The problem considered is the realization of the kets \( |(l_1 l_2)LM>\), which appear in the usual Clebsch-Gordan transformation for integral angular momenta, when one realizes the uncoupled ones, \(|l_1 m_1>\) and \(|l_2 m_2>\), by solid harmonics. It is shown that the first named kets are correspondingly realized by homogeneous and harmonic polynomials, defined in a 6-dimensional Euclidean space, and belonging to the set of hypersolid harmonics which carry the so-called "most degenerate" irreducible representations of the 6-dimensional rotation group.

Considera-se o problema da realização dos estados \(|(l_1 l_2)LM>\) que compa-recem na transformação usual de Clebsch-Gordan, quando os estados desa-coplados \(|R m>\) e \(|t R m_2>\) são realizados pelos sólidos harmônicos. Mostra-se que os ket \(|(l_1 l_2)LM>\) são correspondentemente realizados por polinômios homogêneos e harmônicos, definidos em um espaço euclidiano a seis dimensões, pertencentes ao conjunto dos hipersólidos harmônicos que portam as representações ditas as "mais degeneradas" do grupo das rota-ções a seis dimensões.

1. INTRODUCTION

In the literature of angular momentum, Refs.1-4, and group theory for physicists, Refs.5-9, Clebsch-Gordan coefficients are introduced via the well-known unitary transformation which involves abstract "state vectors". One of the present authors has wondered for years why a realization of the \(|(l_1 l_2)LM>\) kets, appearing in that expansion for
integral values of the angular momenta, \( l_1, R \) were not explicitly exhibited in specialized monographs. Indeed, if one is able to give the very well-known realization of the \( \text{SO}_3 \) generators, in terms of particle position vector and its gradient, and realize the uncoupled kets, \( |l_1 m_1> \) and \( |l_2 m_2> \), present in the C-G expansion, by the usual solid harmonics, one should also be able to produce the corresponding realization of the \( |l_1 l_2 LM> \) kets. This is the purpose of the present paper.

Section 2 is devoted to show the relation of the realization sought for with the homogeneous and harmonic polynomials, in six variables, which carry the "most degenerate" irreducible representations (irreps) of \( \text{SO}_6 \), the 6-dimensional rotation group. In order to make it comprehensible how the realization of the \( |l_1 l_2 LM> \) kets comes up, in the C-G expansion, we summarize the essential steps needed in constructing the irreducible polynomial bases of \( \text{SO}_6 \). Details on the construction of those bases can be found in a paper by the present authors on the nonrelativistic quantum mechanical three-body problem.

2. THE MOST DEGENERATE REPRESENTATION OF \( \text{SO}_6 \) AND ITS RELATION TO THE \( |l_1 l_2 LM> \) KET

Starting from the beginning, we write down the Clebsch-Gordan transformation:

\[
|l_1 m_1> |l_2 m_2> = \sum_{LM} <l_1 m_1 l_2 m_2 |(l_1 l_2 LM> |(l_1 l_2 LM> , \quad (2.1)
\]

restricting ourselves to integral angular momenta. The uncoupled kets, we realize them by solid harmonics \( Y_{l_1 m_1} (x) = |x|^{l_1} Y_{l_1 m_1} (\mathbf{x}) \), and \( Y_{l_2 m_2} (y) \), whose arguments, \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \), are independent 3-vectors in 3-dimensional Euclidian space. Each of these infinite sets of solid harmonics carry all single-valued irreps of \( \text{SO}_3 \), and are labelled by the invariant (Casimir) operators \( L^2 (x) (L^2 (y)) \) and \( L^3 (x) (L^3 (y)) \), which correspond to \( \text{SO}_3 \) and its \( \text{SO}_2 \) subgroup, respectively. Here, \( L(x) = -i \mathbf{x} \times \nabla_x \), and mutatis mutandis for \( L(y) \). Since \( \mathbf{x} \) and \( \mathbf{y} \) are independent variables, the commutators of \( L^3 (x) \) and \( L_k (y) \) vanish.
For the coming discussion, it is convenient to reverse the C-G transformation, rewriting it under the form

\[ Y_{(L_1, L_2)}LM(x, y) = \sum_{m_1, m_2}^{L_1, L_2} \langle L_1, m_1 | L_2, m_2 \rangle Y_{L_1, m_1}(x) Y_{L_2, m_2}(y) \quad (2.2) \]

and we are here adhering to the usual convention by which C-G coefficients are real. In (2.2), the lefthand side denotes the realization we are going to produce. In the righthand side, we have the product of two solid harmonics, in distinct, and independent, variables. These products carry all irreps of the direct product group, so, \( \times \) \( SO_3(y) \), with generators \( L(x) \) and \( L(y) \), respectively. It is then natural to imagine that the functions \( Y_{(L_1, L_2)}LM(x, y) \) are, in some way, related to the polynomial bases of the group \( SO_6(x, y) \), of rotations in a 6-dimensional Euclidian space, \( E_6 \) of vectors \( \mathbf{r} = (x, y) \), i.e., \( (x)_{i} = x_{i} \) and \( (y)_{i+3} = y_{i}, \ i = 1, 2, 3 \). This because the above direct product group is a subgroup of the larger orthogonal group, which can provide all labels present in \( Y_{(L_1, L_2)}LM \).

Before aiming at the center of the target, we briefly indicate the main steps in producing the polynomial bases which carry the irreps of \( SO_6(x, y) \).

The fifteen generators of \( SO_6(x, y) \) can be realized by the antisymmetric operators, which in compact a notation we write as

\[ \Lambda_{i,j}^{\alpha \beta} = \frac{1}{2} \left( x_{i}^{\alpha} \frac{\partial}{\partial x_{j}^{\beta}} - x_{i}^{\beta} \frac{\partial}{\partial x_{j}^{\alpha}} \right); \quad (2.3) \]

\( x_{i}^{2} = x_{i}^{2} \), \( x_{i}^{2} = y_{i} \), \( \alpha, \beta = 2 \), \( i, j = 1 \). They indeed fulfill the commutation relations of the \( SO_6 \) algebra. If we then construct the invariant operators, we find that they produce a single, independent, label, we shall call it \( \lambda \); the simplest invariant operator which provides this label is the one quadratic in the generators, namely \( J_{2} = \Lambda_{i,j}^{\alpha \beta} \Lambda_{i,j}^{\alpha \beta} \). In the realization (2.3), it is given by
where \( \mathbf{r} \) stands for a 6-dimensional vector, in \( \mathbb{E} \), components \( x, y; \mathbf{r}^2 = x^2 + y^2 \); \( \nabla \) and \( \nabla^2 \) (which equals \( \nabla^2 + \nabla^2 \)) are the corresponding gradient and Laplace operators.

Since the generators, (2.3), are homogeneous operators of zeroth degree, in \( x \) and \( y \), they preserve the degree of homogeneous polynomials in those variables, \( \text{i.e.,} \) they transform homogeneous polynomials into homogeneous polynomials of the same degree, and this shows that homogeneous polynomials, in \( x, y \), carry representations of SO \((x, y)\). These representations are, however, in general reducible. To select the irreducible polynomials, we have only to impose their harmonicity in \( \mathbb{E} \), \( \text{i.e.,} \) \( \nabla^2 \psi(x, y) = 0 \), since then they will be eigenfunctions of the invariant operator \( J_2 \), (2.4), with eigenvalues \((1/2)\lambda(\lambda+4)\):

\[
J_2 \psi^\lambda(x, y) = \frac{1}{2} \lambda(\lambda+4) \psi^\lambda(x, y). \tag{2.5}
\]

In (2.5), \( \lambda \) is the degree of homogeneity extracted by the operator \( \mathbf{r} \cdot \nabla \), according to Euler's theorem; the notation \([\lambda]\) means that we are dealing with polynomials which carry irreducible representations. SO\(_6\) has rank three, but their irreps carried by polynomials are labelled by a single parameter, \( \Lambda \), with integral values starting from zero. That is why these representations are sometimes called "most degenerate". In classifying the constituents of a \([\lambda]\)-basis, we have perforce chosen that chain of SO\(_6\)(\(x, y\)) subgroups which would provide the labels that appear in the \(|(\ell_1, \ell_2)LM\rangle\) ket, namely,

\[
\begin{align*}
SO_6(x, y) & \supset SO_3(x) \times SO_3(y) \supset SO_3(L) \supset SO_2(M). \tag{2.6} \\
& \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\lambda & \downarrow \ell_1 \quad \ell_2 \quad L \quad M
\end{align*}
\]
In (2.6), \( R \) and \( R \) correspond to the invariant operators \( L^2(x) \) and \( L^2(y) \) of \( SO_3(x) \) and \( SO_3(y) \), respectively, with \( L(x) = -i x \times \frac{\partial}{\partial x} \) and \( L(y) = -i y \times \frac{\partial}{\partial y} \), the quantum numbers \( L \) and \( M \) are associated with the Casimir operators of \( SO_3(L) \) and \( SO_3(M) \), which are \( L^2 \) and \( L_3 \), with \( L_3 = L(x) + L(y) \), and \( L_3 \) its 3rd. component.

In the construction of the irreducible \([\lambda]\)-polynomials, the hypersolid harmonics of \( E_6 \), we start by coupling the usual solid harmonics, \( Y_{\ell_1 m_1}^{(x)} \) and \( Y_{\ell_2 m_2}^{(y)} \), via \( C-G \) coefficients, obtaining in this way homogeneous and harmonics polynomials, in \( E_6 \), of degree \( \ell_1 + \ell_2 \) in the six variables, \( x \) and \( y \). These hypersolid harmonics provide the realization of the \( \{L_{\ell_1 \ell_2}^{LM}\} \) ket we were searching for. They are eigenfunctions of \( L^2(x) \), \( L^2(y) \), \( L_2 \) and \( L_3 \); since their degree is \( \ell_1 + \ell_2 \), we have, of course, \( A = R_1 + R_2 \). We shall come back to them at the very end.

We think it worthwhile to terminate this paper by showing how the other irreducible \([\lambda]\)-polynomials are constructed, in the belief that having the whole set of \([A]\)-bases, one shall be able to see the \( Y_{\ell_1 \ell_2}^{(x,y)} \)'s against the background of all \([\lambda]\)-polynomials.

The hypersolid \( Y_{\ell_1 + \ell_2}^{(x,y)} \) do not exhaust the set of hypersolid harmonics, in \( E_6 \), since if we multiply

\[
y_{\ell_1 \ell_2}^{LM} (x,y) = \sum_{m_1 m_2} c_{\ell_1 m_1 \ell_2 m_2}^{LM} \psi_{\ell_1 m_1}^{(x)}(x) \psi_{\ell_2 m_2}^{(y)}(y)
\]

by a polynomial (so far unknown) in the variables \( x^2 \) and \( y^2 \), hereafter denoted by \( B_{\ell_1 \ell_2}^{(x^2,y^2)} \), which is homogeneous in the variables \( x, y \), of degree \( A-R-\% = 2n \), \( n = 0, 1, 2, \ldots \), we obtain new hypersolid harmonics by imposing the requirement of harmonicity upon the expression obtained after multiplication of (2.7) by the \( B_{\ell_1 \ell_2}^{(x^2,y^2)} \). Since the \( B_{\ell_1 \ell_2}^{(x^2,y^2)} \)'s are functions of \( x^2, y^2 \), they are invariant under \( SO_3(x) \) and \( SO_3(y) \) and, therefore, under \( SO_3(L) \) as well. This means that, in making the Ansatz
(in which $N_{\ell_1,\ell_2}$ are normalization constants introduced for professional reasons) we did not spoil the condition, for the $\psi^{(\ell_1,\ell_2)}$'s, of being eigenfunctions of $L^2(x)$, $L^2(y)$, $L_3$ and $L_\omega$. The functions (2.8) are harmless homogeneous harmonic polynomials of degree $\lambda=\ell_1+\ell_2+2n$. The requirement of harmonicity is enough to determine the $B^{(\lambda)}_{\ell_1,\ell_2}$'s up to a multiplicative constant, and the normalization constants $N_{\ell_1,\ell_2}$ have been chosen as to make the $B^{(\lambda)}_{\ell_1,\ell_2}$'s, up to the factor $x^{2n}$, coincide with Jacobi's polynomials:

$$B^{(\lambda)}_{\ell_1,\ell_2}(x^2, y^2) = x^{2n} rac{\rho_n^{(\ell_1+1/2, \ell_2+1/2)} \left( \frac{y^2 - x^2}{x^2 + y^2} \right)}{\rho_n^{(\ell_1+1/2, \ell_2+1/2)}},$$

(2.9)

$n = (1/2)(A-R, -R)$ being the degree of the Jacobi polynomials, and $r = (x^2+y^2)^{1/2}$; Ref. 10.

The Ansatz (2.8), with (2.9), gives all hypersolid harmonics, in $E$, the carriers of the "most degenerate" representation of $SO_6$. One can verify that the labels $\lambda, \ell_1, \ell_2, L, M$ are related by the following branching laws:

$$\ell_1+\ell_2 = \lambda, \lambda-2, \ldots, \left\{^0_1 \right\}; \quad \ell_1+\ell_2 \geq L \geq \left| \ell_1-\ell_2 \right|, L \geq M \geq -L$$

We are at the end of our tether. If we now take $n = 0$, in (2.8), we obtain, disregarding the multiplicative constant $N_{\ell_1,\ell_2}$, the realization which has been revealed in (2.7), a subset of the polynomial bases of $SO_6(x, y)$. 

$$y^{(\lambda)}_{\ell_1,\ell_2}(x, y) = N_{\ell_1,\ell_2} \frac{\rho_n^{(\ell_1, \ell_2)}(x^2, y^2)}{\rho_n^{(\ell_1, \ell_2)}},$$

(2.8)
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