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Energy Levels of the Schrodinger Equation for a Linear plus Coulomb-like Potential*

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A computer integration of the Schrodinger equation for a pair of particles, with reduced mass $m/2$, interacting by a potential of the form $V(r) = V_0 + \lambda r - \gamma/r$ was performed for several values of the parameter $a = m\gamma/(m\lambda)^{1/3}$ in the range $0.29 \leq a \leq 0.65$. Other physically interesting quantities such as $\langle 1/r \rangle$ and $4\pi|\psi(0)|^2$ where also computed for the first eigenstates in S-, P- and D-waves.

Realizamos a integração, por computador, da equação de Schrodinger para um sistema de duas partículas, de massa reduzida $m/2$, interagindo por um potencial da forma $V(r) = V_0 + \lambda r - \gamma/r$, para diversos valores do parâmetro $a = m\gamma/(m\lambda)^{1/3}$, no intervalo $0.29 \leq a \leq 0.65$. Calculamos também outras grandezas fisicamente interessantes tais como $\langle 1/r \rangle$ e $4\pi|\psi(0)|^2$, para as primeiras excitações em ondas S, P e D.

1. INTRODUCTION

In the present work we report some of the results of a computer integration of the Schrödinger equation for a system of two particles, of reduced mass $m/2$, interacting by a potential of the form

$$V(r) = V_0 + \lambda r - \frac{\gamma}{r} \quad . \tag{1.1}$$

This type of potential has been suggested by theoretical investigations of non-abelian gauge theories as applied to the gauge model of strong interactions[1]. In this context, the linear part of $V(r)$ represents the confining, long-range, part of the interaction between the consti-

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tuents quarks, whereas the Coulomb-like attractive part of it comes from the short-range gluon exchange, as expected in asymptotically free gauge theories.

The eigenfunctions of the Schrödinger equation for the potential (1.1) constitute the starting point of a recent and rather successful perturbative treatment of the meson masses (especially of those recently discovered vector mesons forming the ψ-family) in which the Fermi-Breit interaction terms, arising from gluon exchange, are taken into account in first order[2].

Having in mind further applications to hadron spectroscopy, we perform a computer integration of the corresponding radial Schrödinger equation, for several values of the parameter $a = m\gamma/(m\lambda)^{1/3}$ in a suitably chosen range, namely, $0.29 \leq a \leq 0.65$, in order to examine the sensitivity of the solutions on the parameters involved, for the various physically interesting cases. We also compute other relevant physical quantities such as $\langle 1/r \rangle$ and $4\pi|\psi(0)|^2$ for the lower excitations in S-, P- and D-waves.

As is well known, the case $\gamma = 0$ (that is, the pure linear case) admits exact solutions only for S-wave states[3]. Since the case $\gamma = 0$ is also of some interest[4], we also solve it for the first few excitations in P- and D-waves.

The computer rnethod of intergration we used was the "matching method"[5] together with Nurnerov's formula[6].

In Section 2, we state the problem and present the main results obtained. A brief discussion of the results is given in Section 3.

2. THE PROBLEM AND MAIN RESULTS

The radial Schrödinger equation for a system of two particles, with reduced mass $m/2$, in the CM-system and in terrns of the dimensionless variable ($\hbar=c=1$)

$$\rho \equiv \beta r = (m\lambda)^{1/3} r, \tag{2.1}$$

reads

$$\left[\frac{d^2}{d\rho^2} - \frac{L(L+1)}{\rho^2} + \frac{\alpha}{\rho} + \varepsilon - \rho\right] u(\rho) = 0 . \tag{2.2}$$

The eingenfunctions for bound-states obey the usual boundary conditions

$$u(0) = u(\infty) = 0 . \tag{2.3}$$

In (2.2), ε and α are dimensionless quantities defined by

$$\varepsilon = \frac{m(E-V_0)}{\beta^2} , \tag{2.4}$$

$$\alpha = \frac{m\gamma}{\beta} , \tag{2.5}$$

where E is the energy eigenvalue.

For the expectation value of $1/r$, we have, by using (2.1),

$$\left\langle \frac{1}{r} \right\rangle = \int_0^\infty \frac{u^2(r)}{r} dr = \int_0^\infty \frac{u^2(\frac{\rho}{\beta})}{\rho} d\rho . \tag{2.6}$$

As

$$\int_0^\infty u^2(r)\, dr = \frac{1}{\beta} \int_0^\infty u^2(\tfrac{\rho}{\beta})\, d\rho = 1 ,$$

upon defining

$$\bar{u}(\rho) = \frac{1}{\sqrt{\beta}} u(\tfrac{\rho}{\beta}) , \tag{2.7}$$

we can rewrite (2.6) as

$$< \frac{1}{r} > = \beta \int_0^\infty \frac{\bar{u}^2(\rho)}{\rho} \, d\rho \equiv \beta \, K_1 \, . \tag{2.8}$$

The quantity K_1, defined by Eq. (2.8) is tabulated for the pure linear case in Table 1 and plotted against a, in Figs. 4-6.

Also of interest is the value of the quantity $4\pi|\psi(0)|^2$.

The value of $\psi(\vec{r})$ at the origin is different from zero only for S-waves. In this case, we have

$$\begin{aligned} \psi(0) &= \frac{1}{\sqrt{4\pi}} \lim_{r \to 0} \frac{u(r)}{r} \\ &= \frac{\beta}{\sqrt{4\pi}} \lim_{\rho \to 0} \frac{u(\frac{\rho}{\beta})}{\rho} \\ &= \frac{\beta^{3/2}}{\sqrt{4\pi}} \lim_{\rho \to 0} \frac{\bar{u}(\rho)}{\rho} \, , \end{aligned} \tag{2.9}$$

where we used Eqs. (2.1) and (2.7). Defining

$$D = \lim_{\rho \to 0} \frac{\bar{u}(\rho)}{\rho} \tag{2.10}$$

we can write

$$\psi(0) = \frac{\beta^{3/2}}{\sqrt{4\pi}} \, D \, , \tag{2.11}$$

or

$$4\pi|\psi(0)|^2 = \beta^3 \, D^2 \, . \tag{2.12}$$

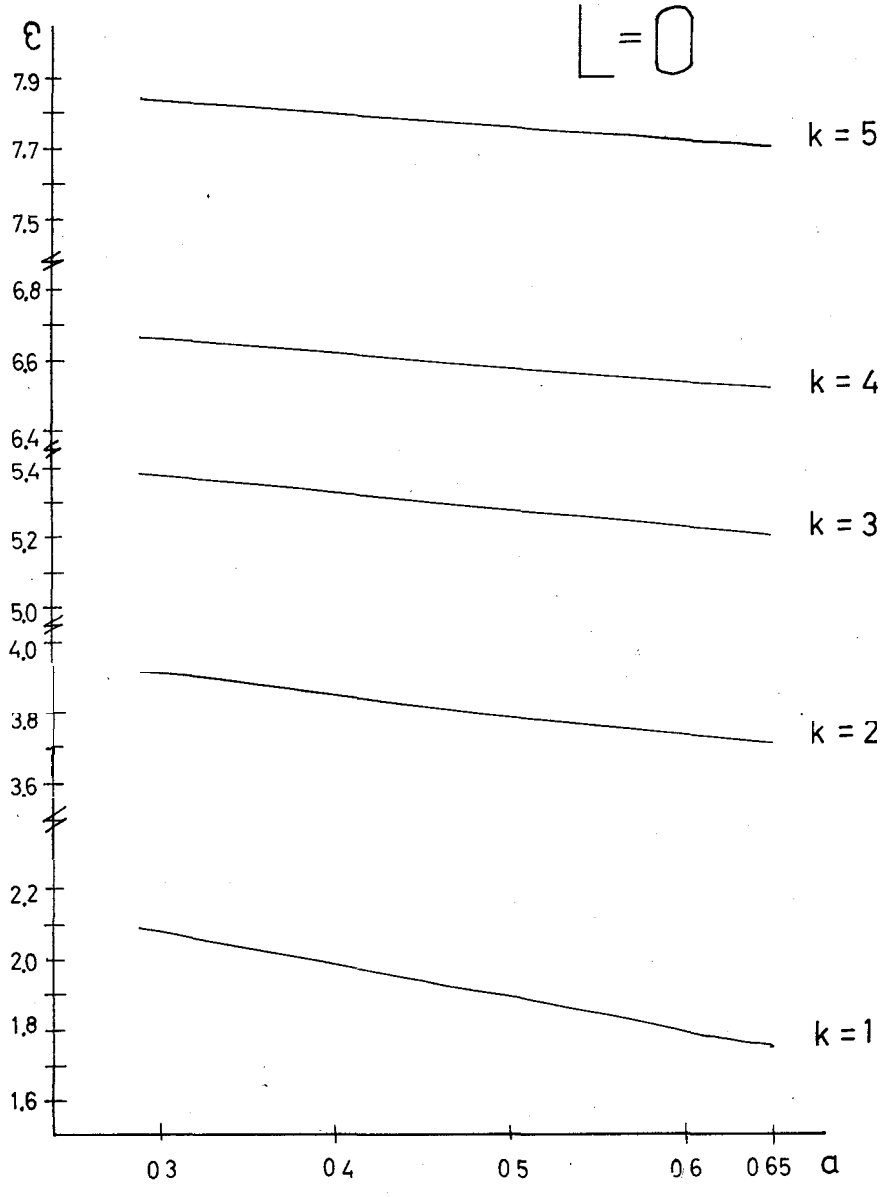

Fig. 1 - The eigenvalues ε_k of Eq. (2.4) as functions of the parameter a Eq. (2.5), for the first five excitations (k=1, ..., 5) in S-wave.

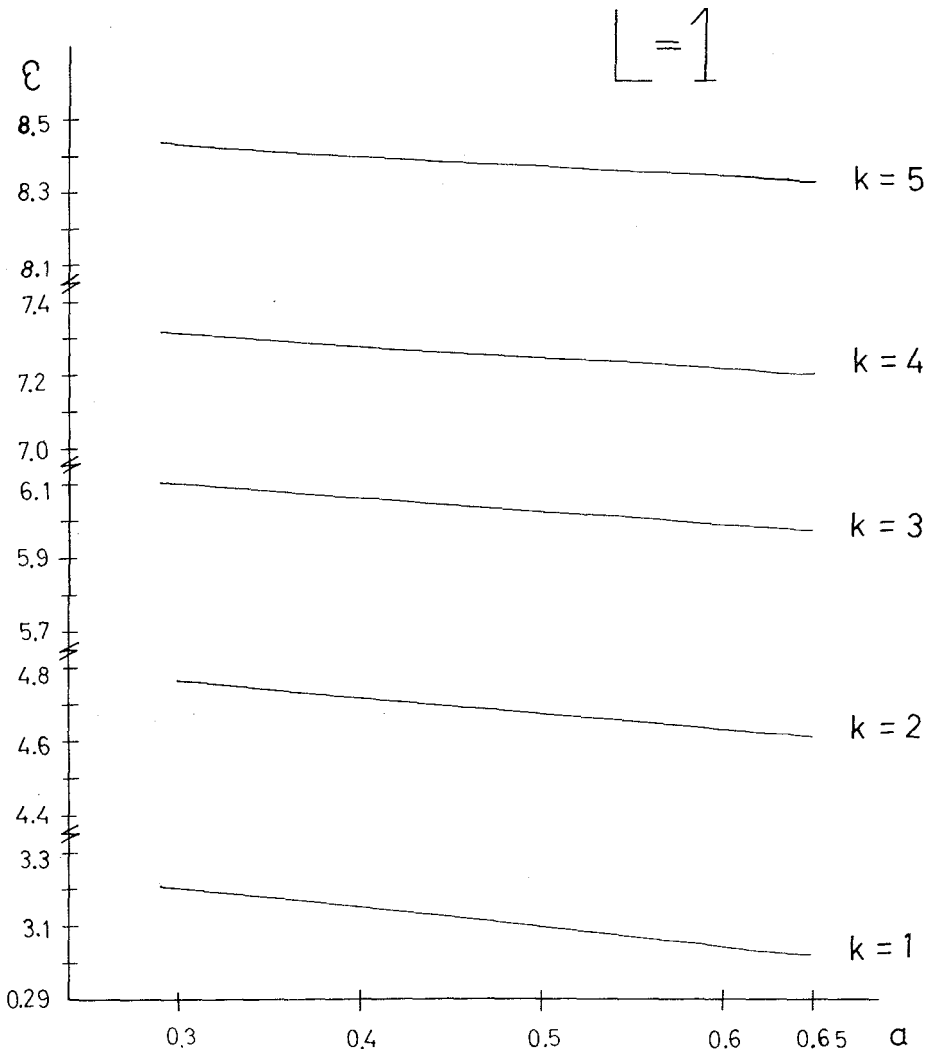

Fig.2 - The same as for Fig.1, in P-wave.

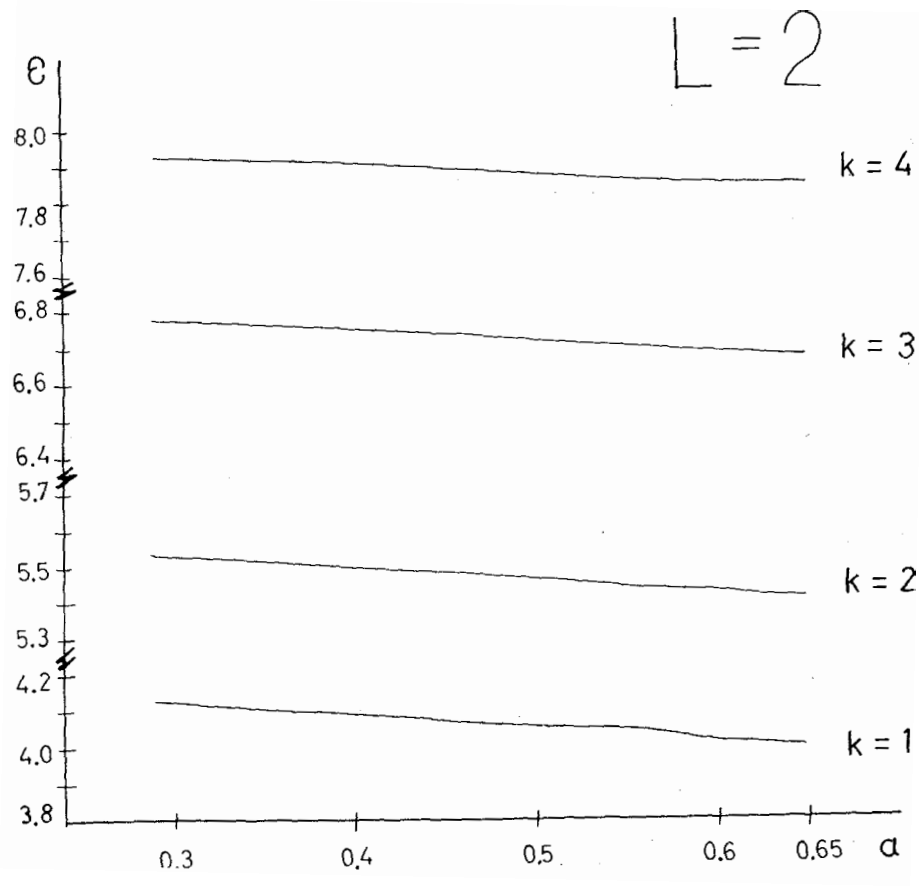

Fig.3 - The same as for Fig.1, in D-wave (k=1, 2, 3, 4).

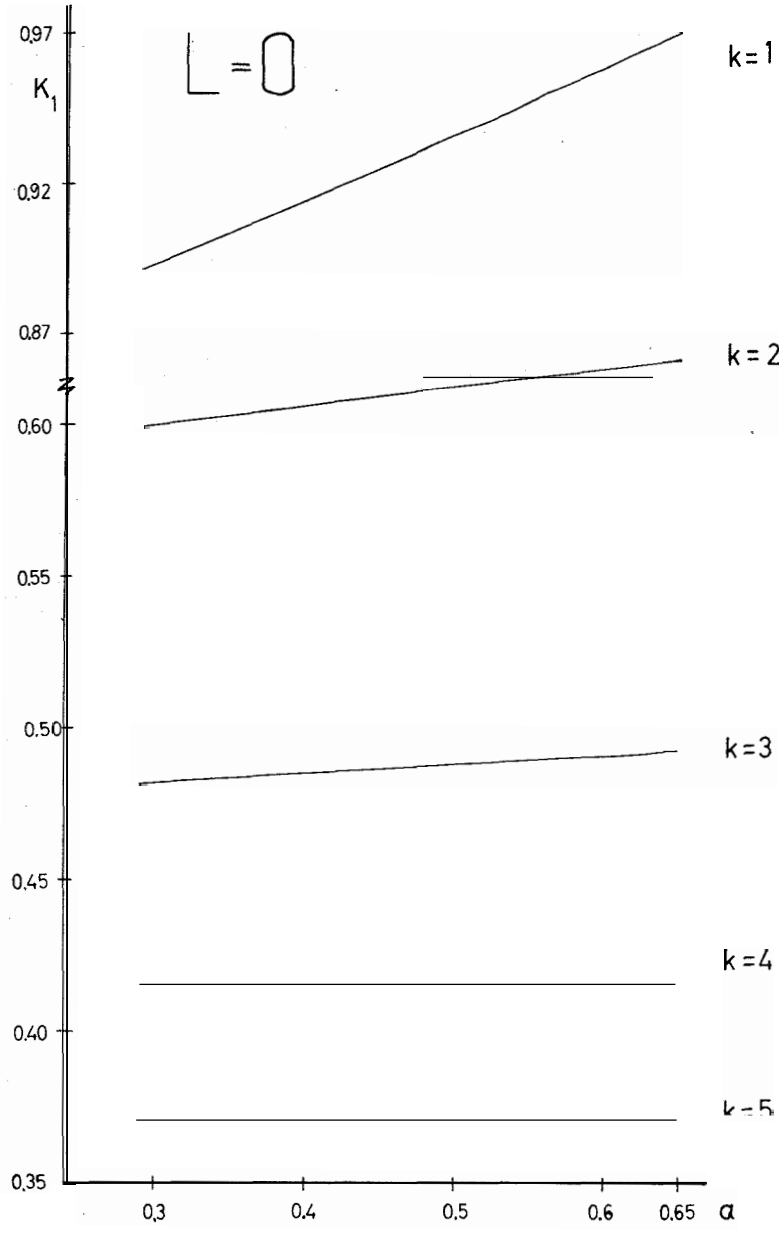

Fig.4 - The quantity K_1, defined by Eq.(2.8) as function of the parameter a, Eq.(2.5), for the first five excitations in S-wave.

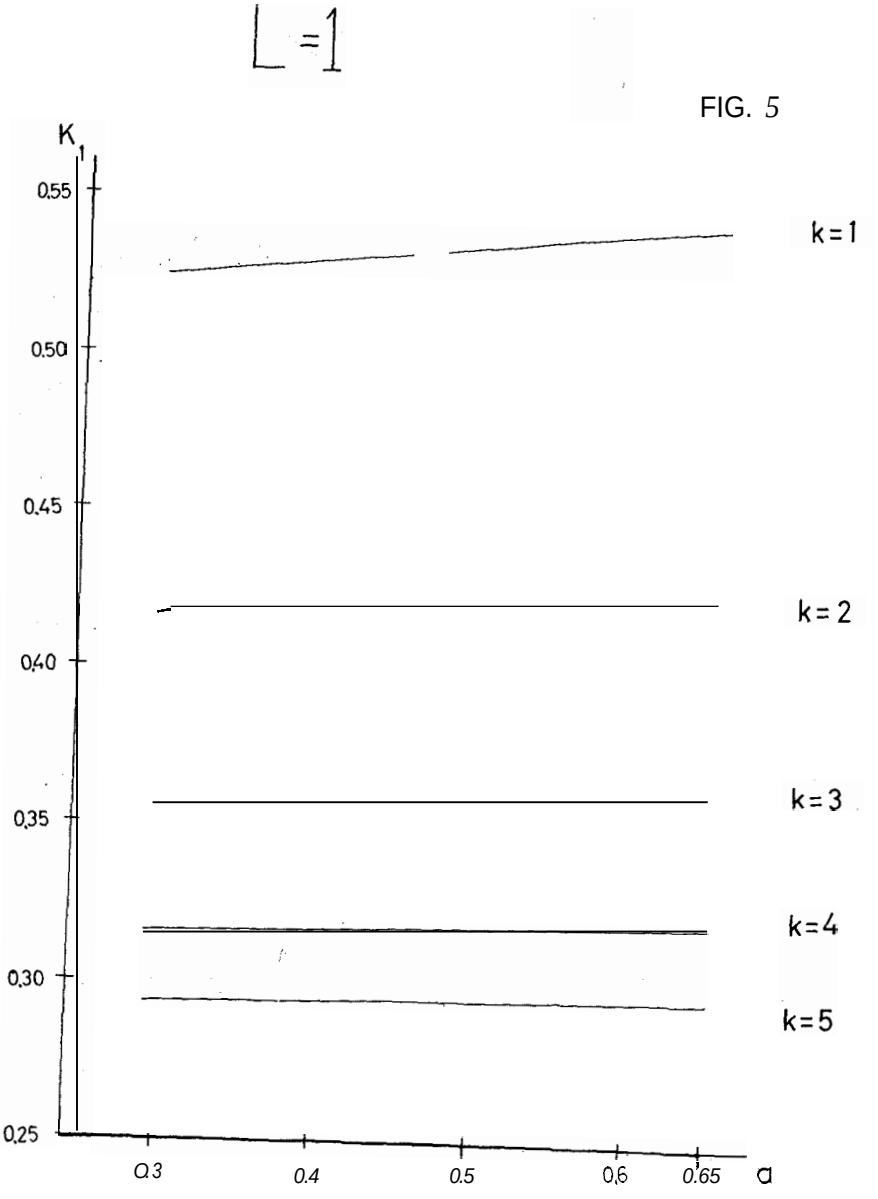

Fig.5 - The same as for Fig.4, in P-wave.

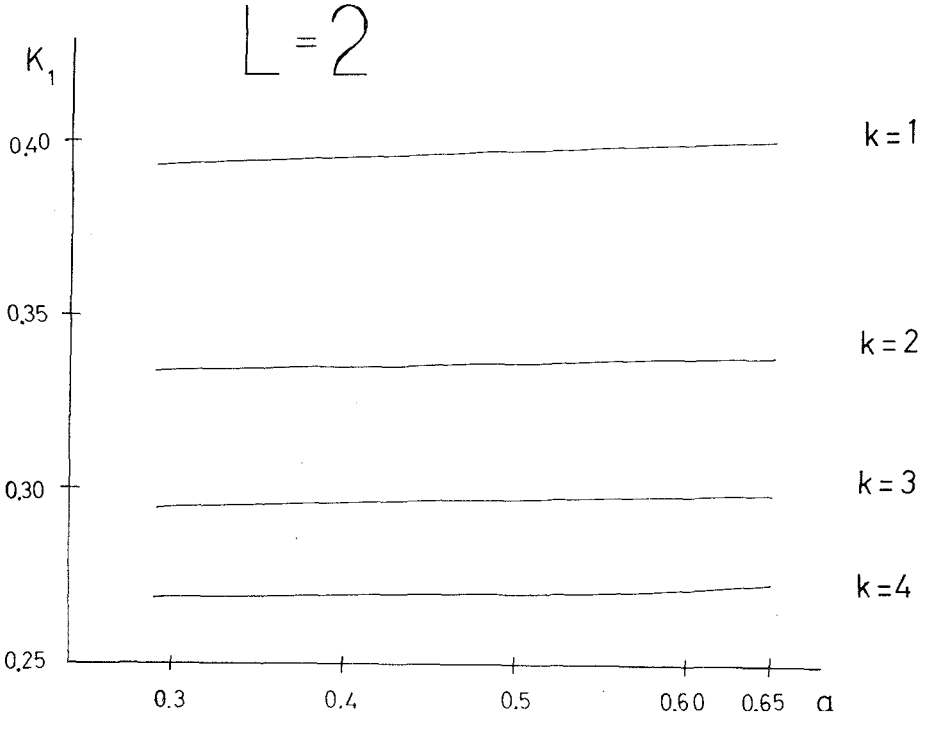

Fig.6 - The same as for Fig.4, in D-wave.

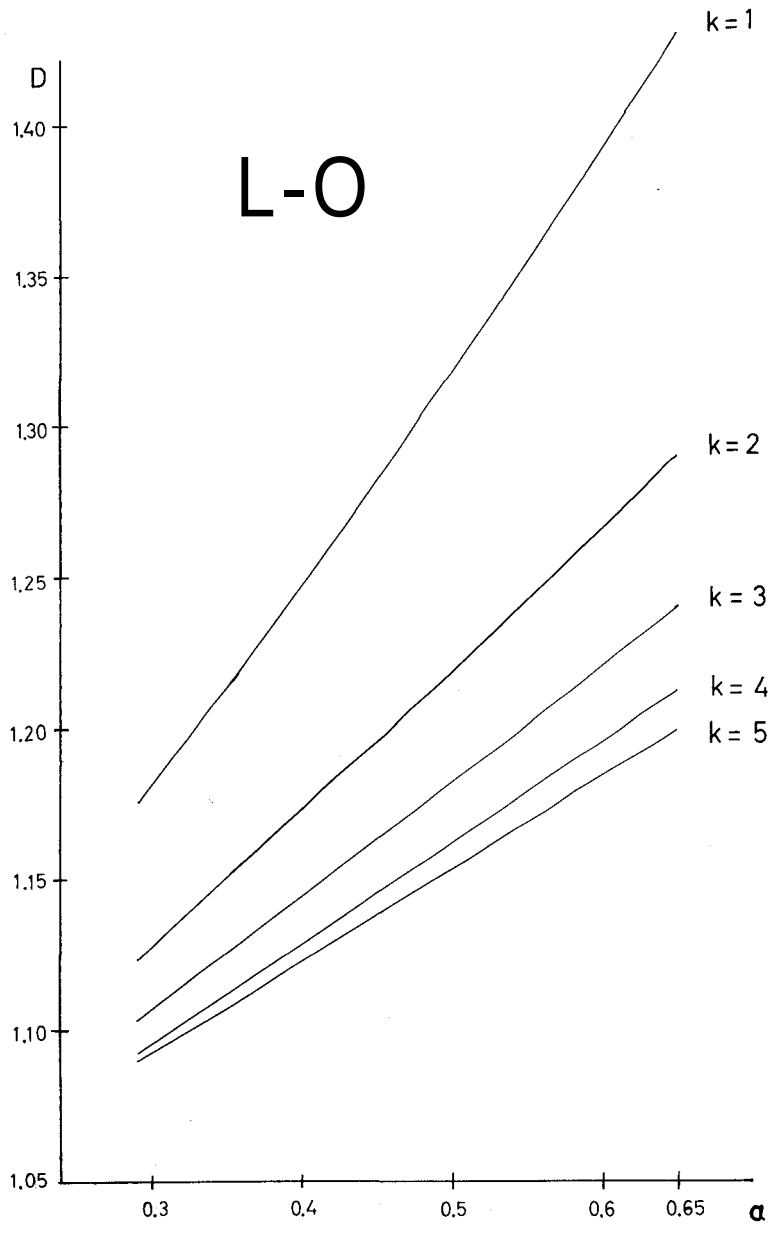

Fig.7 - The quantity D, defined by Eq. (2.10) as function of the parameter *a*, Eq. (2.5), for the first five excitations in s-wave.

We notice that , for the pure linear case, D=1, and we recover a well-known result[3].

We tabulate, for each S-wave excitation, the quantity D, from which the value of $4\pi|\psi(0)|^2$ can be obtained by means of Eq.(2.12).

After these elementary considerations, we pass to the presentation of the results. .

i) We first consider the case $\gamma = a = 0$, corresponding to a pure linear potential. The eigenvalues ε_k for the first few excitations in S-, P- and D-waves are tabulated in Table 1, together with the corresponding values of the quantity K_1, defined by Eq. (2.8).

ii) For the general case $\gamma \neq 0$, we present our results in graphical form, by plotting E (Eq.(2.4)) as a function of a (Eq. (2.5)), in the interval $0.29 \leq a \leq 0.65$, for the first excitations in S-, P- and D-waves (Figs. 1,2 and 3, respectively).

iii) We also plot, in Figs. 4,5 and 6, the quantity K_1 (Eq.(2.8)) as a function of the parameter a, for the first excitations in S-, P- and D-waves, respectively. Finally, in Fig.7, the quantity D (Eq.(2.10)) is plotted against a for the first excitations in S-wave.

3. A SHORT DISCUSSION OF THE RESULTS

We presented in the previous section, the results of our computer integration of Eq. (2.2) for the case $\gamma = a = 0$ and for values of the parameter a, defined by (2.5), in the interval $0.29 \leq a \leq 0.65$.

For the case α=0, the values obtained for ε_k corresponding to S-waves, agree with those known from the exact solution[3] [i.e., ε_k is minus the k^{th} zero of Ai(ρ)].

For the general case $\alpha \neq 0$, the ε_k are, in a very good approximation, linear functions of the parameter a, in the interval considered, as can

be seen from Figs. 1-3 (Note 7). We notice that, if we extrapolate the "straight lines" to the value $a=0$, we recover, in a good approximation, the values for ε_k given in Table 1. The slopes of the lines are seen to decrease (in absolute value) for increasing k (higher excitations). Hence, as is clear for instance from Fig.1 for S-waves, the lines become more and more parallel to the a-axis, tending to the asymptotic value $E_k \simeq [3\pi(4k-1)/8]^{2/3}$, as predicted from the pure linear case[3]. This means that the effect of the Coulomb-like term become less and less important for higher excitations.

The same linear-like behavior was obtained for the quantities K_1, Eq. (2.8). Hence, $\langle 1/r \rangle$ increases almost linearly with a, in the range considered. The slope of the "straight lines", for a given value of L, also decreases for higher excitations, as can be clearly seen from Figs. 4,5 and 6.

The values of the quantity $D(a)$ for S-wave excitations shown in Fig. 7 also exhibit a linear increase with a and also the slopes decrease for higher excitations. We finally notice that the lines in Fig. 7 converge to the point $D(0)=1$, as expected from the exact result in the pure linear case[3].

$L = 0$		$L = 1$		$L = 2$	
ε_k	K_1	ε_k	K_1	ε_k	K_1
2.33811	0.83487	3.36125	0.51223	4.24818	0.38763
4.08795	0.58217	4.88445	0.41102	5.62971	0.33066
5.52056	0.47231	6.20762	0.35289	6.86888	0.29321
6.78801	0.40820	7.40566	0.31443	8.00965	0.26770
7.94413	0.36565	8.51527	0.29256	9.09670	0.25266

Table 1 - The eigenvalues E_k of Eq. (2.4) and the quantity K_1, defined by Eq. (2.8) for the first five excitations ($k=1,\ldots,5$) in S-, P-and D-wave, in the pure linear case ($\gamma=0$).

REFERENCES AND NOTES

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7. The almost linear behavior of ε_k in the interval considered might be predicted from first order perturbation theory taking $-\gamma/r$ as a perturbation. Hence, $\varepsilon \simeq \varepsilon_{lin} - a \langle 1/\rho \rangle \approx \varepsilon_{lin} - a[K_1(0) + aK_1'(0)]$, where we used Eq. (2.8). It can be easily seen that the quadratic term is much smaller than the linear and therefore the almost linear behavior follows.