is used in this case.

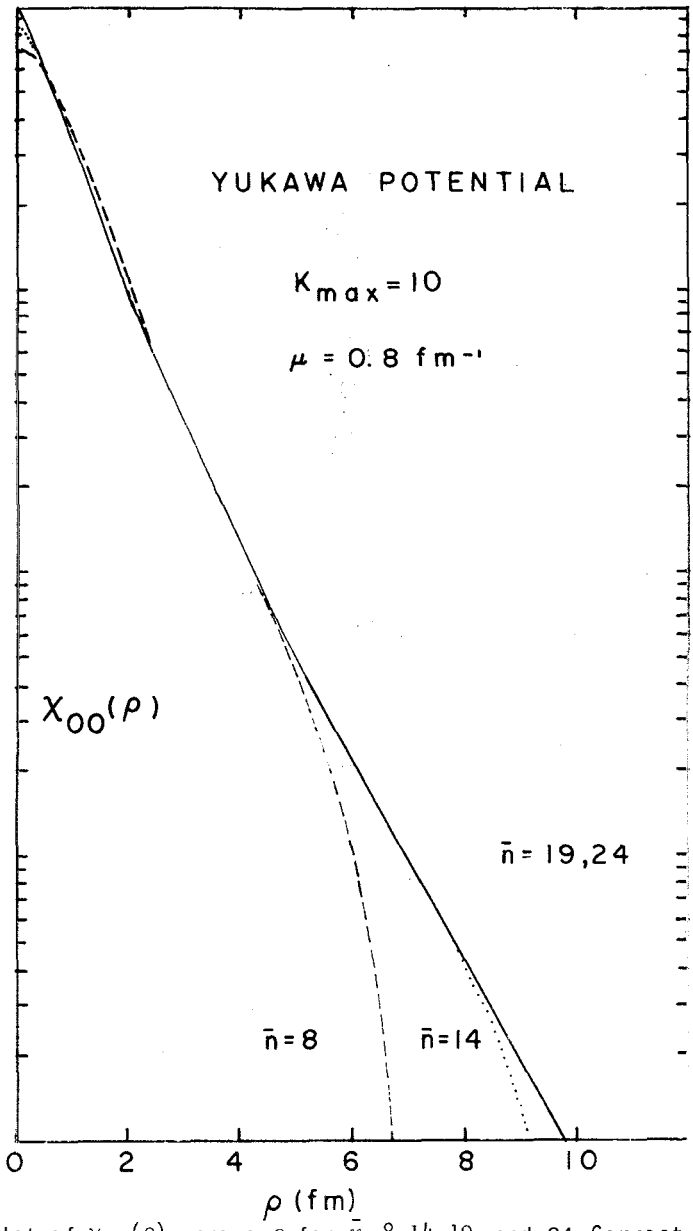

FIG.2. Plot of $\chi_{00}(\rho)$ versus ρ for $\bar{n}=8,14,19$ and 24. Correct asyrnptotic behavior of $\chi_{00}(\rho)$ is also achieved for $\bar{n}\sim 20$. The Yukawa potential was used in this case.

To finalize this Section, a calculation is done to obtain the elastic electron scattering form factor for ^{3}H. It is easy to prove that for an L=0 symmetric state,

$$F_{00}(q^2) = \sum_{K\nu, K'\nu'} \int d\Omega_6 \, v_{K\nu}(\Omega_6) v_{K'\nu'}(\Omega_6) \times$$

$$\times \int_0^\infty d\rho \, \rho^5 \, \chi_{K\nu}(\rho) j_0(\sqrt{2/3}\, q\rho \sin\theta) \chi_{K'\nu'}(\rho) . \qquad (3.2)$$

Numerically, the order of integration is very important in this expression since the function $j_0(\sqrt{2/3}\, q\rho \sin\theta)$ oscillates rapidly for large q and ρ. We chose to use the Filon method (Tranter, 1955) to perform first the integral over ρ and subsequently Gaussian quadrature to integrate over the angles.

We can also derive an approximate form for Eq. (3.2). Using the expansion for the spherical Bessel function, we find that

$$<F_{00}^{app}(q^2) \simeq 1 - \frac{1}{6} q^2 <r^2> , \qquad (3.3)$$

where

$$<r^2> = \frac{2}{3} <\rho^2 \sin 2\theta> \equiv \frac{1}{3} <\rho^2>$$

and < > stands for the expectation value over the whole phase space. The final form factor has to be multiplied by the proton form factor to take into consideration the proton dimensions (Jansen *et al.*, 1966). Fig. 3 shows a plot of the elastic scattering electron form factor and Table 5 summarizes it. Once again the tail correction is not found necessary in this calculation.

4. APPROXIMATE METHODS

There exist many approximate methods to deal with the three body system.

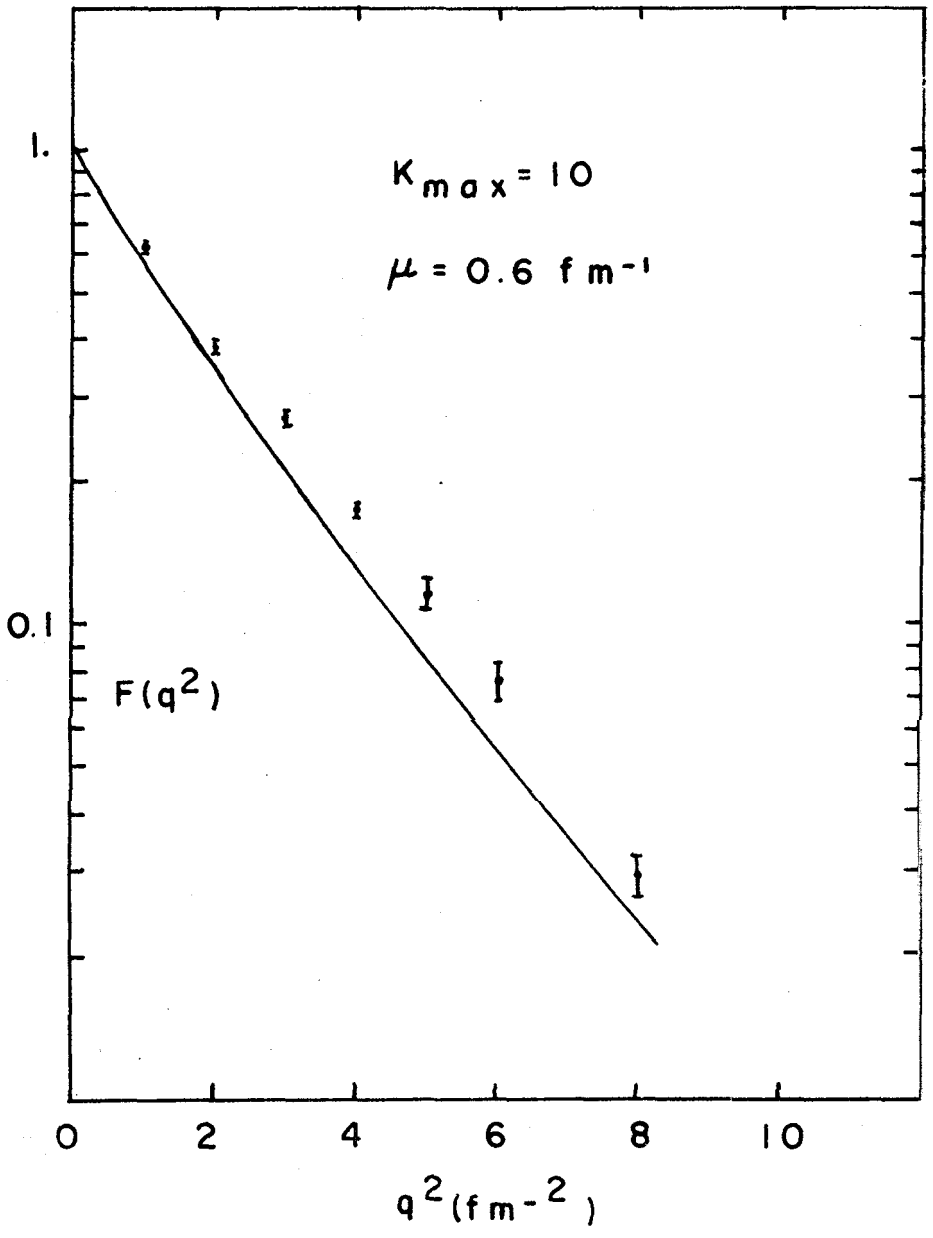

FIG.3. Elastic electron form factor for ^{3}H, using the Volkov nucleon--nucleon potential. The shape of the curve is essentially the same for $\bar{n} > 15$. The experimental points are from Collard *et al.* (1965).

TABLE 5

Elastic electron form factor for H^3 using Volkov potential and taking $\mu = 0.6$ fm^{-2}

q^2 (fm^{-2})	$F_{00}(q^2)$						$F_{00}^{app}(q^2)$	
	K	$\bar{n}$	K	$\bar{n}$	K	$\bar{n}$	K	$\bar{n}$
	10	8	10	24	12	24	12	24
0.	1.000		1.000		1.000		1.000	
1.	.565		.560		.560		.445	
2.	.336		.333		.332		-.000	
3.	.207		.206		.205		-.363	
4.	.131		.130		.130		-.661	
5.	$.843\times10^{-1}$		$.838\times10^{-1}$		$.837\times10^{-1}$			
6.	.550		.547		.547			
7.	.362		.361		.360			
8.	.239		.239		.238			
9.	.158		.158		.158			

Recently, Bhaduri and Nogami (1976) have looked at the three body atomic system using an older method due to Feshback and Rubinow (1955). Earlier, McMillan (1965) compared this method with the one due to Morpurgo (1952). These methods are much simpler to use than the K-harmonic approach, although they remain approximate, while the latter can be considered as exact if sufficient hyperspherical functions and GHOF are used in the solution. It is our intention to compare these approaches in various physical situations to establish a criteria of validity for the two approximate methods.

a) Morpurgo Method

One assumes that the wave function of the three-boson system is a function of a single symmetric variable R

$$R = \left[\frac{2}{3}(r_{12}^2 + r_{13}^2 + r_{23}^2)\right]^{1/2} , \qquad (4.1)$$

where r_{ij} is the interparticle distance between particles i and j. The expectation value of the three-body Hamiltonian H $\langle\phi|H|\phi\rangle$, may then be reduced to a single integral over the variable R, and application of the condition $\delta[\langle\phi|H|\phi\rangle/\langle\phi|\phi\rangle] = 0$ yields the equation (Morpurgo, 1952)

$$-\frac{\hbar^2}{m}\frac{1}{R^5}\left(\frac{d}{dR}R^5\frac{df}{dR}\right) + \left[\frac{48}{\pi}\int_0^1 dz\, z^2\sqrt{1-z^2}\,V_{12}(Rz)\right]f = Ef(R) . \qquad (4.2)$$

If one makes the transformation $\rho = \sqrt{2}R$, the above equation reduces to

$$-\frac{\hbar^2}{2m}\frac{1}{\rho^5}\frac{d}{d\rho}\left(\rho^5\frac{df}{d\rho}\right) + \left[\frac{48}{\pi}\int_0^1 dz\, z^2\sqrt{1-z^2}\,V_{12}(\sqrt{2}\,\rho z)\right] f(\rho) = Ef(\rho),$$

which is exactly the differential equation for $\chi_{00}(\rho)(\equiv f)$ (Vallières et *al.*, 1976) in the *K*-harmonic approach. The integral can be easily identified with Eq. (3.1):

$$V_{00,00}^{\text{eff}}(\rho) = -\left(\frac{2m}{\hbar^2}\right)\int_0^{\pi/2} v_{00}^2\cos^2\theta\sin^2\theta d\theta . 3V_{12}(\sqrt{2}\,\rho\cos\theta).\int d\hat{\xi}\, d\hat{\eta}$$

$$= -\left(\frac{2m}{\hbar^2}\right)\frac{3\times 16}{\pi}\int_0^{\pi/2} d\theta\cos^2\theta\sin^2\theta .\, V_{12}(\sqrt{2}\,\rho\cos\theta), \qquad (4.4)$$

with $\cos\theta$ identified as z in Eq. (4.3).

b) Feshbach-Rubinow Method

Everything is done similarly to the Morpurgo method but R is now taken as

$$R = \tfrac{1}{2}\,(r_{12} + r_{13} + r_{23}) \qquad (4.5)$$

and the differential equation is now given by

$$\frac{1}{R^5}\frac{d}{dR}\left(R^5\frac{df}{dR}\right) + \left[-\left(\frac{2m}{\hbar^2}\right) 12 \int_0^1 dz\, z^2(1-z+\frac{1}{6}z^2)V_{12}(Rz)\right]f + \kappa^2 f = 0, \qquad (4.6)$$

where

$$\kappa^2 = \left(\frac{2m}{\hbar^2}\right) \frac{7}{15} E \ .$$

Eq. (4.6) can resemble the $K=0$, K-harmonics equation (2.6), if one calls, the term in parenthesis, $V^{\text{eff}}_{00,00}(R)$. Certainly, this is only a technical procedure, which is not the case for Morpurgo method.

As a check, we could reproduce, using the Feshbach-Rubinow method, same of the tesults found in McMillan's paper (1965).

To test the accuracy of the Feshbach-Rubinow method, we calculate the binding energy of the model triton for both the Volkov and the Yukawa potentials. We compare in Table 6 these results with the ones obtained by the Morpurgo method (hyperspherical, $K=0$) and by the full hyperspherical calculation. We see that, for the Yukawa potential, the Feshbach--Rubinow method gives a binding energy comparable to that of the full hyperspherical approach, while it does worse than the Morpurgo approach for the Vo'lkov potential. We conclude from the calculations using these

TABLE 6

Comparison of the approximate methods

Method	Volkov		Yukawa	
	$\|B\|$ (MeV)	μ (fm^{-1})	$\|B\|$ (MeV)	μ (fm^{-1})
Morpurgo ($K=0$)	7.7352	0.6	2.5221	0.6
Feshbach-Rubinow	6.5853	0.3	3.7994	0.3
Hyperspherical Calculation	8,4847	0.6	3.7782	0.8

two extreme classes of potentials, that the Feshbach-Rubinow method is a good approximation for purely attractive potentials. A similar situation has been pointed out by Bhaduri and Nogami (1976) for the helium atom and helium like ions.

5. CONCLUSION

The use of the generalized harmonic oscillator functions (GHOF) as a basis for the expansion of the radial functions, in the K-harmonics approach, reduces the difficult problem of solving a coupled system of differential equations to one of standard matrix diagonalization. Many advantages of this approach are described in the text; its main drawback, namely its wrong asymptotic behavior, is analysed carefully through the 'model triton' for various potentials and shown not to matter the least in practical situations. Nevertheless, for a more refined calculation one could always use the method for tail correction proposed earlier (Vallières et *al.*, 1976).

The potentials we have used here are obviously not very realistic, our purpose being to show the feasibility of the method rather than describe in detail the physical ^3H. Certainly, the GHOF could be used in conjunction with more sophisticated nucleon-nucleon potentials or for an arbitrary number of particles (Vallières et *al.*, 1976). We have also developed a method to handle unequal mass (three-body) systems (Coelho et *al.*, 1976).

A more realistic application of the techniques used in this paper has been made by the authors (Vallières, 1976) to the C^{12} nucleus. The alpha particle model was considered to represent C^{12} with special interest in the structure of the 0^+ excited state.

In the second half of this paper, we also examine Morpurgo and Feshbach-Rubinow methods. The former is shown to be equivalent to the $K=0$ hyperspherical approach; the latter is found to be a very accurate approximate method for the case of purely attractive potentials.

We would like to thank Professors Y. Nogami and D.W.L. Sprung of McMaster University for their continuous interest in this problem. We also much appreciated many discussions with Dr. H. Bando of McMaster University. One of us (HTC) acknowledges the agreement NRC(Canada)-CNPq (Brazil) for his stay at McMaster University, and the Duck Acme Pharm for encouragement. We acknowledge the efficiency of the McMaster computer center.

APPENDIX

We give in this Appendix, the proper definition of the GHOF as well as some of their properties, enabling one to calculate the matrix elements in Eqs. (2.6)-(2.8).

Generalized Harmonic Oscillator Function

$$R_{nK}(\xi) = \left[\frac{2n!}{\Gamma(n+K+\frac{3N-3}{2})}\right]^{1/2} \xi^K \exp(-\tfrac{1}{2}\xi^2)\, L_n^{K+\frac{3N-5}{2}}(\xi^2) .$$

Differential Equation

$$\left[\frac{d^2}{d\xi^2} + \frac{3N-4}{\xi}\frac{d}{d\xi} + 2E_{nK} - \xi^2 - \frac{K(K+3N-5)}{\xi^2}\right] R_{nK}(\xi) = 0,$$

where $E_{nK} = 2n+K+\frac{1}{2}(3N-5)$ and, physically, $\xi = \mu\rho$ (ρ is the hyperradius, and μ a scaling parameter); N is the number of particles.

Norm

$$\int_0^\infty d\xi\, \xi^{3N-4} |R_{nK}(\xi)|^2 = 1 .$$

Recurrence Relation

$$R_{n+1,K}(\xi) = \left[(2n+K+\frac{3N-3}{2} - \xi^2) R_{n,K}(\xi) -\right.$$

$$- (n^2 + nK + \frac{n}{2}(3N-5))^{1/2} R_{n-1,K}(\xi)\Big]$$

$$\times \left[(n+1)(K + n + \frac{3N-3}{2})\right]^{1/2} ,$$

$$R_{0K}(\xi) = \left[\frac{2}{\Gamma(K + \frac{3N-3}{2})}\right]^{1/2} \xi^K \exp(-\tfrac{1}{2}\xi^2) .$$

Derivatives

$$\frac{d}{d\xi} R_{nK}(\xi) = (\frac{K}{\xi} - \xi) R_{nK}(\xi) - 2\sqrt{n}\, R_{n-1,K+1}(\xi),$$

$$\frac{d^2}{d\xi^2} R_{nK}(\xi) = \left[-(\frac{K}{\xi^2}) - 1 + (\frac{K}{\xi} - \xi)^2\right] R_{nK}(\xi) -$$

$$- 2\sqrt{n}\,(\frac{2K+1}{\xi} - 2\xi) R_{n-1,K+1}(\xi) + 4\sqrt{n(n-1)} R_{n-2,K+2}(\xi)$$

$$= \left[\xi^2 - (4n + 2K + 3N-3) + K(K+3+N-5)/\xi^2\right] \times$$

$$\times R_{nK}(\xi) - \frac{3N-4}{\xi} \frac{d}{d\xi} R_{nK}(\xi) .$$

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