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Stability of the Direct and Inverse Problems in One-Dimensional Scattering Theory

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The one-dimensional scattering problem is to relate the potential q in the operator

$$E \equiv -\partial^2 + q(x), \quad -\infty < x < \infty ,$$

to the so-called scattering data associated with E. It is known that to a certain class of potentials there corresponds a certain class of scattering data, and conversely. In this paper, we investigate the sense in which this correspondence and its inverse are stable. This question is interesting *per se* and is particularly important for numerica1 or experimental approaches to these problems.We introduce appropriate metrics in certain classes of potentials and of scattering data, and prove continuity results for both the direct and inverse mappings.

O problema do espalhamento uni-dimensional *é* relacionar o potencial q no operador

$$E \equiv -\partial^2 + q(x); \quad -\infty < x < \infty,$$

com os chamados dados de espalhamento associados com E. Sabe-se que a uma certa classe de potenciaís corresponde uma certa classe de dados de espalhamento, e reciprocamente. Neste artigo, investigamos em que sentido esta correspondência e sua inversa são estáveis. Essa questão tem interesse por si mesma e *é* particularmente relevante para o tratamento numérico ou experimental destes problemas. Introduzimos métri-

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cas apropriadas em certas classes de potenciais e de dados de espalhamento, e obtemos resultados de continuidade tanto para o problema direto como para o problema inverso.

1. INTRODUCTION

The one-dimensional scattering problem appears not only within the context of quantum mechanics, cf. Landau[1], but also in the mathematical description of a series of other physical phenomena. These include the reflection of electromagnetic waves by various media - such as a plasma, cf. Szu *et al.*[2], the ionosphere, cf. Kay[3], a dieletric slab, cf. Portinari[4] - or the propagation of waves in transmission lines, cf. Colin[5]. One-dimensional scattering plays a role also in the study of long water waves in a channel, due to its relationship to the Korteweg--deVries equation, cf. Gardner et *al.*[6].

The results contained in this paper give partial answers to the question raised by Sabatier in Ref. 5, narnely, "how stable are scattering problems, and in which sense". This question is a basic concern to anyone who is either seeking numerical approaches to these problems, or dealing with experimental data. In our case, we were led to this investigation in the course of our search for an efficient numerical procedure for the one-dimensional inverse scattering problem. The numerical results we obtained are described in the second part of Ref.7 and will be published in the near future.

The organization of this paper is as follows: In Section 2, we recall some basic results on one-dimensional scattering and introduce the terminology that will be used throughout. In Section 3, we state and prove some continuity results for both the direct and inverse problems. We also point out how to get similar results for the radial problem, and in particular how to restate a previous result by Marchenko[8]. For that, we use the correspondence between the properties of the potential q and of the reflection coefficients r^+ and r^-, as described in Theorem 4. Section 4 is devoted to the proof of some technical lemmas needed in Section 3.

2. BACKGROUND

In this Section, we explain briefly the basic facts of one-dimensional scattering, introducing at the same time the notation and terminology we adopted.

We will use the standard notations:

(i) For $1 \leq p \leq \infty$, $L^p(D)$ $\equiv$ space of numerical functions f defined on D and such that

$$\| f \|_p \equiv \left(\int |f(x)|^p \, dx\right)^{1/p} < \infty, \quad \text{if } 1 \leq p < \infty,$$

or

$$\| f \|_\infty \equiv \sup_{x \in D} |f(x)| < \infty ;$$

(We shall usually take for D the real line $\mathbb{R}$ or the positive axis $\mathbb{R}^+$.)

(ii) $C_0(D)$ $\equiv$ space of continuous numerical functions defined on D that vanish at ∞.

(iii) $W^{1,p}(\mathbb{R}^+)$ $\equiv$ space of numerical functions defined on $\mathbb{R}^+$ that are differentiable almost everywhere and such that

$$\| f \|_p + \| f' \|_p < \infty ;$$

Consider now the stationary Schrödinger operator

$$E \equiv -\partial^2 + q , \quad \partial \equiv d/dx ,$$

on $L^2(\mathbb{R})$. We assume that the real *potential*, $q(x)$, tends to zero sufficiently fast as $x \to \pm\infty$. Then, it is known that the continuous spectrum of E comprises the positive axis $\mathbb{R}^+$, and has multiplicity 2. In addition, there may exist a finite number of negative eigenvalues λ_j.

To each point k^2, k real, in the continuous spectrum, one can associate a two-dimensional space of generalized *eigenfunctions* $y(x,k)$. These are solutions of the eigenvalue equation

$$- y'' + qy = k^2 y . \tag{2.1}$$

They do not belong to $L^2(\mathbb{R})$ but can be shown to be bounded. Indeed, there exist constants

$$A_\pm \equiv A_\pm(y) , \; B_\pm \equiv B_\pm(y) ,$$

such that

$$\lim_{x \to \pm\infty} \left[y(x,k) - \{A_\pm e^{ikx} + B_\pm e^{-ikx}\} \right] = 0 , \tag{2.2}$$

i.e., the solutions of (2.1) behave asymptotically as the solutions of the unperturbed equation $y'' + k^2 y = 0$.

Conversely, given any pair of constants (α, β), there exists a unique solution of (2.1) for which

$$A_+ = \alpha , \quad B_+ = \beta ;$$

of course, A_-, B_- may also be prescribed arbitrarily.

The pairs of constants (A_-, B_+) and (A_+, B_-) are called the *incoming* and *outgoing components* of the solution $y(x,k)$. This terminology is motivated as follows:

If y is a solution of (2.1), then

$$u(x,t) \equiv e^{-ikt} \, y(x,k)$$

is a time-harmonic solution of the perturbed wave equation

$$u_{tt} - u_{xx} + qu = 0 .$$

The terms

$$A_\pm \, e^{ikx} \, e^{-ikt} = A_\pm \, e^{ik(x-t)}$$

and

$$B_{\pm}\, e^{-ikx}\, e^{-ikt} = B_{\pm}\, e^{-ik(x+t)}$$

represent waves moving to the right or to the left, respectively. We call a wave *incoming* if it moves from ∓ ∞ towards the origin, and *outgoing* if it moves away from the origin towards ± ∞ .

It turns out that the incoming components (A_-, B_+) determine the outgoing components (A_+, B_-) uniquely. The operator relating them is givem by a 2x2 matrix called the *scattering matriz* and denoted by :

$$S \begin{pmatrix} A_- \\ B_+ \end{pmatrix} = \begin{pmatrix} A_+ \\ B_- \end{pmatrix} \quad . \tag{2.3}$$

The S-matrix depends only on k, i.e., $S \equiv S(k)$, and not on the particular solution y being considered.

We can attribute a physical meaning to the elements $s_{ij}(k)$ of S:

Let ψ_- denote the solution of (2.1) which is a wave of unit amplitude coming in from $-\infty$, i.e., $A_- = 1$, $B_+ = 0$. (The existence of such ψ_- can be proved rigorously.) Then by (2.3) the outgoing components of ψ_- are

$$A_+ = s_{11} \quad , \quad B_- = s_{12} \quad .$$

Now, $s_{11} = A_+$ is the amplitude of a wave travelling to the right, i.e., in the same direction as the above incoming wave, while $s_{12} = B_-$ is the amplitude of a wave travelling in the opposite direction. The latter is thus *reflected*, while the former is *transmitted*. Accordingly, we will call

$s_{11}(k)$: *transmission coefficient from the Zeft,*
$s_{21}(k)$: *reflection coefficient from the Zeft,*
$s_{12}(k)$: *reflection coefficient from the right,*
$s_{22}(k)$: *transmission coefficient from the right.*

The matrix S has the following important properties, valid for real k:

(i) $s_{11} = s_{22}$; (ii) S is unitary; (iii) $s_{ij}(-k) = \overline{s_{ij}(k)}$;
(iv) $s_{11}(k)$ can be continued to the half-plane $\mathrm{Im}\, k > 0$ as a nonvanishing meromorphic function with simple poles at the points $i\kappa_j$, where $-\kappa_j^2 = \lambda_j < 0$ are the eigenvalues of E;
(v) $\lim_{|k|\to\infty} s_{11}(k) = 1$; $\lim_{|k|\to\infty} s_{ij}(k) = 0$, for $i \neq j$.

Property (i) allows us to refer to $s_{11} = s_{22}$ as the transmission coefficient. From now on, we will denote it by $t(k)$, the reflection coefficients from left and right by $r^-(k)$ and $r^+(k)$, respectively, while $r(k)$ will stand for either $r^-(k)$ or $r^+(k)$. Thus, in this notation,

$$S = \begin{pmatrix} t(k) & r^+(k) \\ r^-(k) & t(k) \end{pmatrix} . \tag{2.4}$$

Based on the unitarity of the S-matrix and on the uniqueness result mentioned after (2.2), we see that a solution of (2.1) is uniquely determined by any one of the pairs of *asymptotic* components

$$(A_-, B_+) , (A_+, B_-) , (A_-, B_-) , (A_+, B_+) .$$

The unitarity of S also implies that

$$t\, \overline{r^-} + r^+\, \overline{t} = 0 , \tag{2.5}$$

$$|r^-|^2 + |t|^2 = 1 , \tag{2.6$_-$}$$

and

$$|t|^2 + |r^+|^2 = 1 . \tag{2.6$_+$}$$

Next we observe that S is completely determined by either of its off-diagonal elements and the discrete spectrum $\{-\kappa_j^2\}$ of E. Indeed, equation $(2.6)_\pm$ enables us to get $|t|$ from the knowledge of $|r|$. To determine all of t, we form

$$t(k) = \Pi \left(\frac{k+i\kappa_j}{k-i\kappa_j} \right) \exp \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log (1-|r(s)|^2)}{s - k} ds , \qquad (2.7)$$

for $\text{Im}\, k > 0$, while for k real

$$t(k) = \lim_{\varepsilon \to 0_+} t(k + i\varepsilon) .$$

Knowing t and one of the reflection coefficients, the other reflection coefficient may be computed from (2.5) and thus the S-matrix is determined.

If we know the potential $q(x)$, we can determine r as a function of k, k real. The study of the way the S-matrix depends on $q(x)$ is called the *direct problem.*

In many physical situations, it is difficult, or impossible, to measure q directly, whereas one of the reflection coefficients is suitable to be measured. Bargmann discovered that neither of the reflection wefficients alone contains enough information for the unique determination of the potential q, even if we know the point spectrum of E, cf. Ref. 9. This lack of uniqueness can be overcome if one knows also the *normalizing constants on the right*, m_j^+, or on the *left*, m_j^-. They are introduced, for each eigenvalue $h_j = -\kappa_j^2$, as the inverse of the L^2 norm of the eigenfunctions $\phi_+(x,i\kappa_j)$ or $\phi_-(x,i\kappa_j)$, which are characterized by

$$\phi_{\pm}(x,k) \sim e^{\pm ikx} , \quad x \sim \pm \infty .$$

The *inverse problem* is the search of information about q from the knowledge of one of the reflection coefficients, the corresponding normalizing constants and the point spectrum of E.

Proofs of the results mentioned above may be found in Refs. 7, 10 and 11.

3. STABILITY RESULTS

In what follows, unless otherwise stated, our functions will be real valued and measurable, defined on either the real line $\mathbb{R}$ or on the positive axis $\mathbb{R}^+$. We will use the notation:

$$\| f \|_{(1)} \equiv \int |x\ f(x)|\ dx \ ,$$

$$\| f \|_{(2)} \equiv \sup\ x^2 |f(x)|/2 \ ,$$

where both the integral and the supremum are taken on $\mathbb{R}$ or $\mathbb{R}^+$, whichever the domain of f.

Let P be the set of piecewise continuous functions $q(x)$ defined on $\mathbb{R}$ and satisfying

$$\int_{-\infty}^{\infty} (1 + |x|)\ |q(x)|\, dx < \infty \quad ;$$

a function q in P will be called a *potential*.

Let $R^{\pm}$ be the sets of continuous complex functions $r^{\pm}(k)$ defined on $\mathbb{R}$ and such that:

(i) $r^{\pm}$ have *real* Fourier transforms,

$$F_{\pm}(t) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} r^{\pm}(k)\ e^{\pm 2ikt}\ dk \ , \qquad (3.1)$$

that are absolutely continuous and whose derivatives $F'_{\pm}(t)$ satisfy

$$\pm \int_{a}^{\pm\infty} (1 + |t|)\ |F'_{\pm}(t)|\ dt \leq C_a < \infty \ ,$$

for all real a;

(ii) $|r(k)| < 1$, for $k \neq 0$; if $|r(0)| = 1$, then $r(0) = -1$. (Note that $|r(0)| \leq 1$, by continuity; and by (i), $r(-k) = \overline{r(k)}$, so that $r(0)$ is real);

(iii) $r(k) = O(1/|k|)$, $|k| \sim \infty$.

From now on, when dropping the ± superscripts, we will be referring to either r^+ or r^-, F^+ or F^-, etc. We will refer to the functions in R as *reflection coefficients*.

As remarked in the previous Section, a set of *scattering* data is a triple

$$s^{\pm} \equiv (r^{\pm}, \kappa, m^{\pm}) ,$$

where r is a reflection coefficient, $\kappa \equiv (\kappa_j)$ and $m = (m_j)$ are N-tuples of positive numbers, N being a nonnegative integer and the κ_j's being all distinct. The collection of all such scattering data will be denoted by $S^{\pm}$.

The assumptions made above on the potentials are nearly the weakest that we can make and still have a scattering theory. Indeed, the integrability of q implies the asymptotic behavior of the solutions of (2.1), as described in (2.2), while the existence of the first moment of q implies the finiteness of the point spectrum of E, cf. Ref. 7.

On the other hand, the conditions imposed on the scattering data are also nearly as mild as they could be, as is assured by the following result, cf. Refs. 10, 12:

Theorem 1 (Faddeev). There exists a one-to-one correspondence between the set of potentials P and either set of scattering data S^+ or S^-.

We now state the two main continuity results we obtained. Observe that we restrict ourselves to sets smaller than P and S. Let P^+ be the set of potentials q such that:

(a) each $q \in P^+$ vanishes on some half-line $\{x \leq \alpha\}$;

(b) $q(x) = o(x^{-2})$, $|x| \sim \infty$;

'(c) the operator E associated with q has no point spectrum.

Theorem 2 (Stability of the Direct Scattering Problem)

Take in P^+ the distance corresponding to the norm

$$\| 4 \| \equiv \| q \|_\infty + \| q \|_{(1)} + \| q \|_{(2)} \qquad (3.2)$$

Then the direct scattering mapping $q \to r^+$ is continuous from P^+ to $L^2(\mathbb{R})$.

Remark. To get an analogous result for r_-, introduce the set P^-, requiring instead of condition $(a)_+$:

$(a)_-$ each $q \in P^-$ vanishes on some half-line $\{x \geq \alpha\}$.

In order to state a continuity result for the inverse mapping, let us introduce two new sets:

Let $R_1^\pm$ denote the sets of reflection coefficients $r^\pm$ for which $F'_\pm$ are bounded and satisfy

$$F'_\pm(t) = o(|t|^{-3}), \quad |t| \sim \infty$$

where the $F'_\pm$ are defined in (3.1).

Theorem 3 (Stability of the Inverse Scattering Mapping)

Take in $R_1^\pm$ the distance associated with the norm

$$\| r \| \equiv \|F'\|_\infty + \| F' \|_{(1)} , \qquad (3.3)$$

and in P the distance associated with uniform convergence on compact sets. Then the inverse scattering mappings $R_1^{\pm} \to P$ are continuous with respect to the above metrics.

Remarks. (a) Actually, the mapping $R_1^{+} \to P$ is continuous in the topology of uniform convergence on half-lines $\{x \geq \alpha\}$, while $R_1^{-} \to P$ is continuous in the topology of uniform convergence on $\{x \leq \alpha\}$. (b) The same continuity results hold if we fix an integer N > 0 and consider scattering data $s \equiv (r,\kappa,m)$, with N-tuples κ and m, $r \in R_1$, and take

$$\| s \| \equiv \| F' \|_{\infty} + \| F' \|_{(1)} + \| \kappa \| + \| m \| ,$$

where the norms for κ and m are $\mathbb{R}^N$ norms.

To prove these theorems, we introduce the functions $B_{\pm}(x,y)$. They establish the link between the potential q and the reflection coefficients $r^{\pm}$. The relationship between them and the potential $q(x)$ is expressed by the equations

$$B_{\pm}(x,y) = \pm \int_{x+y}^{\pm\infty} q(t)\, dt + \int_0^y dz \int_{x+y-z}^{\pm\infty} q(t)\, B(t,z)\, dt , \quad \pm y \geq 0, \qquad (3.4)_{\pm}$$

while the so-called Marchenko equations,

$$\Omega_{\pm}(x+y) + B_{\pm}(x,y) \pm \int_0^{\pm\infty} \Omega_{\pm}(x+t+y)\, B_{\pm}(x,t)\, dt = 0, \quad \pm y \geq 0, \qquad (3.5)_{\pm}$$

relate $B_{\pm}$ to the scattering data, since R is defined as

$$\Omega_{\pm}(t) \equiv F_{\pm}(t) + 2 \sum m_j^{\pm}\, e^{\pm 2\kappa_j t} . \qquad (3.6)_{\pm}$$

Equations (3.5) first appeared in Refs. 3, 13.

To study the direct mapping, first we solve (3.4), which is a Volterra equation for $B(x,y)$. Once B is determined, we set $y = 0$ in (3.5), regarding now B as the kernel. The equation we obtain for R is again of Volterra type:

$$\Omega_{\pm}(x) + B_{\pm}(x,0) \pm \int_{x}^{\pm\infty} B_{\pm}(x,t-x)\,\Omega_{\pm}(t)\,dt = 0 . \qquad (3.7)_{\pm}$$

When considering the inverse mapping, we regard R as given and solve equations (3.5) for B, observing that these are a family of Fredholm equations, where x enters as a parameter. Once B is determined, we can get q from (3.4) by setting $y = 0$ and differentiating with respect to x:

$$q(x) = \mp\, \partial_1 B(x,0) . \qquad (3.8)_{\pm}$$

Consequently, we have to analyse the continuity properties of the chain of mappings

$$q \leftrightarrow B \leftrightarrow \Omega \leftrightarrow r .$$

The proof of Theorem 2 is based on Claims 1-3 of Section 4, which give the continuity of $q \to \Omega_+$. Since in the absence of the point spectrum, $\Omega = F$, the Fourier transform of r, we can use the unitarity of the Fourier transform to obtain the continuity of $q \to r_+$.

Theorem 3 is a direct consequence of Claims 4 and 5 in Section 4. We remark that, for the sake of brevity in the statement of this theorem, we required

$$F'(t) = o(|t|^{-3}), \qquad |t| \sim \infty ,$$

instead of the weaker conditions

$$\| F' \|_{(1)} < \infty ,$$

$$F_{\pm}(t) = o(t^{-2}) , \quad t \sim \pm\infty ,$$

which are the ones we actually use, and that hold, for example, if

$$r \in W^{3,1} (\mathbb{R})$$

and

$$\| r \|_{(1)} < \infty .$$

We end this section with two observations. First, results similar to Theorem 2 and 3 can be obtained for the radial problem, since equations analogous to (3.4) and (3.5) hold in this case, cf. Refs. 14,15.

Second, equations (3.4), (3.5) and (3.7) permit us to establish a relationship between some properties of the potential q and those of the reflection coefficient r^{+}, more exactly, of the derivative of its Fourier transform F'_{+}. The proof is based on the following inequalities that hold for x $\geq$ 0, cf. Ref. 10:

$$|\Omega'_{+}(x) - q(x)| < A\,\xi^2(x) , \quad |\Omega'_{+}(x) - q(x)| < A\,\tau^2(x) ,$$

where

$$\xi(x) \equiv \int_x^{a} |q(s)|\, ds , \quad \tau(x) \equiv \int_x^{\infty} |\Omega'_{+}(s)|\, ds .$$

Since

$$\Omega_{+}(t) \equiv F_{+}(t) + 2 \sum m_j\, e^{-2\kappa_j t} ,$$

these inequalities imply

<u>Theorem 4</u>. The potential $q(x)$ satisfies

$$q(x) = O(x^{-n}) , \quad x \sim +\infty ,$$

if and only if

$$F'_+(t) = O(t^{-n}) \ , \ t \sim +\infty \ ,$$

If *E has no point spectrum, then* $q(x) = 0$, *for* $x > A$, *if and only if* $F_+(t) = 0$, *for* $t > A$.

There exist analogous relations relating the behavior of $q(x)$ *and* $F'_-(t)$ *at* $-\infty$; also *for the derivatives* $q^{(j)}(x)$ *and* $F^{(j+1)}(t)$; *and a similar result holds for the radial problem.*

By using Theorem 4, *we can* restate *a previous stability result by Lundina and* Marchenko, *cf. Ref. 8, with hypotheses only on the scattering data, avoiding a priori assumptions on the potentials.*

4. TECHNICAL LEMMAS

In this Section *we will denote by* $L^{(1)}(D)$ or $L^{(2)}(D)$ *the sets of functions* f *on D for which*

$$\| f \|_{(1)} < \infty \quad \text{or} \quad \| f \|_{(2)} < \infty \ ,$$

respectively; $\chi(D)$ *stands for the set of bounded functions* f *that belong to* $L^{(1)}(D)$ *and decay* like

$$f(x) = o(x^{-2}) \ , \qquad |x| \sim \infty \ .$$

We also use the notation $M \equiv \mathbb{R}^+ \times \mathbb{R}^+$.

<u>**Claim 1.**</u> *For any* q *in* $L^1(R^+) \cap L^{(1)}(R^+)$, *the* equation

$$B(x,y) = \int_{x+y}^{\infty} q(t)\ dt \ + \int_0^y dz \int_{x+y-z}^{\infty} q(t)\ B(t,z)\ dt, \quad x,y \geq 0 \qquad (4.1)$$

has a unique solution B. This solution belongs to $C_0(M)$, *its first derivative* $\partial_1 B$ *belongs to* $L^1(M)$, *and the mapping* $S : q \to B$ *is continuous with respect to the norms*

$$|||\, q \,||| \equiv \|\, q \,\|_{(1)} + \|\, q \,\|_{(1)} ,$$

$$|||\, B \,|||_p \equiv \|B\|_\infty + \|\, B(.,0)\,\|_p + \|\,\partial_1 B(.,0)\,\|_1 , \quad 1 \le p \le \infty .$$

Also, if q is continuous, so are $\partial_1 B$ and $\partial_2 B$.

Proof: Consider the operators

$$V_q : \beta \to C(x,y) \equiv \int_0^y a \int_{x+y-z}^{\infty} q(t)\beta(t,z)\, dt ,$$

for q in $L^1(R^+) \cap L^{(1)}(R^+)$. Then,

$$C(x,y) \,/\, \|\,\beta\,\|_\infty \le \int_0^y \left(\int_{x+y-z}^{\infty} |q(t)|\, dt \right) dz = \int_0^y dz \int_{x+z}^{\infty} |q(t)|\, dt$$

$$\le \int_0^\infty dz \int_z^\infty q(t)\, dt = \int_0^\infty |q(t)|\, dt \int_0^t dz = \|\, q \,\|_{(1)} .$$

so that $\|\, V_q B \,\|_\infty \le \|\, q \,\|_{(1)} \|\,\beta\,\|_\infty$; i.e., V_q is a bounded operator on $L^\infty(M)$ which depends continuously on $q \in L^1(R^+) \cap L^{(1)}(R^+)$. Now,

$$\|\, V_q^2 \beta \,\| = \sup_{u,v} \left| \int_0^u dy \int_{u+v-y}^{\infty} dx \; q(x) \int_0^y dz \int_{x+y-z}^{\infty} dt \; q(t) \; \beta(t,z) \right|$$

$$\le \int_0^u dy \int_{v+y}^{\infty} dx\, |q(x)| \left(\int_x^\infty |tq(t)|\, dt \right) \|\,\beta\,\|_\infty$$

$$\le \|\,\beta\,\|_\infty \int_0^\infty x|q(x)| \left(\int_x^\infty t|q(t)|\, dt \right) dx =$$

$$= \left(\| q \|^2_{(1)} /2 \right) \| \beta \|_\infty ,$$

and, in the same way,

$$\| V_q^m \beta \|_\infty \le \left(\| q \|^m_{(1)} / m! \right) \| \beta \|_\infty$$

which implies that on $L^\infty(M)$, $(I+V_q)^{-1}$ exists and

$$\| (I + V_q)^{-1} \| \le \exp \| q \|_{(1)}$$

By the continuity of the inversion mapping on the algebra of bounded invertible operators (with the uniform operator topology), $(I + V_q)^{-1}$ depends continuously on q.

Let

$$Q(x,y) \equiv \int_{x+y}^{\infty} q(t) \, dt .$$

Then $\| Q \|_\infty \le \| q \|_1$, and thus, since (4.1) may be rewritten as

$$B = (I-V_q)^{-1} Q ,$$

the continuity of the mapping S from $L^1(\mathbb{R}^+) \cap L^{(1)}(\mathbb{R}^+)$ to $L^\infty(M)$ does follow.

Now, Q is continuous, and so is $V_q Q$; consequently

$$B = \sum_{m=0} (V_q)^m Q$$

is also continuous, as this series converges in the $L^m(M)$ sense. Since

$$B(x,0) = \int_x^\infty q(t) \, dt , \quad \partial_1 B(x,0) = -q(x) ,$$

we have

$$\| B(.,0) \|_1 \leq \| q \|_{(1)} ,$$

$$\| B(.,0) \|_\infty \leq \| q \|_1 ,$$

$$\| \partial_1 B(.,0) \|_1 = \| q \|_1 ,$$

and therefore the continuity of S in the sense stated above is proven.

To show the other properties of B, let us introduce the functions

$$\xi(x) \equiv \int_x^\infty |q(t)| \, dt ,$$

and

$$\eta(x) \equiv \int_x^\infty |q(t)| \, t \, dt .$$

Then, $|Q(x,y)| \leq \xi(x+y)$, and

$$|(V_q Q)(x,y)| \leq \int_0^y dz \int_{x+y-z}^\infty |q(t)| \, \xi(t+z) \, dt$$

$$\leq \xi(x+y) \int_0^y dz \int_{x+z}^\infty |q(t)| \, dt$$

$$\leq \xi(x+y) \int_x^\infty |q(t)| \, dt \int_0^{t-x} dz \leq \xi(x+y)\eta(x) .$$

By modifying the previous estimates for V_q^m in this fashion, we get

$$|(V_q^m Q)\ (x,y)| \leq \xi(x+y) \quad [\eta(x)]^m/m! \ ,$$

so that the solution of (4.1) satisfies

$$|B(x,y)| \leq \xi(x+y)\ \exp\ \eta(x) \leq \xi(x+y)\ \exp \| q \|_{(1)} \ .$$

This inequality implies that B vanishes at ∞ .

By differentiating (4.1) with respect to x, we get

$$\partial_1 B(x,y) = -q(x+y) - \int_0^y q(x+y-z)\ B(x+y-z,z)\ dz \qquad (4.2)$$

outside a null set. Therefore,

$$\| \partial_1 B \|_1 \leq \int_0^\infty dx \int_0^\infty |q(x+y)|\ dy + \int_0^\infty dx \int_0^\infty dy \int_0^\infty |q(x+y-z)\ B(x+y-z,z)|\ dz$$

$$\leq \| q \|_{(1)} + \exp(\| q \|_{(1)}) \int_0^\infty dx \int_0^\infty \xi(x+y)\,dy \int_x^{x+y} |q(z)|\,dz$$

$$\leq \| q \|_{(1)} + \exp(\| q \|_{(1)}) \int_0^m dx \int_0^w dy \int_y^m |q(t)|\,dt \int_x^\infty |q(z)|\,dz$$

$$= \| q \|_{(1)} + \exp\ (\| q \|_{(1)})\ \| q \|^2_{(1)} \ ;$$

i.e.,

$$\partial_1 B \in L^1(M) \ .$$

Claim 2. If $q \in \chi(\mathbb{R}^+)$, in addition to the conclusions of Claim 1, we have that

$$B \in L^p(M), \quad 2 \leq p \leq \infty \ ,$$

$$\partial_1 B \in L^p \ M \qquad 1 \leq p \leq \infty \ ,$$

and that $S : q \to B$ is continuous with respect to the norms

$$|q| \equiv \| q \|_\infty + \| q \|_{(1)} + \| q \|_{(2)}$$

$$|B|_p \equiv \| B \|_p + \| B(.,0) \|_p + \| \partial_1 B(.,0) \|_p + \| \partial_1 B \|_\infty ,$$

for $2 \leq p \leq \infty$.

Proof: If we consider q_1 and q_2 satisfying the hypotheses, and B_1 and B_2 are their corresponding images by S, we obtain from (4.2) that

$$\| \partial_1 (B_1 - B_2) \|_\infty \leq \| q_1 - q_2 \|_\infty + \|q_2\|_1 \ \| B_1 - B_2 \|_\infty + \| B_1 \|_\infty \| q_1 - q_2 \|_1 .$$

This implies that

$$\partial_1 B \ \in \ L^1(M) \cap L^\infty(M) \ ,$$

and that $\partial_1 B$ depends continuously on q, in the $L^\infty(M)$ norm. We can also conclude that

$$\partial_1 B(x,0) = -q(x) \in L^p(\mathbb{R}^+) \cap L^{(1)}(\mathbb{R}^+), \text{ for } 1 \leq p \leq \infty .$$

All that is left to show is the continuous dependence of B on q , in any $L^p(M)$ norm, for $p \geq 2$. To get this result, let us show first that V_q is a compact operator on $L^1(M)$ which depends continuously on q:

$$\| V_q \beta \|_1 < \int_0^\infty dx \int_0^\infty dy \int_0^y dz \int_{x+y-z}^\infty |q(t) \quad \beta(t,z)| \; dt$$

$$= \int_0^\infty dt \int_0^\infty |q(t) \; \beta(t,z)| \; dz \int_z^{t+z} dy \int_0^{t+z-y} dx$$

$$\int_0^\infty dt \int_0^\infty |q(t) \; \beta(t,z)| t^2/2 dz < \| q \|_{(2)} \| \beta \|_1$$

Now, for $R \to +\infty$,

$$\int_R^\infty dx \int_R^\infty dy \left| \int_0^y dz \int_{x+y-z}^\infty q(t) \; \beta(t,z) \; dt \right|$$

$$\leq \int_R^\infty dt \int_0^\infty dz \; |q(t) \; \beta(t,z)| \int_z^{t+z} dy \int_0^{t+z-y} dx$$

$$= \int_R^\infty t^2 |q(t)|/2 dt \int_0^\infty |\beta(t,z)| dz$$

$$\leq \| \beta \|_1 \sup_{t \geq R} |q(t) t^2/2| \to 0$$

uniformly for $\| \beta \|_1 \leq 1$. In the same way we prove that

$$\lim_{h^2+k^2 \to 0} \| (V_q \beta)(x+h, y+k) - (V_q \beta)(x,y) \|_1 = 0$$

uniformly for $\| \beta \|_1 \leq 1$, so that the set $\{V_q \beta ; \| \beta \|_1 \leq 1\}$ is precompact in $L^1(M)$ by the Kolmogorov-Fréchet Theorem, and therefore V_q is compact on $L^1(M)$.

Assume now that for some $\lambda \neq 0$ and some $\beta \in L^1(M)$, $V_q \beta = \lambda\beta$.

Then since

$$\| V_q\beta \|_\infty \leq \| q \|_\infty \; \| \beta \|_1 ,$$

we conclude that $\beta \equiv 0$. This simplies that $(I - V_q)$ is invertible on $L^1(M)$. Thus, as an operator on either $L^1(M)$ or $L^\infty(M)$, $(I - V_q)^{-1}$ is bounded and depends continuously on $q \in \chi(\mathbb{R}^+)$, so that the same must also hold for $L^p(M)$, $1 < p < \infty$. Now,

$$\left| \int_0^\infty \int_0^\infty f(x,y)\; Q(x,y)\; dx\; dy \right| \leq \int_0^\infty dx \int_0^\infty dy\; |f(x,y)| \int_{x+y}^\infty |q(t)|\, dt$$

$$= \int_0^\infty dx \int_x^\infty dt\; |q(t)| \int_0^{t-x} |f(x,y)|\, dy = \int_0^\infty dt\; |q(t)| \int_0^t dt \int_0^{t-x} |f(x,y)|\, dy$$

$$\leq \int_0^\infty dt\, |q(t)|\, \{ \int_0^t dx \int_0^{t-x} |f(x,y)|^2 dy \}^{\frac{1}{2}} \sqrt{t^2/2} \leq \frac{\| f \|_2}{\sqrt{2}} \| \bar{q} \|_{(1)} \; ,$$

so that $Q \in L^2(M)$. This gives the continuity of the solution B of (4.1) in the sense of the $L^2(M)$ norm. But continuity in the L^2 and L^∞ sense implies continuity for any L^p norm, $2 \leq p \leq \infty$.

<u>Claim 3</u>. If $B = S(q)$ for some $q \in \chi(\mathbb{R}^+)$, S as defined in Claim 1, the equation

$$\Omega(x) + \int_x^\infty B(x,t-x)\; \Omega(t)\; dt + B(x,0) = 0, \; x \geq 0 \; , \qquad (4.3)$$

has a unique solution R. This solution belongs to $L^p(\mathbb{R}^+)$, for $1 \leq p \leq \infty$, and satisfies $\Omega(x) = o(1/x)$, for $x \to +\infty$; it is also differentiable, $\Omega' \in L^2(\mathbb{R}^+)$, and the mapping

$$T : S(\chi) \subset L^2(M) \to L^2(\mathbb{R}^+)$$

$$B \to \Omega ,$$

is continuous.

Remark. Stronger conclusions can be obtained for R', namely that

$$\Omega' \in L^{(1)} (\mathbb{R}^+) \cap C_0(\mathbb{R}^+)$$

which in particular irnplies that

$$\Omega' \in L^p(\mathbb{R}^+), \quad 1 \leq p \leq \infty$$

Proof: Consider the operator

$$V_B: \ \omega \to C(x) \equiv \int_x^m B(x, t-x)\ \omega(t)\ dt, \quad x \geq 0 ,$$

with B in $S(\chi)$. We claim that V_B is of Volterra type on $L^\infty(\mathbb{R}^+)$.
Indeed:

$$|C(x)|/\|\omega\|_\infty \leq \int^\infty \xi(t) \exp \eta(x)\ dt \leq \exp\left(\|q\|_{(1)}\right) \int_x^\infty \xi(t)\ dt$$

$$\leq \|q\|_{(1)} \exp\|q\|_{(1)} ,$$

$$|V_B C(y)|/\|\omega\|_\infty \leq \int_y^\infty \xi(x) \exp \eta(y)\ dx \int_x^\infty \xi(t) \exp \eta(x)\ dt$$

$$\leq \exp\left(2\|q\|_{(1)}\right) \int_y^\infty \xi(x)\ dx \int_x^\infty \xi(t)\ dt$$

$$\leq \left[\| q \|_{(1)} \exp \| q \|_{(1)}\right]^2 /2 ,$$

and in general

$$|V_B^n \omega(x)| / \| \omega \|_\infty < \left[\| q \|_{(1)} \exp \| q \|_{(1)}\right]^n / n! \; .$$

Consequently, a solution for (4.3) exists, is unique, and is given by

$$\Omega = -\Sigma(-V_B)^n B(.,0),$$

being thus a continuous function.

We show now that V_B is bounded if considered as an operator from $L^p(\mathbb{R}^+)$ to $L^q(\mathbb{R})$, where $1 \leq p \leq q \leq \infty$. We already know that V_B is bounded on $L^\infty(\mathbb{R}^+)$. It is also bounded in $L^1(\mathbb{R}^+)$:

$$|(V_B\omega)(x)| \leq \int_x^\infty |\omega(t)| \; \xi(t) \exp \eta(x) \; dt \leq \| \omega \|_1 \exp(\|q\|_{(1)})\xi(x) \; ,$$

so that

$$\int_0^\infty |(V_B\omega)(x)| \; dx \leq \| \omega \|_1 \exp(\| q \|_{(1)}) \int_0^\infty \xi(x) \; dx =$$

$$= \| \omega \|_1 \; \| q \|_{(1)} \exp \| q \|_{(1)} .$$

Also

$$\| V_B\omega \|_\infty \leq \| B \|_\infty \| \omega \|_1 \; .$$

Thus V_B is bounded from L^p to L^q, for $p, q \in \{1, \infty\}$, $q \geq p$. This implies that the same holds for $p, q \in [1, \infty]$, $q \geq p$.

Assume now that $V_B\omega = \lambda\omega$, for $\lambda \neq 0$, $\omega \in L^2$. Then

$$\| \lambda\omega \|_\infty = \| V_B\omega \|_\infty \le \| V_B \|_{2,\infty} \| \omega \|_2 ,$$

so that $w \in L^\infty(\mathbb{R}^+)$, and conseguently $\omega \equiv 0$. Thus V_B is one-to-one on $L^2(\mathbb{R}^+)$. Being V_B the uniform limit of compact operators, we conclude that

$$(I+V_B)^{-1} : L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$$

is defined and bounded. Observe that for $2 \le p < \infty$, $1/p + 1/p' = 1$,

$$\| V_B\omega \|_{p'} = \sup_{\| g \|_p = 1} \int_0^\infty g(x)\,(V_B\omega)(x)\, dx ,$$

and thus,

$$\| V_B\omega \|_{p'} \le \left(\int_0^\infty dx \int_x^\infty |B(x,t-x)|^{p'}\, dt \right)^{1/p'} .$$

$$\cdot \left(\int_0^\infty dx \int_x^\infty |\omega(t)|^p \; |g(x)|^p\, dt \right)^{1/p}$$

$$\le \| B \|_{p'} \; \| \omega \|_p . \tag{4.4}$$

Therefore, by (4.4), as an operator on $L^2(\mathbb{R}^+)$, V_B depends continuously on B. The same also holds for $(I+V_B)^{-1}$, and hence for $\Omega=(I+V_B)^{-1} B(.,0)$.

Let us refine the previous estimates in order to get more information about R. Denoting

$$s(x) \equiv B(x,0) , \qquad \zeta(x) \equiv \exp \eta(x) ,$$

we obtain

$$|s(x)| \le \xi(x)\; \zeta(x),$$

$$|(V_B s)(x)| \leq \int_x^\infty \xi(t)\zeta(x)\xi(t)\zeta(t)\,dt \leq \xi(x)\zeta(x)\zeta(x)\int_x^\infty \xi(t)\;dt$$

$$= \xi(x)\zeta(x)\;[\eta(x)\zeta(x)],$$

$$|(V_B^n s)(x)| \leq \xi(x)\zeta(x)\;[\eta(x)\zeta(x)]^n/n! \quad ,$$

so that

$$|\Omega(x)| = |-\sum_{n=0}^{\infty} (-V_B^n s)(x)| \leq \xi(x)\zeta(x)\;\exp\;[\eta(x)\zeta(x)]\;.$$

Since the function $\zeta(x)\;\exp[\eta(x)\zeta(x)]$ is bounded, and $t\xi(t)\;\varepsilon\;C_0(\mathbb{R}^+)$, we conclude that $t\Omega(t)\;\varepsilon\;C_0(\mathbb{R}^+)$. Since

$$\int_0^\infty |\Omega(t)|\;dt \leq \int_0^\infty dt \int_t^\infty |\Omega'(s)|\;ds = \|\,\Omega'\,\|_{(1)} \;,$$

we have,

$$\Omega\;\varepsilon\;L^p(\mathbb{R}^+)\;,\quad 1 \leq p \leq \infty\;.$$

Now rewrite (4.3) as

$$\Omega(x) + \int_0^\infty B(x,s)\;\Omega(s+x)\;ds + B(x,0) = 0.$$

By differentiation, we get

$$0 = \Omega'(x) + \int_0^\infty B(x,s)\Omega'(s+x)ds + \int_0^\infty \partial_1 B(x,s)\Omega(s+x)ds + \partial_1 B(x,0)$$

$$= \Omega'(x) + \int_x^\infty B(x,t-x)\Omega'(t)dt + \int_x^\infty \partial_1 B(x,t-x)\Omega(t)dt + \partial_1 B(x,0) \;,$$

or, if we denote $A \equiv \partial_1 B$,

$$\Omega' = - (I+V_B)^{-1} [A(.,0) + V_A \Omega] \; .$$

Since $A \in L^2(M)$ and $\Omega \in L^2(\mathbb{R}^+)$, $V_A\Omega \in L^2(\mathbb{R}^+)$. Also, $A(.,0) \in L^2(\mathbb{R}^+)$, so that

$$\Omega' \in L^2(\mathbb{R}^+) \; .$$

The proof is complete.

Claim 4. Let us consider differentiable functions, $R : \mathbb{R} \to \mathbb{R}$ that vanish at ∞, such that

$$\Omega' \in L^{(1)} (\mathbb{R}) \cap L^1(\mathbb{R}) \; .$$

Then, for $1 \leq p \leq q \leq \infty$, the operators

$$G(x,\Omega) \; : \; L^p(\mathbb{R}^+) \to L^q(\mathbb{R}^+) \; ,$$

$$\phi \longrightarrow \psi_x(y) \equiv \int_0^\infty \Omega(x+t+y) \; \phi(t) \; dt \, ,$$

have the following properties:

(a) For any fixed Ω and $p = q$,

$$\lim_{x \to +\infty} \| \; G(x,\Omega) \|_{p,q} = 0 \; ,$$

uniformly in p.

(b) For $q < \infty$, or $p = q = \infty$, G is compact.

(c) Consider the norm

$$\| \Omega \| \equiv \| \Omega' \|_1 + \| \Omega' \|_{(1)} . \tag{4.5}$$

Then

$$\| G(x,\Omega) \|_{p,q} \leq \| \Omega \| , \tag{4.6}$$

for all p, q and x, and $G(x,\Omega)$ depend continuously on the pair (x,Ω), in the sense of any p, q operator norm.

<u>Proof</u>: Let

$$\tau(x) \equiv \pm \int_x^{\pm\infty} |\Omega'(t)| \, dt, \quad \pm x \geq 0 .$$

Then, $|\Omega(x)| \leq \tau(x)$, so that

$$\| \Omega \|_\infty \leq \| \tau \|_\infty \leq \| \Omega' \|_1 ,$$

and

$$\| \Omega \|_1 = \int_{-\infty}^{0} dx \left| \int_{-\infty}^{x} \Omega'(t) \, dt \right| \int_{0}^{\infty} dx \left| -\int_{x}^{\infty} \Omega'(t) dt \right|$$

$$\leq \int_{-\infty}^{0} dx \int_{-\infty} |\Omega'(t)| \, dt + \int_{0}^{\infty} dx \int_{x}^{\infty} |\Omega'(t)| \, dt = \| \tau \|_1$$

$$\leq \| \Omega' \|_{(1)}$$

Therefore,

$$\| \psi_x \|_\infty \leq \| \phi \|_\infty \| C_{x+y} \Omega \|_1 \leq \| \Omega' \|_{(1)} \| \phi \|_\infty ,$$

where C_x is the characteristic function of $[x,\infty)$. Also,

$$\| \psi_x \|_\infty \leq \| C_{x+y} \Omega \|_\infty \ \| \phi \|_1 \leq \| \Omega' \|_1 \| \phi \|_1 ,$$

and

$$\| \psi_x \|_1 \leq \int_0^\infty dy \int_0^\infty |\phi(t)| \ |\Omega(x+t+y)| \ dt$$

$$< \int_0^\infty |\phi(t)| \ dt \int_{x+t}^\infty |\Omega(y)| \ dy < \| \Omega' \|_{(1)} \| \phi \|_1 .$$

Thus, $G(x,\Omega)$ is bounded by $\| R \|$ as an operator from L^p to L^q, for $p \leq q$, $p,q \in \{1,\infty\}$. By the Marcel Riesz Interpolation Theorem, the same holds for $p \leq q$, $p,q \in [1,\infty]$.

Let us show now that

$$\lim_{h \to 0} \| G(x \ h,\Omega) - G(x,\Omega) \|_{p,q} = 0:$$

$$\| \psi_{x+h} - \psi_x \|_\infty = \sup_{y \geq 0} \left| \int_{x+y}^\infty \{\Omega(t+h) - \Omega(t)\} \ \phi(t-x-y) \ dt \right|$$

$$\leq \| \phi \|_\infty \sup_{y \geq 0} \int_{x+y}^\infty |\Omega(t+h) - \Omega(t)| \ dt$$

$$\leq \| \phi \|_\infty \int_x^\infty |\Omega(t+h) - \Omega(t)| \ dt \to 0$$

by the Lebesgue Dominated Convergence Theorem. For $p = 1$,

$$\| \psi_{x+h} - \psi_x \|_1 \leq \int_0^\infty |\phi(t)| \ dt \int_{x+t}^\infty |\Omega(y+h) - \Omega(y)| \ dy$$

$$= \| \phi \|_1 \int_{-\infty}^{\infty} |\Omega(y+h) - \Omega(y)| \, dy \to 0$$

again by the LDC Theorem. A further application of the MRI Theorem gives us the result for any $p = q \in [1,\infty]$. Now, $G(x,\Omega)$ is linear in R. By the estimate (4.6) above, it is a uniformly continuous function of R. Since it is a continuous function of x, for fixed Ω, we conclude that it is a continuous function of the pair (x,Ω).

The compactness of $G(x,\Omega)$, for $q \to \infty$, is a consequence of G being the uniform limit of integral operators on finite intervals:

$$\lim_{M\to\infty} \| G(x,\Omega) - G(x,(1-C_M)\Omega) \|_{p,q} = 0.$$

In the case of $L^\infty(R^+)$, we observe that for $\| \phi \|_\infty \le 1$

$$\| \psi_x' \|_\infty \;\; C \;\; \| \Omega \|_1 \le \| \Omega \|$$

and that

$$|\psi_x(y)| \le \int_y^\infty |\Omega(t)| \, dt \to 0 \text{ , as } y \to \infty$$

uniformly with respect to ϕ.

Claim 5. Consider the family of integral equations

$$B(x,y) + \int_0^\infty \Omega(x+t+y) \, B(x,t) \, dt + \Omega(x+y) = 0 \text{ , } y \ge 0 \text{ ,} \qquad (4.7)$$

for real x. Assume that R is differentiable,

$$\Omega(x) = o(x^{-2}), \quad x \to +\infty,$$

and

$$\| \Omega' \|_{(1)} + \| \Omega' \|_{\infty} \leq \infty .$$

Then:

(a) For each fixed x, there exists a unique solution $B(x,.)$ of (4.7); this solution belongs to the Sobolev spaces $W^{1,p}(R^+)$ for $1 \leq p \leq \infty$, and vanishes at $y = +\infty$.

(b) If Ω' is continuous so is $\partial_1 B(x,y)$.

(c) Let B_a be the restriction of B to $\{x > a,\ y > 0\}$; then the mapping

$$H : \Omega \longrightarrow B_\alpha$$

is continuous with respect to the norms

$$\| \Omega \| \equiv \| \Omega' \|_1 + \| \Omega' \|_{(1)} , \quad \| B_\alpha \| \equiv \sup_{x \geq \alpha} \| B(x,.) \|_{1,p} .$$

Proof: The operators $I + G(x,\Omega)$ are strictly positive in L^2 as is shown in Refr. 7 and 10. Now, assume that $\phi_x \in L^p$, for $p < \infty$ and

$$\phi_x = - G(x,\Omega)\phi_x \quad . \tag{4.8}$$

By (4.6), $\phi_x \in L^\infty$. But if $\phi_x \in L^*$ satisfies (4.8), we can conclude that $\phi_x \in L^2$ and thus $\phi_x \equiv 0$. To prove this last assertion, we only have to observe that for any $\phi \in L^m$,

$$|y\ \psi_x(y)| = |y \int_0^\infty \Omega(x+t+y)\phi(t)\ dt|$$

$$< \| \phi \|_\infty \{ |y+x| \int_{y+x}^\infty |\Omega(t)|\ dt + |x| \int_{y+x}^\infty |\Omega(t)|\ dt \}$$

and this last quantity tends to 0 as $y \to +\infty$.

Now we claim that if

$$F(x,\Omega) \equiv \{I + G(x,\Omega)\}^{-1}$$

then

$$\lim_{\Omega_1 \to \Omega} \{\sup_{x \geq \alpha} \| F(x,\Omega_1) - F(x,\Omega) \|_p \} = 0 .$$

This is a consequence of the continuity of $G(x,\Omega)$ as a function of the pair (x,Ω) and its asymptotic behavior as $x \to +\infty$. Denote

$$(T_x \Omega)\ (y) \equiv \Omega(x+y) \quad ;$$

then

$$B(x,.) = - F(x,\Omega)\ T_x \Omega$$

and therefore

$$\lim_{\Omega_1 \to \Omega} \sup_{x \geq 0} \| F(x,\Omega_1) T_x \Omega_1 - F(x,\Omega) T_x \Omega \|_p$$

$$\leq \lim_{\Omega_1 \to \Omega} \{\sup_{x>0} \| F(x,\Omega_1) \|_p \ \| T_x (\Omega_1 - \Omega) \|_p +$$

$$+ \sup_{x \geq 0} \| F(x,\Omega_1) - F(x,\Omega) \|_p \ \| T_x \Omega \|_p \ \}$$

$$\leq \lim_{\Omega_1 \to \Omega} \{ (\| \Omega \| + 1) \| \Omega_1 - \Omega \| + \| \Omega \|_p \sup_{x \geq 0} \| F(x,\Omega) - F(x,\Omega_1) \|_p \} = 0 .$$

By differentiating (4.7), we get

$$\partial_1 B(x,y) + \int_0^\infty \{\Omega'(x+t+y)B(x,t) + \Omega(x+t+y)\, \partial_1 B(x,t)\}\, dt + \Omega'(x+y) = 0 ,$$

or equivalently

$$\partial_1 B(x,y) = -F(x,\Omega)\{T_x\Omega' + G(x,\Omega')\, B(x,.)\}(y) .$$

By using the estimates in Claim 4, we get that

$$G(x,\Omega')B(x,\cdot) \in L^p$$

and depends continuously on R, in the sense of (4.5), as does $T_x\Omega'$. This completes the proof.

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