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Some Comments on Wilson's Renormalization Group Technique as Applied to Critical Phenomena

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Wilson's renormalization group recursion equations for a spin system are examined and it is found that: a) the convergence of the 4-spin correlation coupling constant (as function of R) is improved as one adds higher order correlations. For $d \leq 4$ this fact may make a solution of the coupling constant recursion formulae feasible; b) only to first order in η is renormalization possible in the case of ηS^6 theory.

As equações de recorrência do método do grupo de renormalização de Wilson para um sistema de spins são examinadas e encontra-se que: a) a convergência da constante de acoplamento da correlação de quatro spins (como função de R) melhora quando são acrescentadas correlações de ordem superior. Para $d \leq 4$, isso pode tornar praticável a solução das fórmulas de recorrência para a constante de acoplamento; b) no caso da teoria ηS^6, a renormalização só é possível até primeira ordem em η.

1. INTRODUCTION

Wilson (Ref.1) has initiated a powerful method to deal with critical phenomena in a spin system (or for that matter collective motion in many-body systems). Starting with Kadanoff's scaling idea (Ref.2), one divides the spin system into blocks of spins. One then introduces the effective Hamiltonian H_L that describes this spin block array, where L is the size of a block and it is larger than unity but smaller than the correlation length. Increasing the size of each of the blocks to $2L$,

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one obtains a new effective interaction, H_{2L}, which describes the new array of larger blocks. Continuing in this fashion, one then ends up with an array of spin blocks of size L_c and a corresponding effective Hamiltonian, H_{L_c}, such that $H_{L_c} = H_{L_c+1}$. This last property of H_{L_c} specifies the critical point associated with the phase transition. The idea behind the renormalization group is to arrive at this "fixed point" (Ref.1).

In this paper, we shall address ourselves to the convergence of the effective coupling constants that enter in the effective Hamiltonian H_L as one increases L. For this purpose we shall utilize, in Section 2, the renormalization group approximate recursion equations as derived by Wilson (Ref.1). Throughout we shall use the same notation as in Ref. 1.

2. THE CONVERGENCE PROBLEM

The recursion equations for a spin system, treated as a nonrelativistic field theory problem, with general interaction between the spins (two -, four -, six -, etc. - spin correlations) taken into account, are as given in Ref.1:

$$H_L = -\frac{1}{2}\kappa_L \int_{\vec{x}} \nabla_{\vec{x}} \left[S_L(\vec{x})\right]^2 - \int_{\vec{x}} P_L\left[S_L(\vec{x})\right],$$

$$Q_\ell(y) = \omega^{-1} P_\ell\left[\left(\frac{2\omega}{\kappa_\ell \rho_0}\right)^{1/2} y\right],$$

$$I_\ell(z) = \int_{-\infty}^{+\infty} dy \exp\{-\left[y^2 + Q_\ell(z+y)+Q_\ell(z-y)\right]\}, \tag{1}$$

$$Q_{\ell+1}(z) = -2^d\left[\{\ln I_\ell(2^{-d/2} y)\right] - \ln I_\ell(0)\},$$

$$\kappa_{\ell+1} = \kappa_\ell \frac{\alpha_\ell}{4},$$

where $L \equiv 2^R$ and it defines, through R, a particular momentum shell in the phase-space decomposition as outlined in Ref.1. Here α_ℓ is a spin scaling factor (renormalization constant); P_L is a polynomial in even powers of the spin field $S_L(\vec{x})$ (note that the order of P_L is not related to L); ω is related to the volume of the ℓ^{th} momentum shell and ρ_0 is the average value of the square of the wave number in the $\ell = 0$ shell.

The approximate solution of these equations for the Gaussian (two-spin correlation) and modified Gaussian (higher order correlations added) were discussed in Ref.1. In five dimensions and higher, the rapid convergence of the higher order correlation coupling constants as functions of R renders the results similar to the Gaussian model. However, for $d \leq 4$ the convergence of these coupling constants (in Ref. 1 only the 4-spin correlation was included) is not guaranteed especially if one deals only with two- and four-spin correlations. This fact comes through the appearance of the factor $\{(2\times2^{-d/2})^n\}\ 2^d$ in the recursion equations for the different coupling constants. Thus for a 4-spin correlation modified Gaussian, one has two coupling constants, namely, r_ℓ and λ_ℓ and the above numerical factors become (n refers to the order of correlation) 4 and 16×2^{-d} respectively. Thus for $d > 5$, the convergente of λ_ℓ ($\lambda_\ell \xrightarrow[\ell\to\infty]{} 0$) is guaranteed whereas for $d \leq 4$ it is not. This result, however, could be modified if one adds higher order correlations (6-spin, 8-spin etc).

For the $d = 4$ case although $16\times2^{-d} = 1$, $\lambda_\ell \xrightarrow[\ell\to\infty]{} \frac{1}{\ell}$, i.e., it is convergent, the fact that the equation determining λ_ℓ involved couplings to other correlations could speed up this convergence. For $d < 4$ ($4 = 3$ to be specific) it is the very presence of these couplings that may render λ_ℓ convergent. This naturally raises the question whether one can calculate the critical point (but not the critical exponents!) for $d < 4$ cases using just the recursion formulae of the coupling constants without resorting to an exact and thus lengthy numerical solution of the set of exact renormalization group equations (Ref.1) by a proper choice of the bare correlation coupling constants.

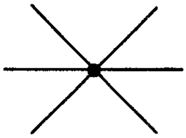
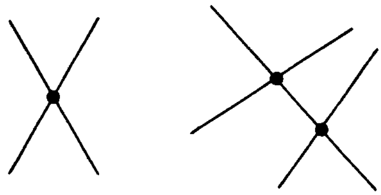


Fig.1 - Three-body scattering due to the **presence** of three-body forces only.



(a) **(b)**

Fig.2 - Figure 2a **corresponds** to two-body scattering, **(2b)** a three-body scattering term due to **two-body** forces.

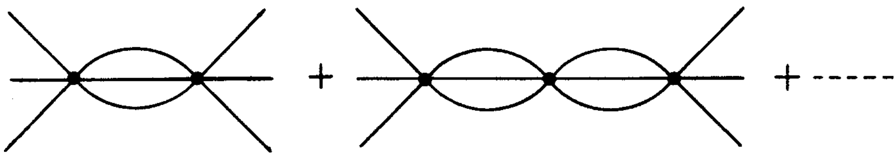


Fig. 3 - Higher-order corrections to the diagram **in** figure 1.

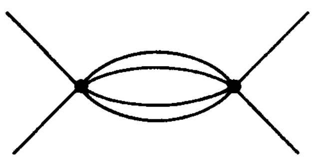


Fig. 4 - Two-body "**scattering**" in S^6- theory.

As an example of the type of recursion equations one might get for the higher order correlations, we have worked out the y^6 case, i.e., we have assumed the following form for $Q_0(y)$:

$$Q_0(y) = r_0 y^2 + \lambda_0 y^4 + \eta_0 y^6 \ .$$

Then recursion equations can be derived if one calculates $I_\ell(z)$ up to third order in λ_0 and first order in η_0.

$$r_{\ell+1} = 4\left(r_\ell + 3\, q_\ell\, \lambda_\ell - 9\, q_\ell^3\, \lambda_\ell^2 + \frac{45}{2}\eta_\ell\, q_\ell^2 - \frac{1980}{16}\, q_\ell^4\, \eta_\ell\, \lambda_\ell + \frac{9\times 321}{32}\, q_\ell^5\, \lambda_\ell^3\right),$$

$$\lambda_{\ell+1} = 16\times 2^{-d}\left(\lambda_\ell - 9\, q_\ell^2\, \lambda_\ell^2 + \frac{15}{2}\eta_\ell\, q_\ell - \frac{7\times 45}{4}\eta_\ell\, \lambda_\ell\, q_\ell^3 + \frac{9\times 111}{8}\, q_\ell^4\, \lambda_\ell^3\right), \qquad (2)$$

$$\eta_{\ell+1} = 64\times 2^{-2d}\left(\eta_\ell - 45\, q_\ell^2\, \eta_\ell\, \lambda_\ell + \frac{9\times 17}{2}\, q_\ell^3\, \lambda_\ell^3\right),$$

$$q_\ell \equiv \frac{1}{1+r_\ell} \ .$$

It is just the terms in η_ℓ, $\eta_\ell\lambda_\ell$, and λ_ℓ^3 that might lead to the convergence of $\lambda_{\ell+1}$ for $d < 4$. This can be generalized to the $A\, y^n$ case.

As we argue below, if none of the lower order correlations are missing, recursion equations can be derived if one calculates only up to first order in A: to see this we would like to make an analogy with the many particle scattering problem. It is hard to envisage two-particle scattering in a system where the only interparticle forces present are three-body forces (in a spin system, 6-spin correlations). However, it is obvious that the reverse is true as shown in Figures 1 and 2.

As is well-known, there is a big difference between Fig. 1 and Fig. 2b, in that the latter corresponds to the scattering of three-particles sequentially, where one deals with a propagator of an "almost-real" particle (it is a virtual particle but approaches its energy shell in the forward direction (Ref.4)). In contrast, Fig. 1 corresponds to the vertex function to zeroth-order in the coupling constant (Ref.3).

These diagrams are basically the only ones that one may deal with in discussing renormalization in a model spin system with a Hamiltonian of the form $aS^2 + bS^4$; thus diagram 1 is the first-order contribution from the S^6 term (perturbation is done in b and c where $a \gg b \gg c$) and diagrarn 2a are the first and second-order contributions from the S^4 term respectively. In the absence of diagram 2a, the only three-body scattering process possible is that of Fig.1 (as the zeroth order).Corrections to this process frorn higher-order contributions are shown in Fig.3. This however, does not exclude the possibility of a two-particle "scattering" in an S^6 theory, namely, via processes as the one shown in Fig.4. A scattering of this type, in the nonrelativistic regime, is not rneaningful. In a spin system, however, it corresponds to the first-order 4-spin vertex function (Ref.3). This however is a second-order diagram in the S^6 coupling constant and it is not allowable in deriving recursion equations of the type discussed (Ref.1).

What happens, then, in the absence of 2a is that recursion formulae, corresponding approximately to the renormalization group equations cannot be obtained. To check this one merely derives these formulaewhen $Q_0(y) = r_0y^2+\eta_0y^6$ (the notation is that of (Ref.1) and one then finds that to first order in η_0 (which is the highest order one can include) a two-body scattering term (4-spin correlations or, S^4) is generated in $Q_1(y)$ thus rendering the procedure of renormalization useless in getting the critical value of the S^2 constant, r_C. Obviously there is a way of describing critical phenomena in a S^6-theory, but is not the renormalization group method (Ref.3). What this implies too is that scaling breaks down since renormalizability is closely associated with scaling (Ref.1) (where scaling is taken to be a more general concept than conventionally treated). This, in a way, confirms Migdal's contention (Ref.3) .

REFERENCES

1. K.G. Wilson, Phys.Rev. B4, (1975) 3174; Ibid *B4*, (1971) 3184; K. G. Wilson and J. Kogut, Phys. Lett. C12 (1974) 75. For application of the recursion equations see, e.g., E.K. Ridel and F.J. Wegner, Phys. Rev. Lett. 29 (1972) 349; Phys. Rev. B7 (1973) 248.
2. L.P. Kadanoff, Physics 2, (1966) 263; L.P. Kadanoff et *al.*, Rev. Mod. Phys. 39, (1967) 395. .
3. A.A. Migdal; Sov. Phys. JETP 32, (1971) 552.
4. R. Dashen and S. Ma, J. Math. Phys. *11*, (1970) 1136.