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The Ciebsch-Gordan Expansion for the Three Dimensional Rotation Group: A Realization of the States Involved

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The problem considered is the realization of the kets $|(\ell_1\ell_2)LM>$, which appear in the usual Clebsch-Gordan transformation for integral angular momenta, when one realizes the uncoupled ones, $|\ell_1m_1>$ and $|\ell_2m_2>$, by solid harmonics. It is shown that the first named kets are correspondingly realized by homogeneous and harmonic polynomials, defined in a 6-dimensional Euclidian space, and belonging to the set of hypersolid harmonics which carry the so-called "most degenerate" irreducible representations of the 6-dimensional rotation group.

Considera-se o problema da realização dos estados $|(\ell_1\ell_2)LM>$ que comparecem na transformação usual de Clebsch-Gordan, quando os estados desacoplados $|\ell_1m_1>$ e $|\ell_2m_2>$ são realizados pelos sólidos harmônicos. Mostra-se que os *ket* $|(\ell_1\ell_2)LM>$ são correspondentemente realizados por polinômios homogêneos e harmônicos, definidos em um espaço euclidiano a seis dimensões, pertencentes ao conjunto dos hipersólidos harmônicos que portam as representações ditas as "mais degeneradas" do grupo das rotações a seis dimensões.

1. INTRODUCTION

In the literature of angular momentum, Refs.1-4, and group theory for physicists, Refs.5-9, Clebsch-Gordan coefficients are introduced via the well-known unitary transformation which involves *abstract* "state vectors". One of the present authors has wondered for years why a realization of the $|(\ell_1\ell_2)LM>$ kets, appearing in that expahsion for

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integral values of the angular *momenta*, ℓ_1, R were not explicitly exhibited in specialized monographs. Indeed, if one is able to give the very well-known realization of the SO, generators, in terms of particle position vector and its gradient, and realize the uncoupled kets, $|\ell_1 m_1>$ and $|\ell_2 m_2>$, present in the C-G expansion, by the usual solid harmonics, one should also be able to produce the corresponding realization of the $|(\ell_1\ell_2)LM>$ kets. This is the purpose of the present paper.

Section 2 is devoted to show the relation of the realization sought for with the homogeneous and harmonic polynomials, in six variables, which carry the "most degenerate" irreducible representations (irreps) of SO_6, the 6-dimensional rotation group. In order to make it comprehensible how the realization of the $|(\ell_1\ell_2)LM>$ kets comes up, in the C-G expansion, we summarize the essential steps needed in constructing the irreducible polynomial bases of SO. Details on the construction of those bases can be found in a paper by the present authors on the nonrelativistic quantum mechanical three-body problem.

2. THE MOST DEGENERATE REPRESENTATION OF SO_6 AND ITS RELATION TO THE $|(\ell_1\ell_2)LM>$ KET

Starting from the beginning, we write down the Clebsch-Gordan transformation:

$$|\ell_1 m_1>|\ell_2 m_2> = \sum_{LM} <\ell_1 m_1 \ell_2 m_2|(\ell_1\ell_2)LM>|(\ell_1\ell_2)LM>, \qquad (2.1)$$

restricting ourselves to integral angular momenta. The uncoupled kets, we realize them by solid harmonics $\mathcal{Y}_{\ell_1 m_1}(\underset{\sim}{x}) = |\underset{\sim}{x}|^{\ell_1} Y_{\ell_1 m_1}(\hat{x})$, and $\mathcal{Y}_{\ell_2 m_2}(\underset{\sim}{y})$, whose argurnents, $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$, are independent 3-vectors in 3-dimensional Euclidian space. Each of these infinite sets of solid harmonics carry all single-valued irreps of SO_3, and are labelled by the invariant (Casimir) operators $\underset{\sim}{L}^2(\underset{\sim}{x})$ $(\underset{\sim}{L}^2(\underset{\sim}{y}))$ and $L_3(\underset{\sim}{x})$ $(L_3(\underset{\sim}{y}))$, which correspond to SO, and its SO_2 subgroup, respectively. Here, $\underset{\sim}{L}(\underset{\sim}{x}) = -i\underset{\sim}{x}\times\nabla_{\underset{\sim}{x}}$, and *mutatis mutandis* for $\underset{\sim}{L}(\underset{\sim}{y})$. Since $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are independent variables, the commutators of $L_j(\underset{\sim}{x})$ and $L_k(\underset{\sim}{y})$ vanish.

For the coming discussion, it is convenient to reverse the C-G transformation, rewriting it under the form

$$Y_{(\ell_1\ell_2)LM}(\underset{\sim}{x},\underset{\sim}{y}) = \sum_{m_1 m_2} \langle \ell_1 m_1 \ell_2 m_2 | (\ell_1 \ell_2) LM \rangle \, Y_{\ell_1 m_1}(\underset{\sim}{x}) \, Y_{\ell_2 m_2}(\underset{\sim}{y}) \qquad (2.2)$$

and we are here adhering to the usual convention by which C-G coefficients are real. In (2.2), the lefthand side denotes the realization we are going to produce. In the righthand side, we have the product of two solid harmonics, in distinct, and independent, variables. These products carry all irreps of the direct product group, $SO_3(\underset{\sim}{x}) \times SO_3(\underset{\sim}{y})$, with generators $\underset{\sim}{L}(\underset{\sim}{x})$ and $\underset{\sim}{L}(\underset{\sim}{y})$, respectively. It is then natural to imagine that the functions $Y_{(\ell_1\ell_2)LM}(\underset{\sim}{x},\underset{\sim}{y})$ are, in some way, related to the polynomial bases of the group $SO_6(\underset{\sim}{x},\underset{\sim}{y})$, of rotations in a 6-dimensional Euclidian space, E_6, of vectors $\underset{\sim}{r} = (\underset{\sim}{x},\underset{\sim}{y})$, i.e., $(\underset{\sim}{r})_i = x_i$ and $(\underset{\sim}{r})_{i+3} = y_i$, $i = 1,2,3$. This because the above direct product group is a subgroup of the larger orthogonal group, which can provide all labels present in $Y_{(\ell_1\ell_2)LM}$.

Before aiming at the center of the target, we briefly indicate the main steps in producing the polynomial bases which carry the irreps of $SO_6(\underset{\sim}{x},\underset{\sim}{y})$.

The fifteen generators of $SO_6(\underset{\sim}{x},\underset{\sim}{y})$ can be realized by the antisymmetric operators, which in compact a notation we write as

$$\Lambda^{\alpha\beta}_{ij} = \frac{1}{2} \left(x^{\alpha}_{i} \frac{\partial}{\partial x^{\beta}_{j}} - x^{\beta}_{j} \frac{\partial}{\partial x^{\alpha}_{i}} \right) ; \qquad (2.3)$$

$x^1_i = x_i$, $x^2_i = y_i$; $\alpha,\beta = 2$ $i,j = 1$, They indeed fullfill the commutation relations of the SO_6 algebra. If we then construct the invariant operators, we find that they produce a single, independent, label, we shall call it λ; the simplest invariant operator which provides this label is the one guadratic in the generators, namely $J_2 = \Lambda^{\alpha\beta}_{ij} \Lambda^{\alpha\beta}_{ij}$. In the realization (2.3), it is given by

$$J_2 = \frac{1}{2}\left[(\underset{\sim}{r}.\underset{\sim}{\nabla})(\underset{\sim}{r}.\underset{\sim}{\nabla}+4) - \underset{\sim}{r}^2\ \underset{\sim}{\nabla}^2\right], \qquad (2.4)$$

where $\underset{\sim}{r}$ stands for a 6-dimensional vector, in E_6 components $\underset{\sim}{x}$, $\underset{\sim}{y}$; $\underset{\sim}{r}^2 = \underset{\sim}{x}^2+\underset{\sim}{y}^2$; ∇ and ∇^2 (which equals $\nabla^2_{\underset{\sim}{x}} + \nabla^2_{\underset{\sim}{y}}$) are the corresponding gradient and Laplace operators.

Since the generators, (2.3) , are homogeneous operators of zeroth degree, in $\underset{\sim}{x}$ and $\underset{\sim}{y}$, they preserve the degree of homogeneous polynomials in those variables, *i.e.*, they transform homogeneous polynomials into homogeneous polynomials of the same degree, and this shows that homogeneous polynomials, in $\underset{\sim}{x}$, $\underset{\sim}{y}$, carry representations of SO_6 $(\underset{\sim}{x},\underset{\sim}{y})$. These representations are, however, in general reducible. To select the irreducible polynomials, we have only to impose their harmonicity in E_6 *i.e.*, $\underset{\sim}{\nabla}^2\ Y(\underset{\sim}{x},\underset{\sim}{y})=0$, since then they will be eigenfunctions of the invariant operator J_2, (2.4), with eigenvalues $(1/2)\lambda(\lambda+4)$:

$$J_2\ Y^{[\lambda]}(\underset{\sim}{r}) = \frac{1}{2}\lambda(\lambda+4)\ Y^{[\lambda]}(\underset{\sim}{r}). \qquad (2.5)$$

In (2.5), λ is the degree of homogeneity extracted by the operator $\underset{\sim}{r}.\underset{\sim}{\nabla}$, according to Euler's theorem; the notation $[\lambda]$ means that we are dealing with polynomials which carry irreducible representations. SO_6 has rank three, but their irreps carried by polynomials are labelled by a single parameter, λ, with integral values starting from zero. That is why these representations are sometimes called "most degenerate". In classifying the constituents of a $[\lambda]$-basis, we have perforce chosen that chain of $SO_6(\underset{\sim}{x},\underset{\sim}{y})$ subgroups which would provide the labels that appear in the $|(\ell_1\ell_2)LM\rangle$ ket, namely,

$$\begin{array}{ccccccccc} SO_6(\underset{\sim}{x},\underset{\sim}{y}) & \supset & SO_3(\underset{\sim}{x}) & \times & SO_3(\underset{\sim}{y}) & \supset & SO_3(\underset{\sim}{L}) & \supset & SO_2(M) \\ \| & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \lambda & & \ell_1 & & \ell_2 & & L & & M \end{array} \qquad (2.6)$$

In (2.6), ℓ_1 and ℓ_2 correspond to the invariant operators $\underset{\sim}{L}^2(\underset{\sim}{x})$ and $\underset{\sim}{L}^2(\underset{\sim}{y})$ of $SO_3(\underset{\sim}{x})$ and $SO_3(\underset{\sim}{y})$, respectively, with $\underset{\sim}{L}(\underset{\sim}{x}) = -i\underset{\sim}{x} \times \nabla_{\underset{\sim}{x}}$ and $\underset{\sim}{L}(\underset{\sim}{y}) = -i\underset{\sim}{y} \times \nabla_{\underset{\sim}{y}}$; the quantum numbers L and M are associated with the Casimir operators of $SO_3(\underset{\sim}{L})$ and $SO_2(M)$, which are $\underset{\sim}{L}^2$ and L_3, with $\underset{\sim}{L} = \underset{\sim}{L}(\underset{\sim}{x}) + \underset{\sim}{L}(\underset{\sim}{y})$, and L_3 its 3rd. component.

In the construction of the irreducible $[\lambda]$-polynomials, the hypersolid harmonics of E_6 we start by coupling the usual solid harmonics, $Y_{\ell_1 m_1}(\underset{\sim}{x})$ and $Y_{\ell_2 m_2}(\underset{\sim}{y})$, via C-G coefficients, obtaining in this way homogeneous and harmonics polynomials, in E_6, of degree $\ell_1+\ell_2$ in the six variables, x_i and y_i. These hypersolid harmonics provide the realization of the $|(\ell_1\ell_2)LM\rangle$ ket we were searching for. They are eigenfunctions of $\underset{\sim}{L}^2(\underset{\sim}{x})$, $\underset{\sim}{L}^2(\underset{\sim}{y})$, $\underset{\sim}{L}^2$ and L_3; since their degree is $\ell_1+\ell_2$, we have, of course, $\lambda = \ell_1 + \ell_2$. We shall come back to them at the very end.

We think it worthwhile to terminate this paper by showing how the other irreducible $[\lambda]$-polynomials are constructed, in the belief that having the whole set of $[\lambda]$-bases, one shall be able to see the $Y^{[\ell_1+\ell_2]}_{(\ell_1\ell_2)LM}(\underset{\sim}{x},\underset{\sim}{y})$'s against the background of all $[\lambda]$-polynomials.

The hypersolids $Y^{[\ell_1+\ell_2]}$ do not exhaust the set of hypersolid harmonics, in E_6 since if we multiply

$$Y^{[\ell_1+\ell_2]}_{(\ell_1\ell_2)LM}(\underset{\sim}{x},\underset{\sim}{y}) = \sum_{\substack{m_1\\ m_2}} \langle \ell_1 m_1 \ell_2 m_2 | (\ell_1\ell_2)LM\rangle \, Y_{\ell_1 m_1}(\underset{\sim}{x}) \, Y_{\ell_2 m_2}(\underset{\sim}{y}) \tag{2.7}$$

by a polynomial (so far unknown) in the variables $\underset{\sim}{x}^2$ and $\underset{\sim}{y}^2$, hereafter denoted by $B^{[\lambda]}_{\ell_1\ell_2}(\underset{\sim}{x}^2,\underset{\sim}{y}^2)$, which is homogeneous in the variables $\underset{\sim}{x}$, $\underset{\sim}{y}$, of degree $\lambda-\ell_1-\ell_2 = 2n$, $n = 0,1,2\ \ldots,$ we obtain new hypersolid harmonics by imposing the requirement of harmonicity upon the expression obtained after multiplication of (2.7) by the $B^{[\lambda]}$'s. Since the $B^{[\lambda]}$'s are functions of $\underset{\sim}{x}^2$, $\underset{\sim}{y}^2$, they are invariant under $SO_3(\underset{\sim}{x})$ and $SO_3(\underset{\sim}{y})$ and, therefore, under $SO_3(\underset{\sim}{L})$ as well. This means that, in making the *Ansatz*

$$y^{[\lambda]}_{(\ell_1\ell_2)LM}(\underset{\sim}{x},\underset{\sim}{y}) = N^{[\lambda]}_{\ell_1\ell_2}\; B^{[\lambda]}_{\ell_1\ell_2}\;(\underset{\sim}{x}^2,\underset{\sim}{y}^2)\,.$$

$$\cdot \sum_{\substack{m_1\\ m_2}} \langle \ell_1 m_1 \ell_2 m_2 | (\ell_1\ell_2)LM\rangle \; Y_{\ell_1 m_1}(\underset{\sim}{x})\; Y_{\ell_2 m_2}(\underset{\sim}{y}) \;, \qquad (2.8)$$

(in which $N^{[\lambda]}_{\ell_1\ell_2}$ are **normalization** constants introduced for professional reasons) we did not spoil the condition, for the $Y^{[\lambda]}$'s, of being eigenfunctions of $\underset{\sim}{L}^2(\underset{\sim}{x})$, $\underset{\sim}{L}^2(\underset{\sim}{y})$, $\underset{\sim}{L}^2$ and L_3. The functions (2.8) are harmless homogeneous harmonic polynomials of degree $\lambda=\ell_1+\ell_2+2n$. The requirement of harmonicity is enough to determine the $B^{[\lambda]}$'s up to a multiplicative constant, and the normalization constants $N^{[\lambda]}_{\ell_1\ell_2}$ have been chosen as to make the $B^{[\lambda]}$'s, up to the factor r^{2n}, coincide with Jacobi's polynomials:

$$B^{[\lambda]}_{\ell_1\ell_2}(\underset{\sim}{x}^2,\underset{\sim}{y}^2) = r^{2n}\, P_n^{(\ell_1+1/2,\ell_2+1/2)}\left(\frac{\underset{\sim}{y}^2-\underset{\sim}{x}^2}{\underset{\sim}{x}^2+\underset{\sim}{y}^2}\right) \;, \qquad (2.9)$$

n = (1/2) (A-R,-%,) being the degree of the Jacobi polynomials, and $r = (x^2+y^2)^{1/2}$; Ref. 10.

The *Ansatz* (2.8), with (2.9), gives *all* hypersolid harmonics, in E, , the carriers of the "most degenerate" representation of SO, One can verify that the labels λ, ℓ_1, ℓ_2, L, M are related by the following branching laws:

$$\ell_1+\ell_2 = \lambda,\ \lambda-2,\ldots,\ \{^0_1 \;\; ; \;\; \ell_1+\ell_2 \geq L \geq |\ell_1-\ell_2|,\ L\geq M\geq -L$$

We are at the end of our tether. If we now take $n = 0$, in (2.8), we obtain, disregarding the multiplicative constant $N^{[\lambda]}_{\ell_1\ell_2}$, the realization which has been revealed in (2.7), a subset of the polynomial bases of $SO_6(\underset{\sim}{x},\underset{\sim}{y})$.

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